Course on Principles of Communication Systems – Part 1
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Lecture 41
Module 7
Quantization, Mid-Rise Quantizer, PDF and Power of Quantization Noise, Quantization noise power versus Quantizer Resolution

Hello welcome to another module in this massive open online course. So we are looking at quantization and we have started looking at uniform quantization, alright. We have we looked at various aspects of quantization quantization converts analog sample value, alright maps an analog sample value to a set of discrete set of values, alright which can be represented by bits to be transmitted over a channel or for storage we have started looking at a uniform quantizer, uniform quantizer with an odd number of quantization levels odd number of quantization intervals which results in a mid-trade quantizer.

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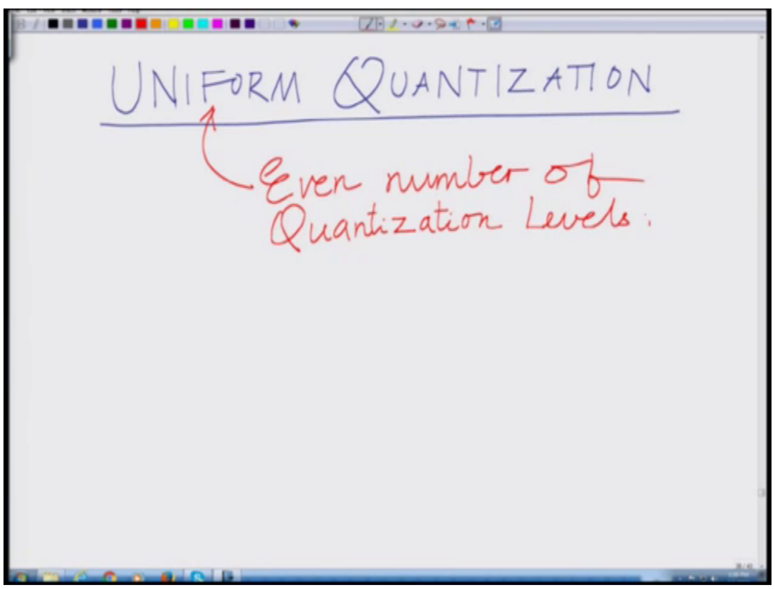

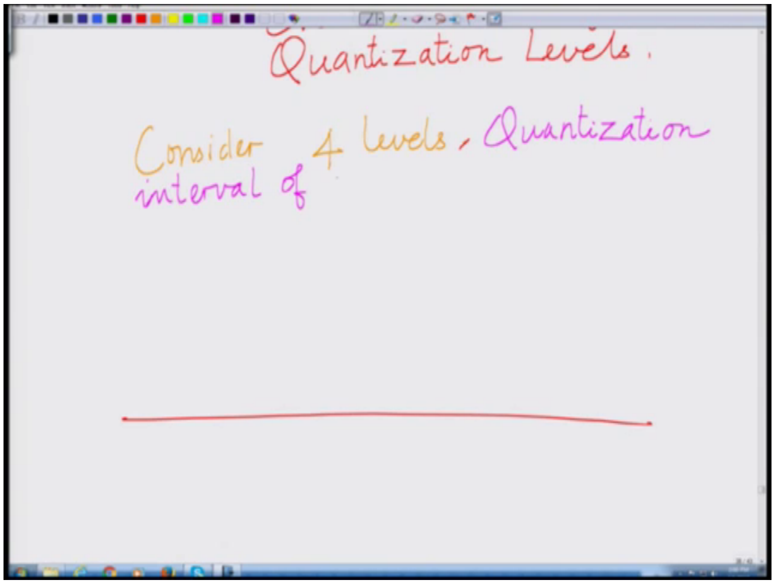

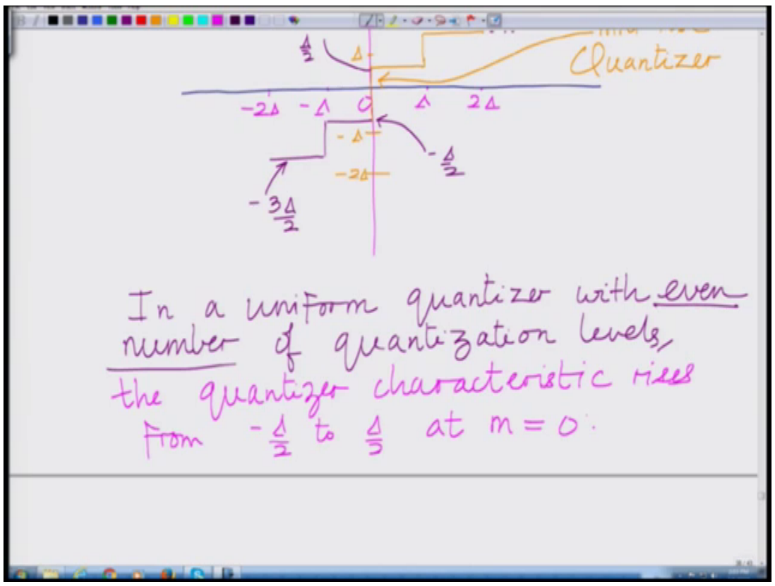

So let us now look at uniform quantizer with an even number of quantization levels, alright so we continuing to look at uniform quantizer uniform quantization previously we have seen uniform quantization with an odd number of quantization levels, so let us now look at uniform quantization with an even number of quantization levels, okay. And therefore uniform quantization with an even number of even number of quantization levels, okay let us say can we consider 4 quantization levels and therefore we will have a quantizer consider 4 quantization levels comma and quantization interval of width delta, okay let us write that quantization interval along with a quantization interval of width delta, okay.

So therefore we will have 4 quantization intervals each of width delta so naturally they will be from 0 to delta, delta to 2 delta minus delta to 0, minus 2 delta to minus delta these are the 4 quantization levels and (obi) and of course in 0 to delta if you are mapping if you are considering uniform quantization therefore you are mapping it to the midpoint that is delta by 2, okay. So you are mapping it to let say this is your delta, this is 2 delta, this is minus delta, this is minus 2 delta se we are mapping it to delta by 2 and then delta to 2 delta it we are mapping it to 3 delta by 2, okay.

So this is basically delta by 2, 3 delta by 2 and of course in minus delta to 0 you are mapping it to minus delta by 2, minus 2 delta to minus delta you are mapping it to minus 3 delta by 2, okay. So basically this is your delta by 2 this level is basically 3 delta by 2 this is minus delta by 2 and this is minus 3 delta by 2 alright. And therefore naturally you can see that in the middle, that is at if you look at it in the middle that is at 0 right, so if you look at it in the middle that is at 0 correct, we have the quantizer characteristic which is rising at 0, alright.

