Principles of Communication- Part I
Professor Aditya K. Jagannathan
Department of Electrical Engineering
Indian Institute of Technology Kanpur
Module No 6
Lecture 37
Ideal Impulse Train Sampling, Reconstruction of Original Signal from Samples, Sinc Interpolation

Hello welcome to another module in this massive open online course, so we are looking at the sample sampling of analog signals. In this module let us start looking at also the reconstruction aspect that is from the (recons) that is from the samples of the sampled signal how can we reconstruct the original signal?

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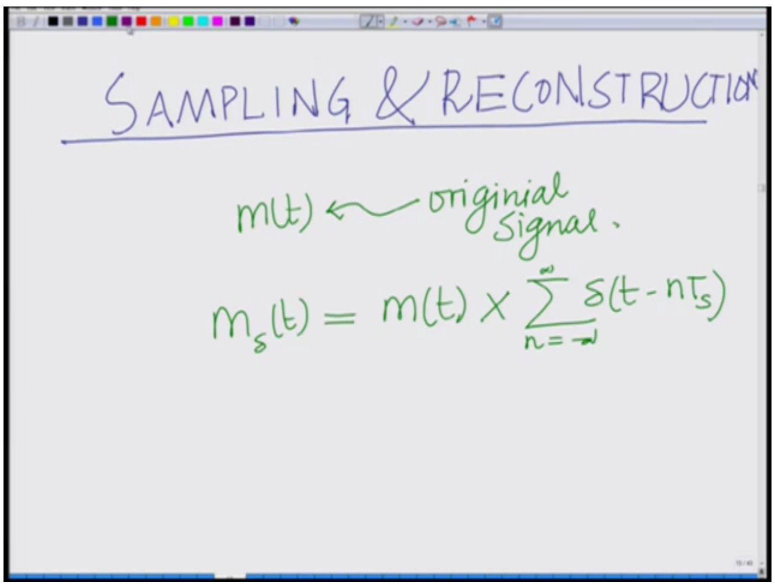

Alright, so let us look at so we are looking at sampling let us also start looking at reconstruction that is how do you reconstruct? That is how do you reconstruct? That is if you want to reconstruct the original samples how do you reconstruct the original? How do you reconstruct the original signal from the sample? So we want to look at the reconstruction aspects, so sampling and reconstruction now we have already seen that the sampled signal if m(t) is the original signal this is the your original signal, now the sampled signal m delta t is given by multiplying this by an impulse train that is m(t) times summation n equal to minus infinity to infinity Delta T minus nTs this is your impulse train, correct?

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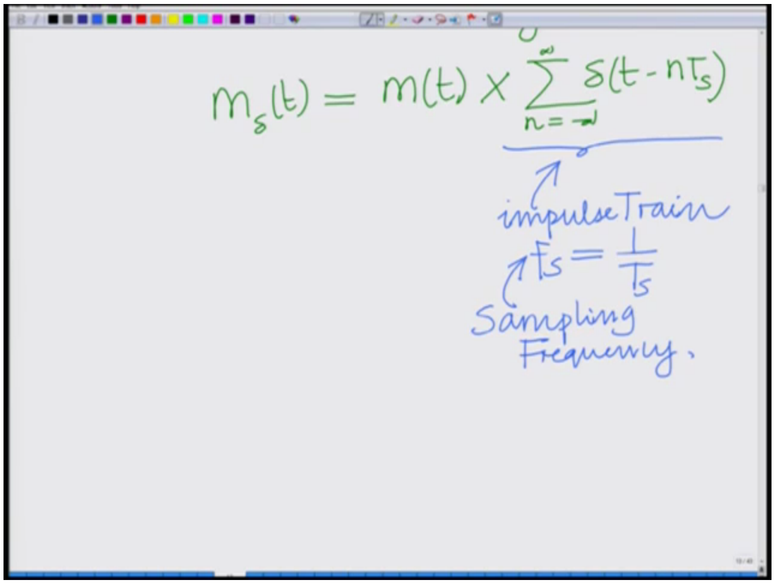

So this is your impulse train which is a series of impulses at every multiple of the sampling time interval Ts, okay where a sampling frequency Fs equals 1 over Ts, okay. Fs is the sampling frequency this is our sampling frequency and now we are multiplying m(t) by this (impul) (tra) impulse train that is m(t) times summation n equal to minus infinity to infinity delta t minus nTs and that each impulse at nTs, right? Picks the value of the original signal m at nTs.