Therefore it rises from minus delta by 2 to delta by 2, therefore this is known as a mid-rise quantizer, alright. So we see when we have an even number of quantization levels the quantizer characteristic rises at delta equal to 0 from minus delta by 2 to delta by 2 in a uniform quntizer with an even number of quantization levels this is termed as a midrise quantizer. So in a uniform quantizer with an even number of quantization levels in a uniform with even number that is with an even number of quantization levels the quantizer characteristic rises from minus delta by 2 to delta by 2 at x equal 0 or basically the input equal to 0 or at m equal to 0.

This is therefore termed as a mid-rise quantizer, that is this is therefore this is therefore termed as a mid-rise this is therefore termed as a mid-rise quantizer. This is therefore a mid-rise quantizer it rises in the middle the middle is at m equal to 0 it rises from minus delta by 2 to delta by 2 therefore this is termed as mid-rise quantizer. So they have two different kinds of uniform quantizers one with an odd number of quantization levels which is flat in the middle at delta (equa) at m equal to 0 that is the mid trade quantizer uniform quantizer with an even number of quantization levels in which the characteristic rises from minus delta by 2 to delta by 2 at m equal to 0 this is termed as the mid-rise quantizer, okay.

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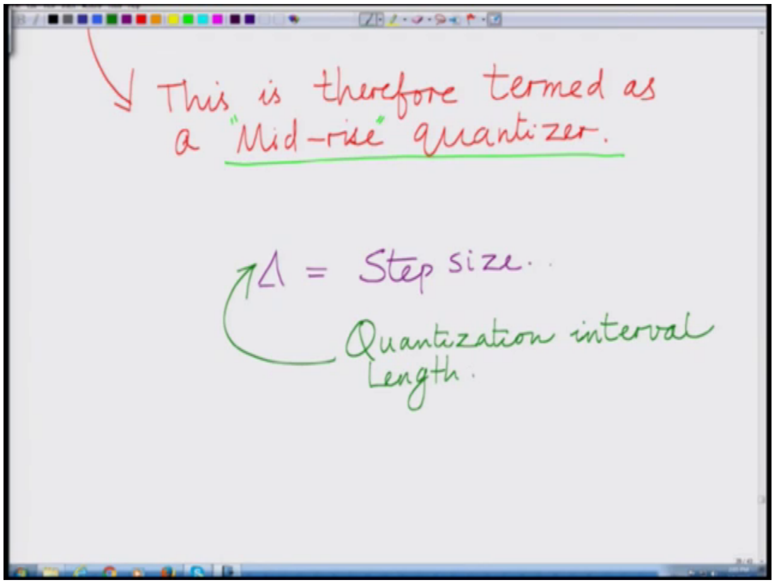

Now, let us also look at some other aspects of quantization, for instance we have delta equal to the step size that is also the basically this is also your quantization interval this is also known as the step size, step size of quantization this is also known as the quantization interval, okay.

So you can think of this as the so quantization interval length or basically length of the quantization intervals step size is a basically the step from one quantization interval to other or basically one quantization level to another, alright. So this is the step size delta and this is uniform in a uniform quantizer, okay.

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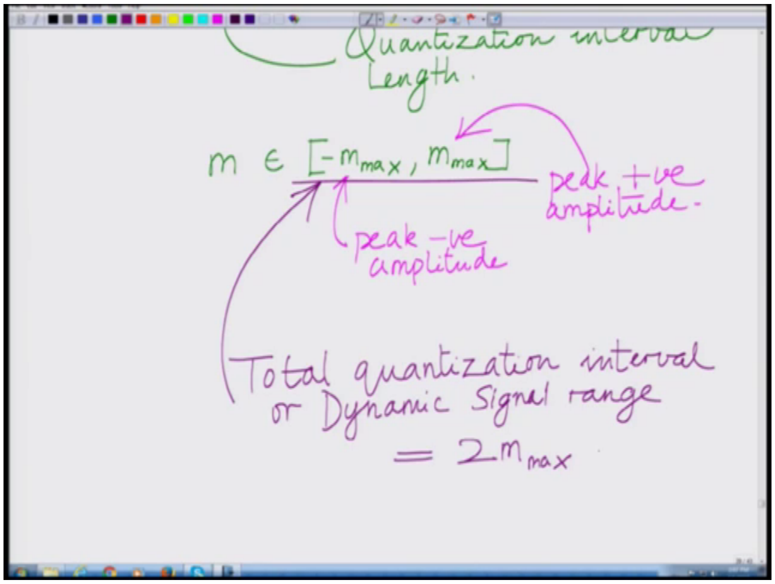

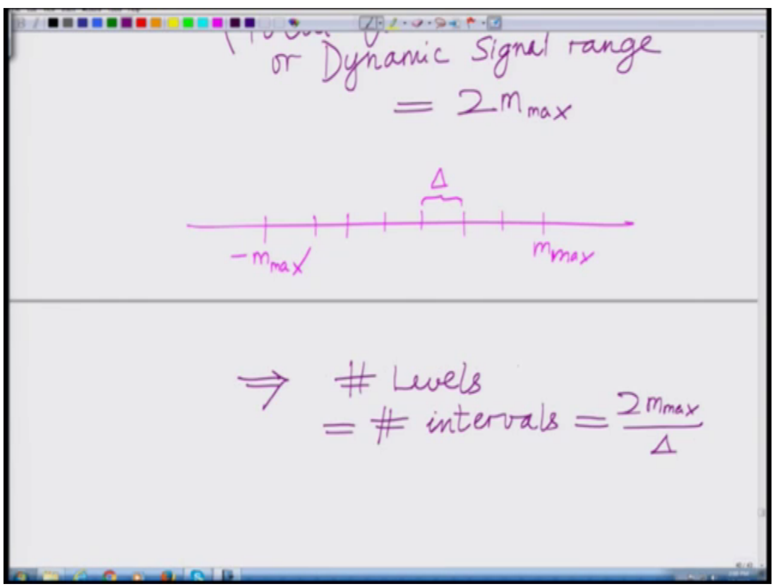

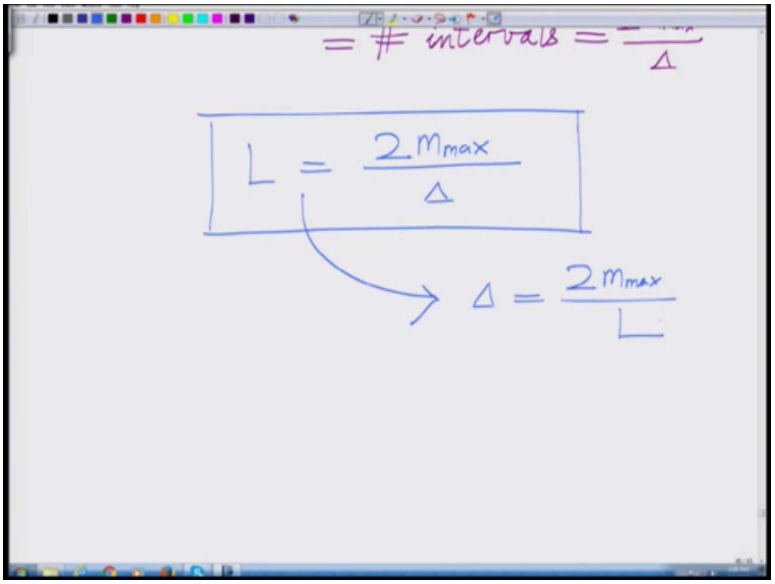

Now, let us say the number of levels equals now let us say signal m spans from, well let us say spans from minus m max to m max so this is maximum amplitude or peak negative amplitude let us put it that way and we are assuming to it symmetric and this is your peak positive amplitude.

m max is your peak positive amplitude and let us say so therefore total interval therefore total quantization interval therefore total width of quantization interval or we can say the dynamic signal range so total quantization interval or basically your dynamic signal range that is equal to twice m max, okay m max minus minus m max that is twice m max that is the total dynamic signal range that is the width, alright. Now this is being quantized with quantization interval that is quantization steps as delta which means so now so therefore you have twice m max this is your dynamic range, so you have minus m max over here you have m max over here and you are dividing this into uniform intervals of width delta this is your step size.

So naturally this implies that the number of levels that is number of quantization levels equals number of intervals equals 2 m max divided by delta. So let us say number of levels L, okay so let us say the number of quantization levels L which are element of this discrete set we have L equals 2 m max by delta for this uniforms quantizer, okay. So delta (ek) so L equal (2 m) 2 max by delta or alternatively this implies also that this also means that delta equals 2 m max divided by L that is the width of the interval that is a step size is twice m max the dynamic range divided by the total number of quantization levels, okay that is the width of each uniform interval, okay.

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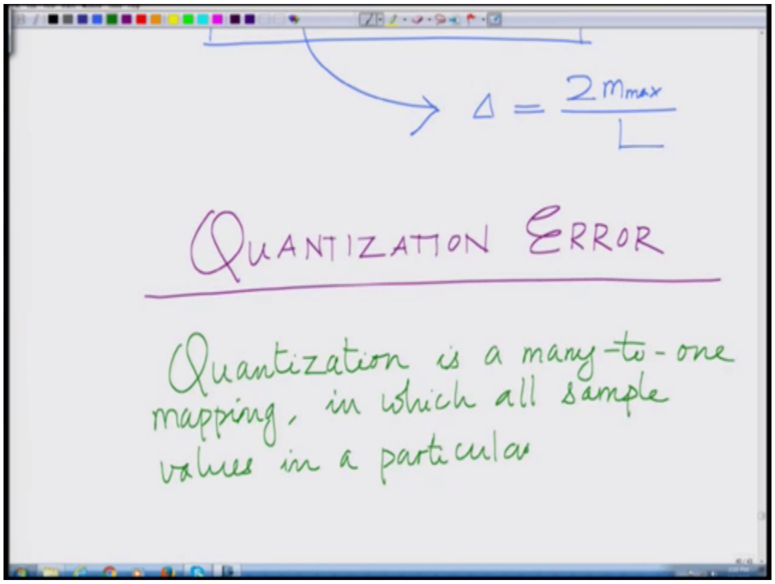

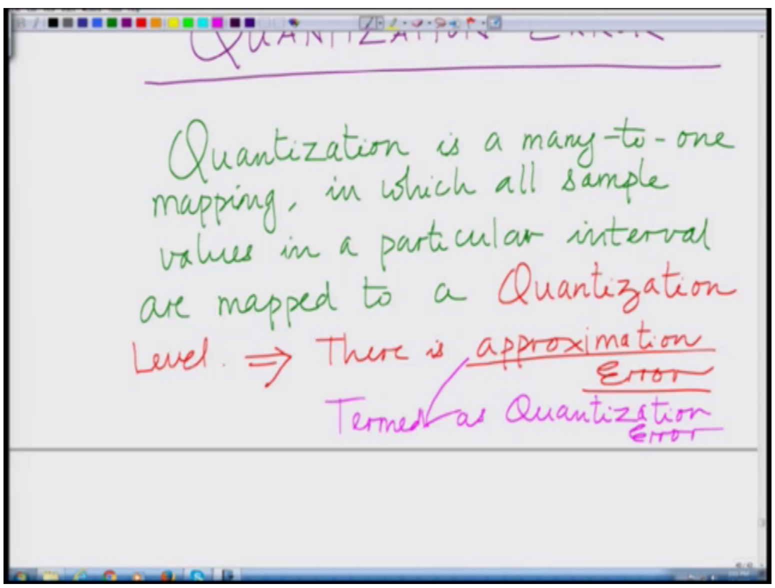

Now, let us look at the notion of quantization error, okay and this is very important matric which determines the performance of the quantization of the quantizer which is a quantization error. So we have said that all samples in a particular quantization interval, correct a particular quantization interval are mapped to a particular quantization level which are many to one mapping, alright. Which means there are several values which are mapped to an approximate quantization level, therefore there is going to be quantization level, alright.

So quantization is a many to one mapping in which all sample values in a particular interval are mapped to a quantization level, this implies that there is error. When there is an approximation error quantization is nothing but approximation, correct? Which is basically your (quantiza) termed as the (quanti) there is an approximation error in this approximation which is termed as this is termed as quantization error, alright because we are mapping a large number of samples approximating them by a single quantized value that leads to an error that is termed as quantization error.

And the context of uniform quantization we already seen that for instance we look at the quantization interval 0 to delta for the previously considered quantizer with even number of quantization levels all the samples in 0 sample values in 0 to delta are mapped to minus minus delta by 2. So naturally if the sample is 0 we will have 0 to delta are mapped to delta by 2, so naturally if the sample is 0 we are going to have a quantization error, right? Which is basically the quantization level minus m delta by 2 minus 0 which is delta by 2 or if it is delta which is mapped to delta by 2 then we are going to have a quantization error of delta minus 2 to minus delta which is minus delta by 2.

So basically you have a quantization error with spans, right? For every particular interval which spans from minus (del) minus delta by 2 to delta by 2.