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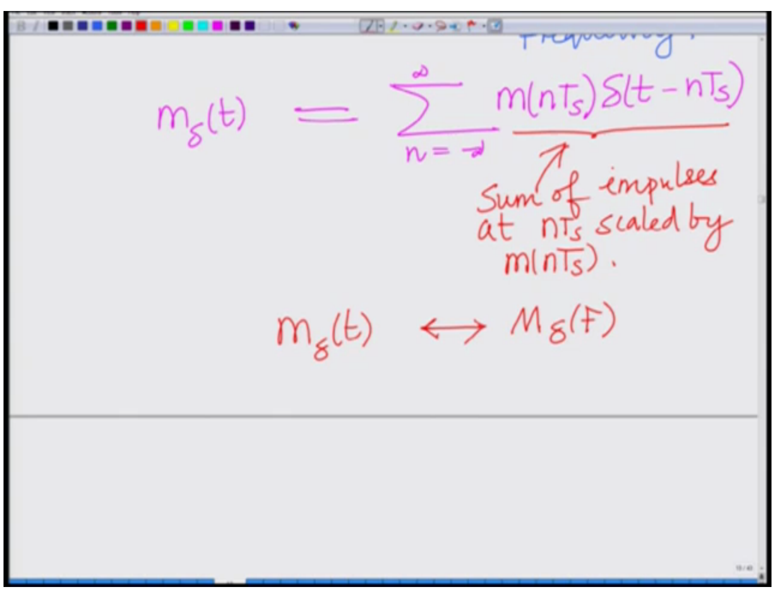

So we have from this from this impulse sampling what we are going to get is your m delta T equals summation n equal to minus infinity to infinity m of nTs times delta t minus nTs that is impulse at nTs scaled by the function nTs, so impulse so what is this? Summation of impulses at nTs scaled by impulse at nTs scaled by the function that is m at function at nTs. Now also when we looked at the frequency (respo) we looked at the Fourier transform of this that is if m delta t has a Fourier transform has a Fourier transform m delta F.

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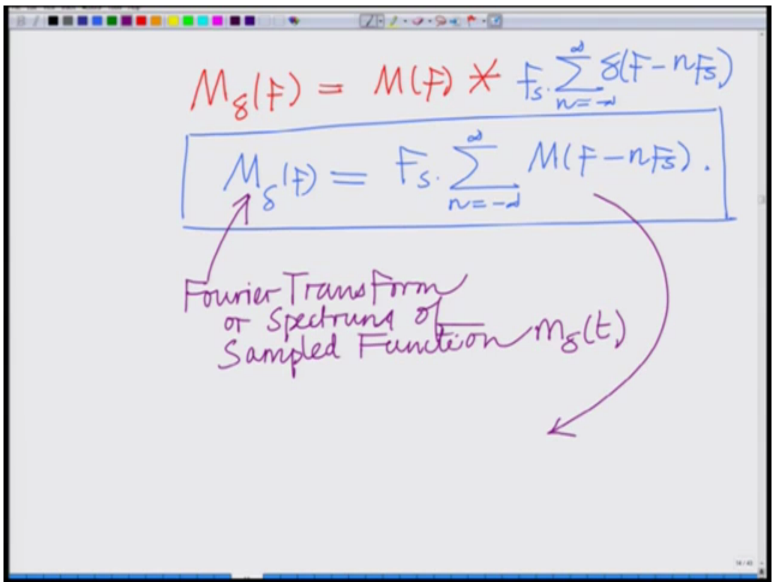

We said that m delta F is basically M of F convolved with since in the time domain we are multiplying in the frequency domain we have to convolved with the frequency response of the impulse train in the frequency for your transfer of the impulse train we have already seen is summation is Fs summation n equal to minus infinity to infinity delta F minus nFs which is equal to Fs summation n equal to minus infinity to infinity M of F minus Fs.

So this is your M delta F this is the Fourier transform of the sample function, okay. This is the Fourier transform of your sampled function or basically the spectrum the spectrum of the sampled function m delta t. And if you write is what you will see? Is that this is Fs so Fs is a scaling factor and MF the original spectrum MF shifted to each nFs that is MF minus nFs shifted every multiple of the sampling frequency and the sum of all these spectral copies.