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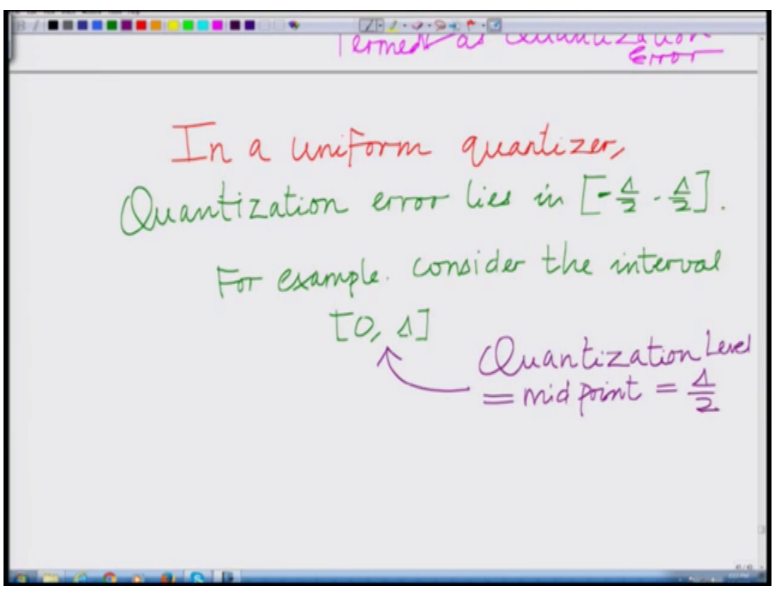

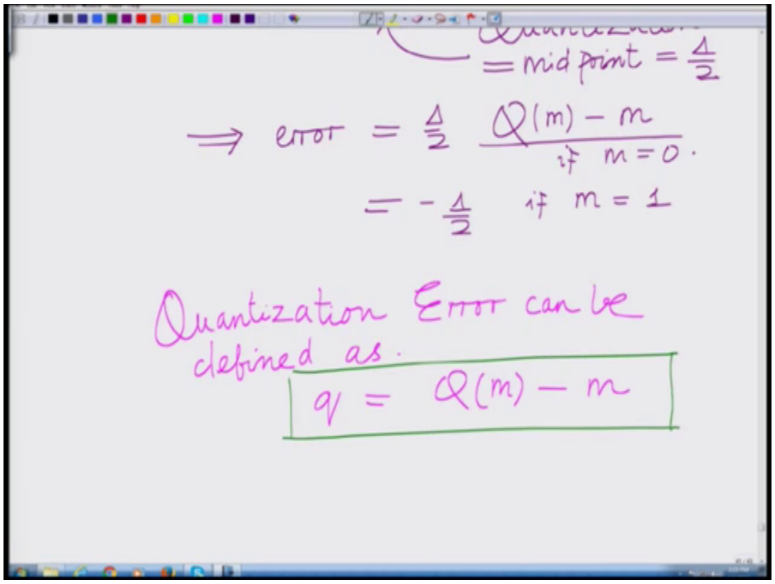

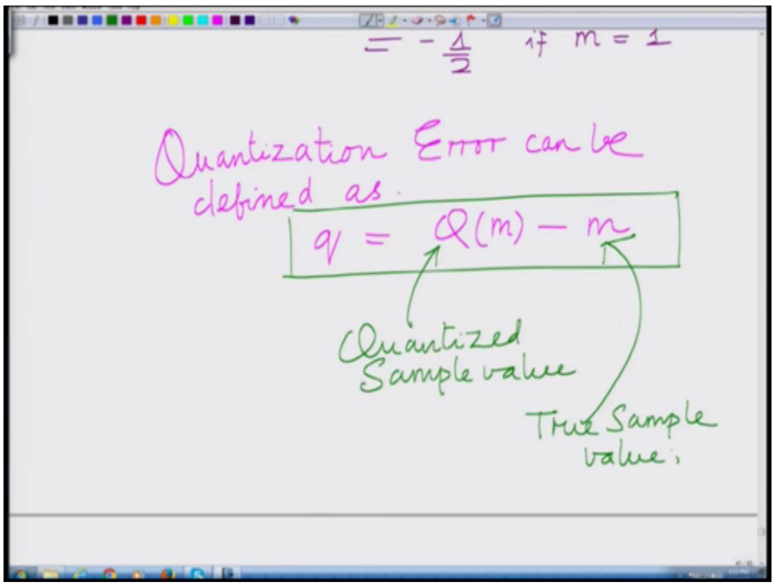

So in a uniform quantizer, in a uniform quantization, quantization error the quantization lies in minus delta by 2 to delta by 2, correct? For example, consider the interval 0 to delta, okay the quantization level corresponding quantization level equal to the equals the mid-point that is equal to delta by 2 implies the error equals well delta by 2 where we say error equals let us say we do not the quantization the error equals delta by 2 that is your Q(m) minus m.

So Q(m) is delta by 2 minus m is 0 if m equal to 0 and error equal to minus delta by 2, so this is our definition of error quantization error Q(m) minus m if m equal to 0 is delta by 2 is minus delta by 2 if m equals 1 or basically close to 1 it is minus delta by 2. So the quantization error can be defined as your basically q equals Q(m) minus m that is your quantization error, that is

Q(m) is a quantized sample value, m is the actual true sample value m is the true sample value, so Q(m) is the is your quantized sample value, this is your m equals your true sample value, m equals your true sample value, okay.

And therefore we have seen that q lies between minus delta by 2 to delta by 2 in particular a simple model can be developed by assuming the quantization error to be uniformly distributed between minus delta by 2 to delta by (())(17:38). So to develop a simple model for the because this quantization error is random, right? Depending on what is the actual sample value it can lie anywhere between minus delta by 2 to delta by 2 but the probability with which it takes a value in any particular range in minus delta by 2 to delta by 2 is a random quantity, alright?

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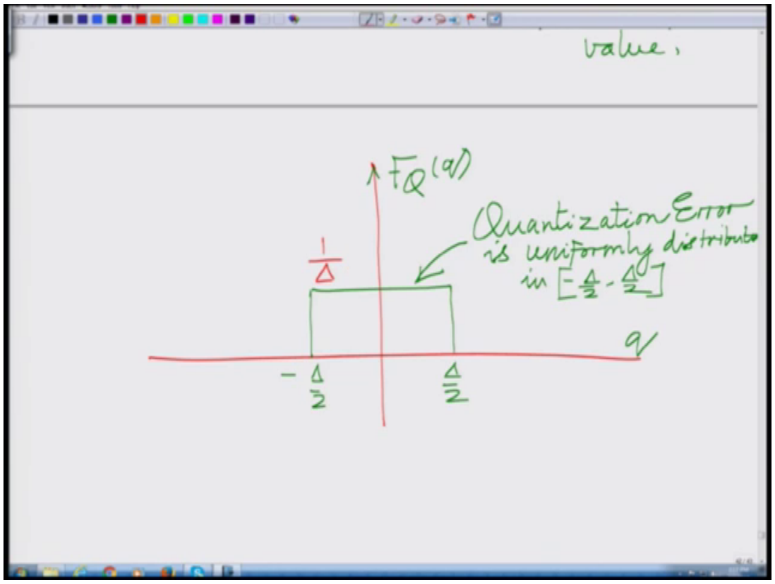

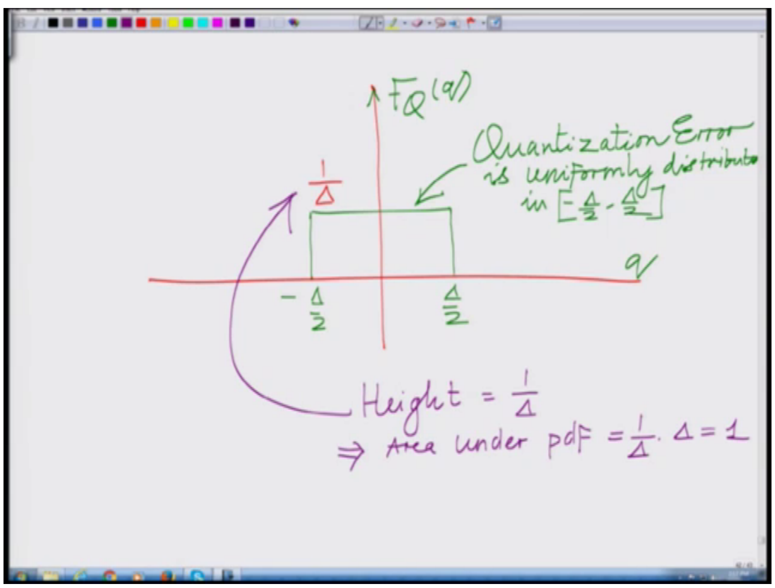

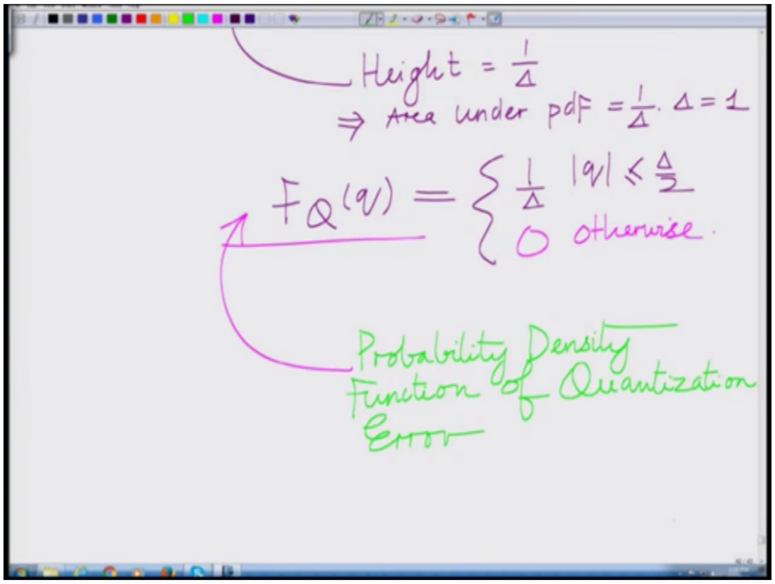

So to make things simple, we are assuming that this quantization error is uniformly distributed in minus delta by 2 to delta by 2. So to make things simple we are assuming quantization error is uniformly distributed in minus delta by 2 to delta by 2 error let us say this is your q, this is your q, this is the (prob) probability density function (f) so quantization error is uniformly distributed in minus delta by 2 to delta by 2 the interval minus delta by 2 to delta by 2.

Naturally, the height of this uniform pulse that is this uniform probability density function is 1 over delta, that is 1 over the length of the interval, okay height is equal to 1 over delta, okay that is a probability density function 1 over the length of the interval, okay. So therefore if you look at the area under this curve 1 over delta into delta it is 1, so height equal to 1 over delta area under PDF so height equal to 1 over delta equal to 1 over delta implies area under PDF that is total probability equal to 1 over delta into delta equal to 1, which is the property that the PDF is satisfied.

And therefore and by the way fQ(q) this is equal to 1 magnitude of q less than or equal to delta by 2 sorry this is equal to 1 by delta magnitude of question less than delta by 2 0 otherwise, okay and fQ(q) is this is termed as the probability density function probability density function this is your probability density function of the quantization error, this is the probability density function of the quantization error, okay. So it is an uniform probability density function and minus delta by 2 to delta by 2 which means it is height of 1 over delta in minus delta by 2 to delta by 2, alright and fQ(q) is basically the probability density function which is the uniform probability density function, okay.

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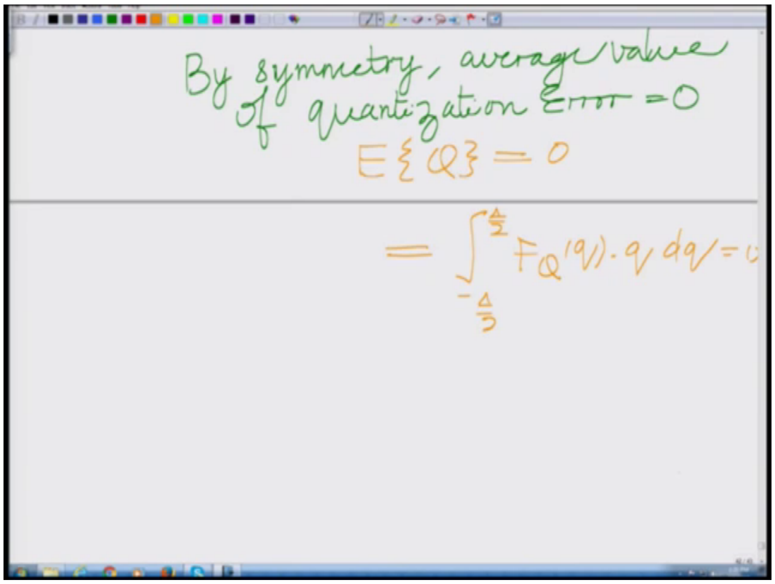

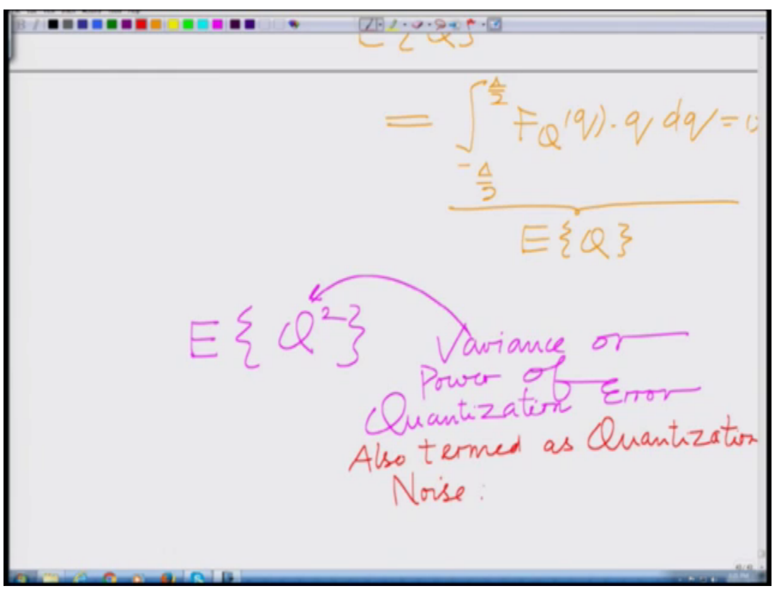

Now, let us see what is a power or the variance of the now obviously if you can look at if you look at the mean the mean now one can see of course from symmetry that is minus delta by 2 to delta by 2 is uniformly distributed.