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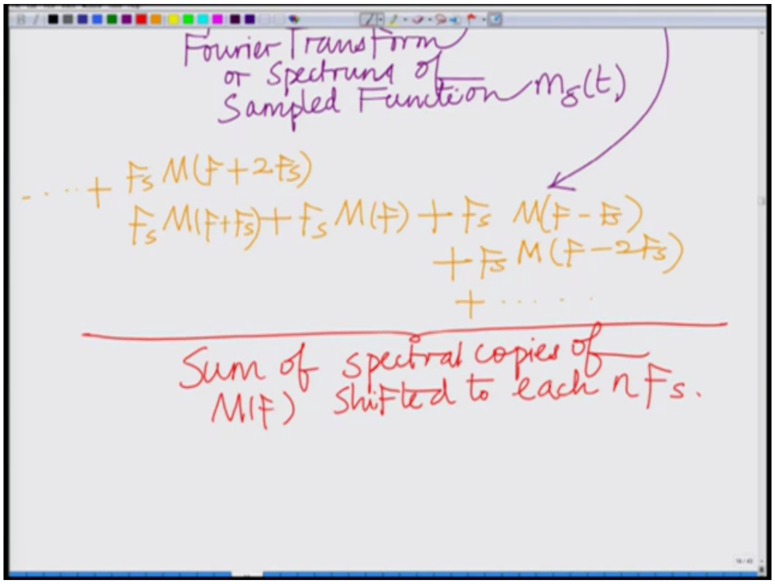

So what you have is very interesting you have Fs times M of F Fs times M of well, F minus Fs plus Fs times M of F minus 2Fs plus so on plus also Fs that is this is copy shifted to minus Fs, Fs M of F plus Fs, Fs M of F plus Fs plus Fs M of F plus 2Fs there is copy shifted to minus 2 and so on, alright. So this is basically the sum of spectral copies sum of spectral copies shifted to of MF shifted to each nFs where n shifted to each nFs where n is any integer Fs is the sampling frequency. So therefore this has a very beautiful frequency domain interpretation if you look at it in the frequency domain what we have is you have your original spectrum.

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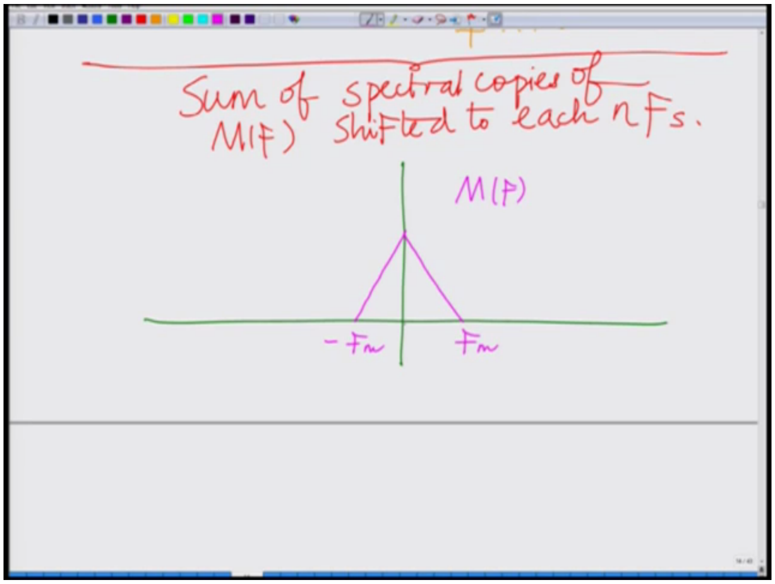