Therefore by symmetry mean is equal to 0 the average value of quantization error is equal to 0 by symmetry average value that is if you look at this quantity expected value of Q that is 0 which is also defined as I think you can check this that is minus delta by 2 to delta by 2 fQ(q) times q dq that is equal to 0 because this integral correct this integral is symmetric correct? So fQ(q) minus delta by 2 so this is basically the mean this is your expected value of the quantization error Q.

Now we want to find expected value of Q square what is the variance or what is the power of this quantization error of quantization noise, so we need to find the variance or power of quantization error also termed as quantization noise also termed as quantization and the reason for this being that this quantization causes an error which act as a noise in the signal, so if you have a signal and quantized version of the signal quantization version version has error, that results in noise, right? So that is quantization noise.

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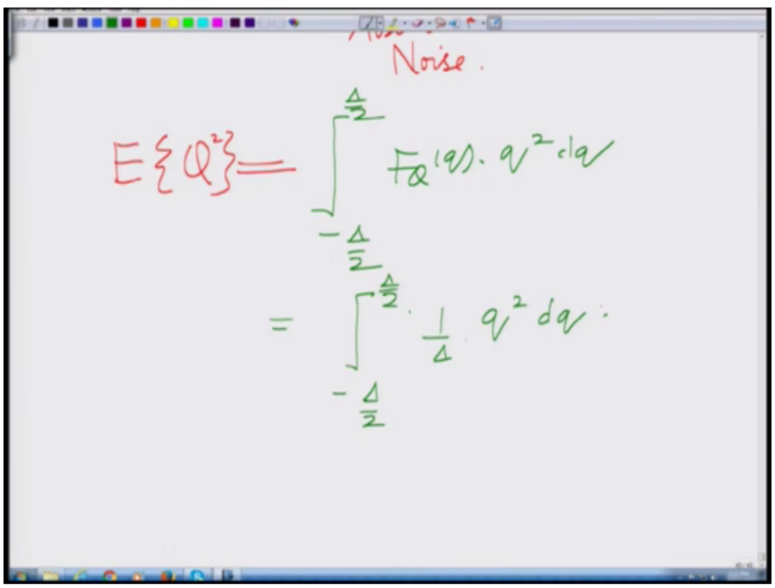

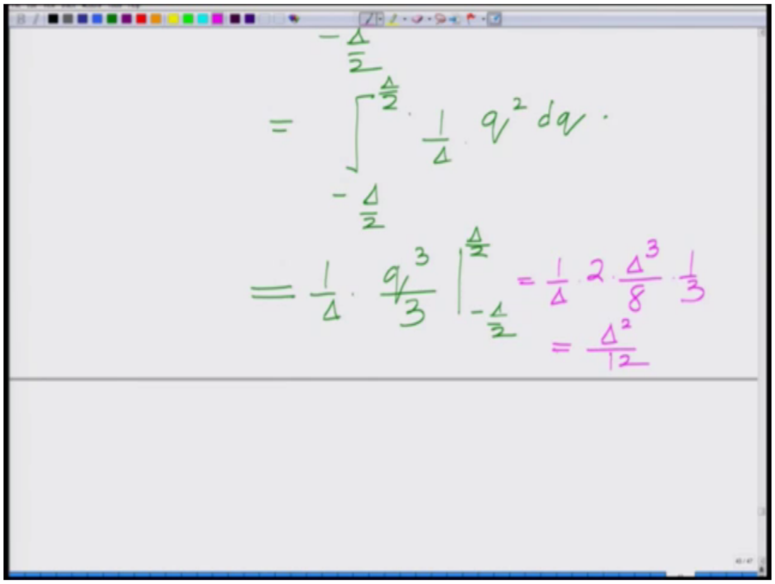

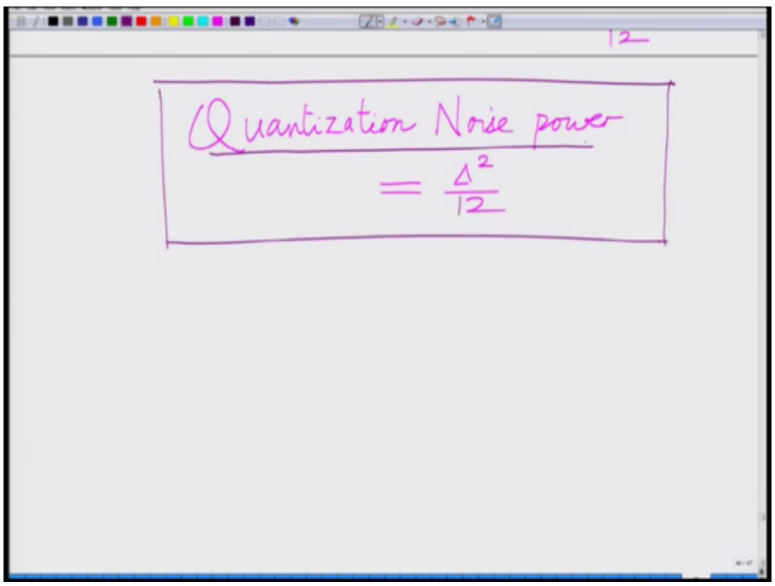

Now what is a variance of this quantization noise? That is basically your expected q square that is basically integral minus delta by 2 to delta by 2 that is defined as fQ(q) q square dq which is equal to integral minus delta by 2 to delta by 2 1 over delta q square dq that is equal to basically that is equal to that is integral minus delta by 2 to delta by 2 1 over delta q square dq which is equal to 1 over delta integral minus delta by 2 to delta by 2 q square dq that is q cube divided by 3 evaluated from minus delta by 2 to delta by 2 and that is equal to 1 over delta times twice delta cube twice delta cube divided by 8 into 1 by 3 which is basically equal to delta square divided by 2.

So the power of quantization so quantization noise power with this uniform distributed quantization noise, so quantization noise power equals delta square by 12 so this is our assumption of quantization noise power that is delta square by 12, okay. And now so this is your quantization noise power so are have quantization noise, right? The quantization error which causes quantization noise also termed as quantization noise that has a noise power delta square by (())(25:33).

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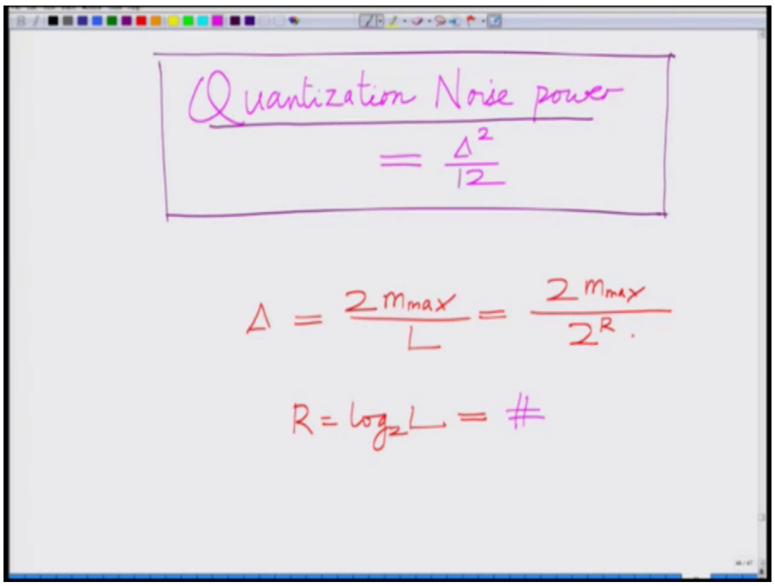
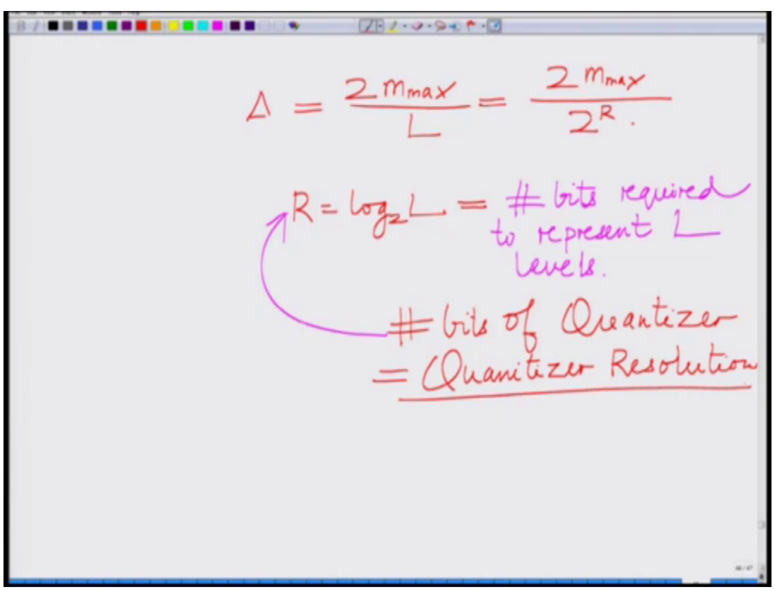




= Quantizer Resolution

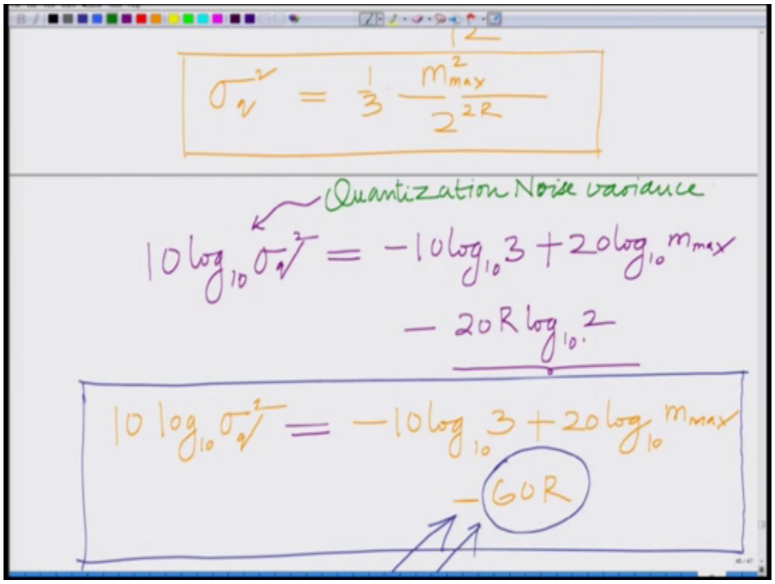
$$\sigma_q^2 = \frac{\Delta^2}{12}$$

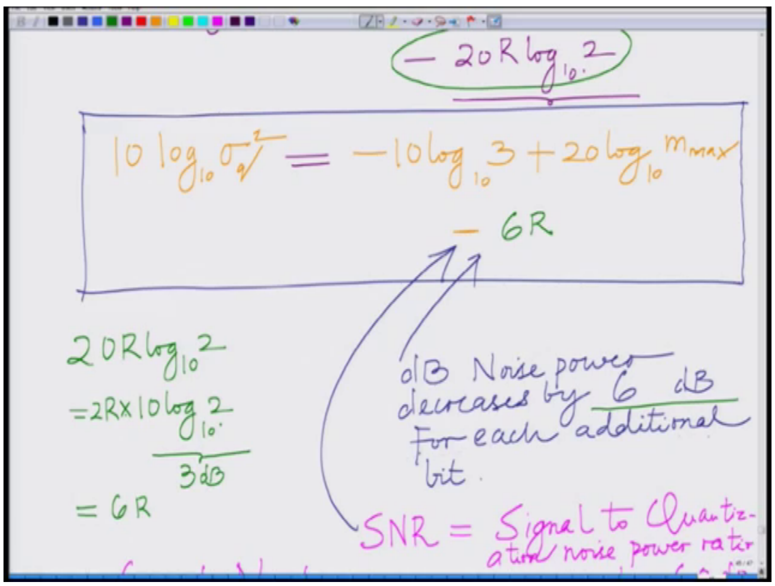
$$= \frac{(2 m_{max})^2 / 2^{2R}}{12}$$

$$\boxed{\sigma_q = \frac{1}{3} \frac{m_{max}^2}{2^{2R}}}$$

$$10 \log_{10} \sigma_q^2 = -10 \log_{10} 3 + 20 \log_{10} m_{max} - 20 R \log_{10} 2$$

$$\boxed{10 \log_{10} \sigma_q^2 = -10 \log_{10} 3 + 20 \log_{10} m_{max} - 60R}$$





Now, let us look at the let us again reformulate delta delta equals well 2 m max divided by L which is equal to well 2 m max divided by 2 to the power of R where R equal to log2 to the base L equals number of bits alright because if you have L levels for instance let say we have L equal to 8 levels we need 3 bits, alright? All possible combinations of 3 bits 3 binary bits information bits gives us 2 to the power of 3 that is 8 levels, alright? So if you have L levels we need logL to the base 2 that is R bits.