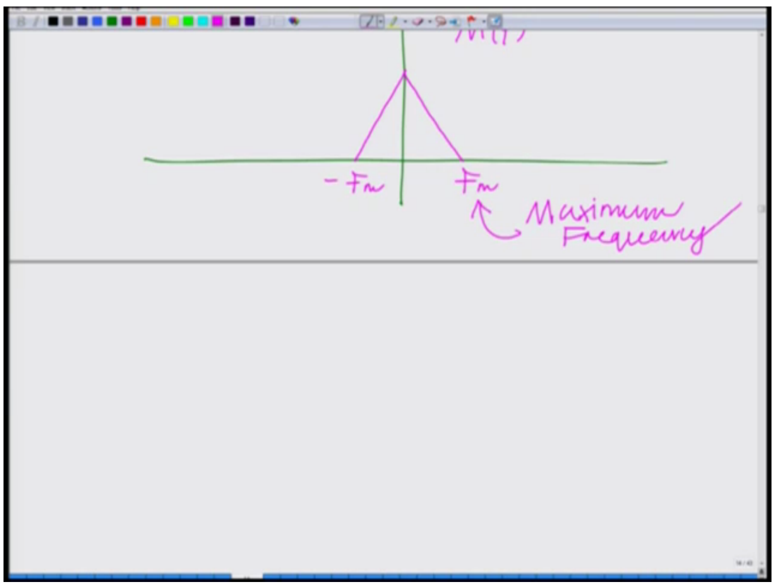

If you have your original spectrum that is MF which is has maximum frequency Fm let us say minimum frequency that is from minus Fm to Fm that is its support or where it is nonzero is from minus Fm to Fm the maximum frequency is Fm, this is the maximum frequency and therefore now what you are doing is basically you have Fs times MF.

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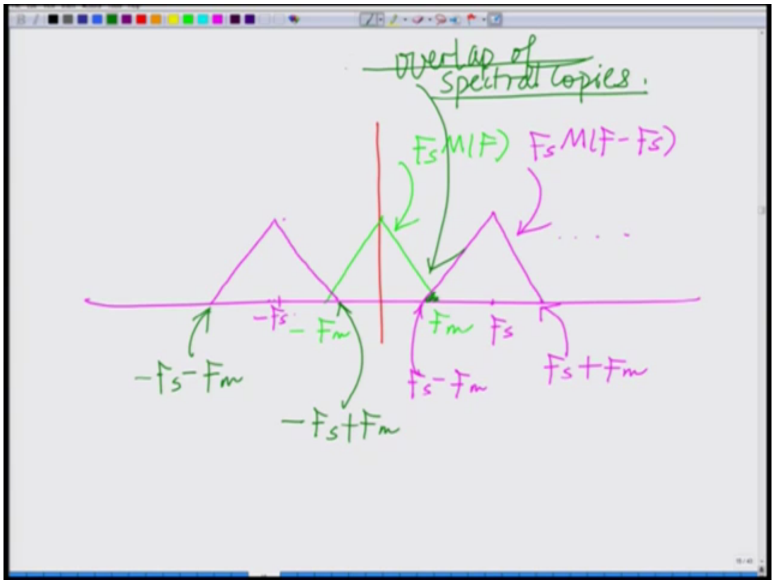

So the sampled signal m delta F will be well, Fs times MF so this is F of M Fm minus Fm, so this is Fs times MF then you have another copy at Fs shifted by Fs which can be this is Fs, so naturally this point is going to be Fs minus Fm this point is going to be Fs plus Fm this is your Fs times M of F minus Fs, so on your going to be have copies Fs M of F minus 2Fs copy at 2Fs and also similarly you can have a copy at that is if this is your minus Fs this is minus Fs, so naturally this point is going to be minus Fs plus Fm, this point is going to be minus Fs minus Fs and now you can see there is an overlap here there is a possibility of overlap of spectral copies and this overlap of spectral copies this leads to distortion that is what we have said overlap of spectral copies overlap leads to distortion, okay.

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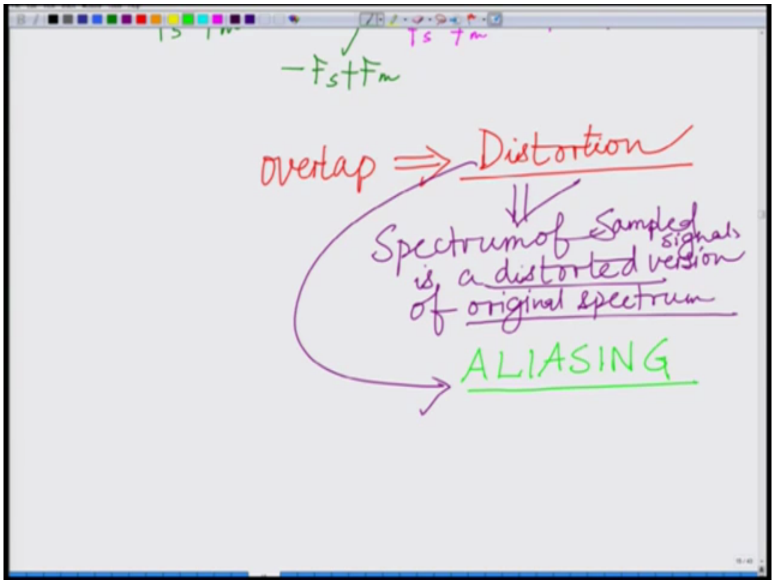

Distortion means that the sampled spectrum is a distortion distorted version of the original spectrum that is m delta F that is sampled spectrum what does this implies? Sampled spectrum for I should not say sampled spectrum rather I should say spectrum of sampled signal we are not sampling the spectrum but rather the spectrum of sampled signal spectrum of sampled signal is a distorted version of the original spectrum and this distortion is termed as this is very important this distortion is termed as Aliasing.