So we need R equal to logL to the base 2 this is equal to number of bits required to represent L levels to represent the L levels R equal to logL to the base 2, okay. So R equals number of bits of the quantized they are also termed as quantizer revolution number of bits of quantizer that is number of bits output where each sample also termed as equal to quantizer resolution, okay so this is also termed as a resolution of the quantizer, okay.

Now we also know that sigma q square that is a quantization error sigma q square equals delta square by well 12 but we have delta equals 2 m max by 2 to the power of R, so this is substituting that value for delta this is 2 m max square divided by 2 to the power of R divided by well 12 which is equal to well 1 by 3 m max square divided by 2 to the power to 2 R, so quantization error variance sigma q square I think we can also write this here that is your quantization noise power is sigma say quantization noise power is sigma q square equal to delta square by 12, now we have sigma q square is 1 over 3 m max where m max is the peak amplitude of the signal sample m max square divided by 2 to the power of 2 R.

And therefore now if we take the consider it in db quantization noise variance 10 log 10 sigma square that is equal to well minus 10 log 10 3 plus well 20 log 10 20 log 10 to the base m max minus 2 R that is 20 R minus 20 R minus 2 log 10 minus 20 R log 10 to the base 10 10 log to the base minus 20 R log minus 20 R or plus rather 2 to the power of L equals 2 to the power of 2 R so divided by 2 to the power of 2 R so sigma square minus 20 yeah 20 R log 10 to the base 2.

Now log 10 to the base 2 we know this is equal to 3 so this is basically you can write this as minus 10 log 3 to the base 10 plus 20 log m max to the base 10 correct minus 20 into 3 that is basically minus 20 log 10 to the base 2 to the base 10 is approximately 3 so minus 20 into 3 minus 60 R this is your noise power. So if you look at this this is the noise power so you can see the noise power decreases by so you can see that the noise power, alright the dB noise power decreases by 60 R that is for each additional noise bit it decreases by 60 dB so dB noise power.

So let us look at the quantization noise variance that is sigma q square is 1 over 3 m max square divided by 2 R. Now let us look at the quantization noise this is the quantization noise variance in dB this is the quantization noise variance that is 10 log 10 sigma q square we have said that is equal to minus 10 log 10 that is 10 log 10 sigma q square equals minus 10 log to the base 10 3 plus 20 log to the base 10 m max minus 20 log 20 R alright, so when you take log of this factor 2 to the power of minus 2 to the power of 2 R 2 R in the denominator which becomes 2 to the power of minus 2 R so that gives us minus 20 R log to the base 10.

Now 10 log 2 to the base 10, so I can write this as basically 20 log R to the base 20 log 20 R log to the base 20 (lo) so 20 R log 2 to the base 10 equals well that equals 2 R into 10 log 2 to the base 10 into 10 log 2 to the base 10 and 10 log 2 to the base 10 is 3 dB, okay. So this is basically your 3 dB so this is equal to 6 times R, so we have here a factor minus 6 times R now therefore the noise variance you can see that this noise variance sigma square dB noise variance decreases by a factor of 6 dB, alright decreases by a factor of 6 dB for each additional bit.

That is when R increases by 1, alright minus 6 R which is the noise variance decreases by a factor of decreases by a minus 6. That is a noise variance decreases by minus 6, therefore the signal to noise power ratio improves by a factor of 6 dB.

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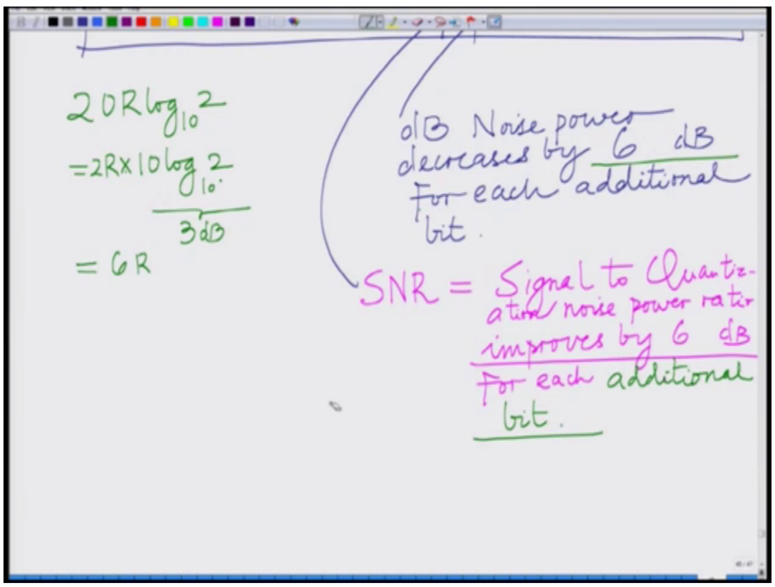

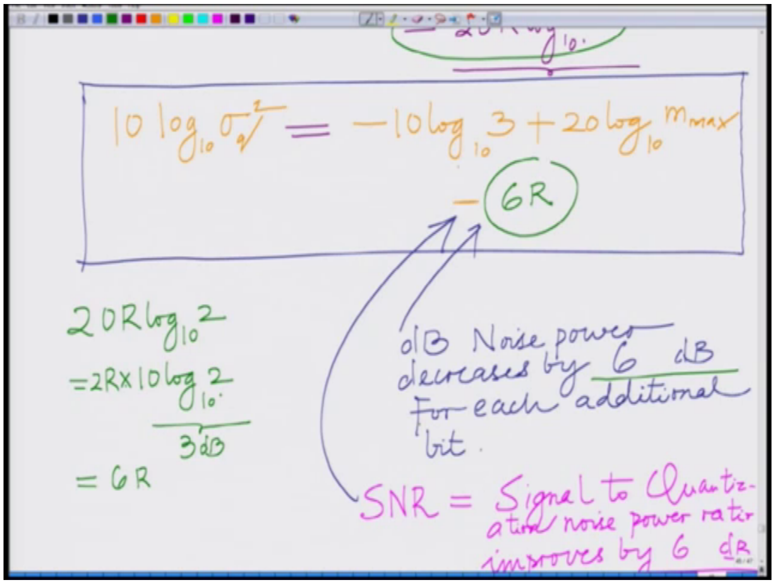

So SNR signal to noise power signal to quantization noise power ratio improves by a factor of 6 dB for each bit or for each additional bit for each additional and therefore what we have is that looking at this expression, correct? When looking at this expression and you look at this term minus 6 R you will realize that for each additional bit the dB noise variance dB quantization noise variance decreases by minus 6 dB, therefore the signal to quantization noise power ratio improves by a factor of 6 dB for each additional bit, that is for each additional bit in the quantizer that is as the quantizer resolution increases by 1 bit, alright.