We have said that this distortion is termed as Aliasing that is these different spectral copies alias, okay. And the condition for aliasing, remember we have also derived the condition for aliasing, aliasing occurs when Fs minus Fm is less than Four Square.

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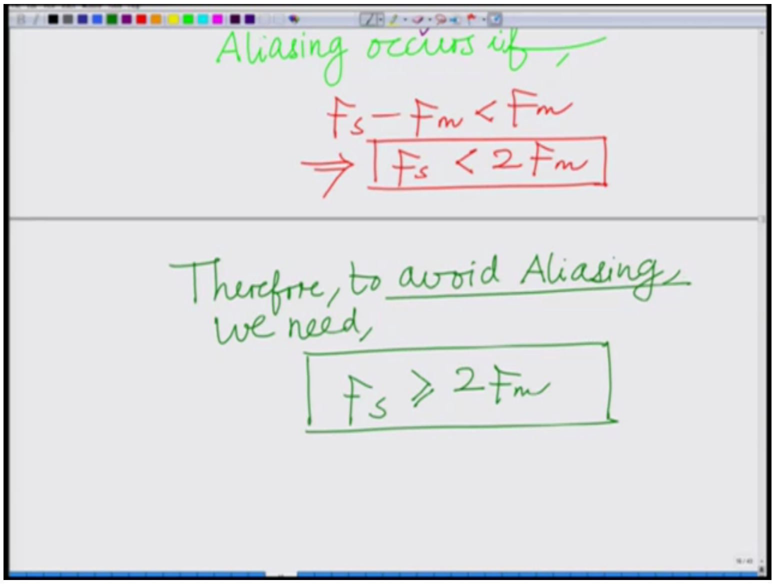

So aliasing occurs when occurs if Fs minus Fm is less than Fm that implies Fs less than 2Fm that is sampling frequency is less than twice the maximum frequency, so this is the condition for aliasing therefore to avoid aliasing we need we need Fs greater than or equal to 2Fm that is the minimum sampling frequency to avoid aliasing or the distortion caused by aliasing is Fs greater than equal to, to avoid aliasing we need Fs greater than equal to 2 Fm where Fm is the maximum frequency.

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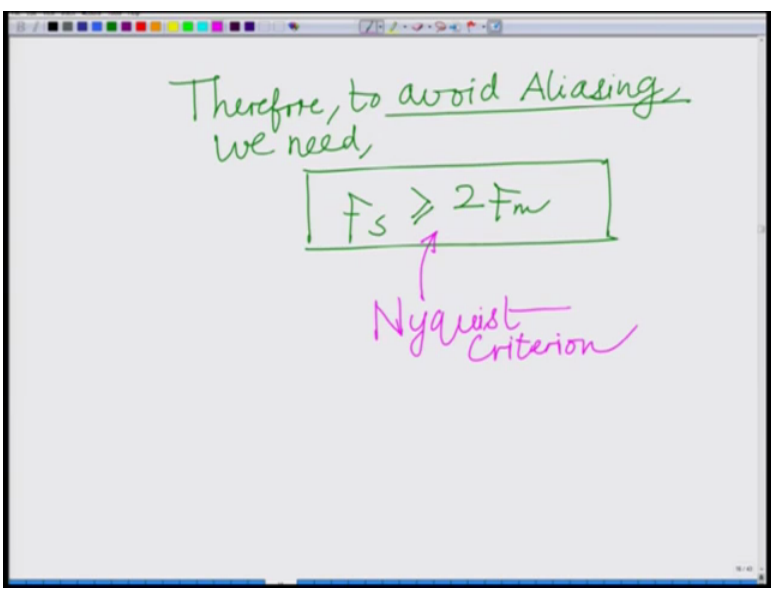

And this is known as the Nyquist criterion this is known as the Nyquist sampling theorem or this is known as the Nyquist, this is known as the Nyquist criterion.

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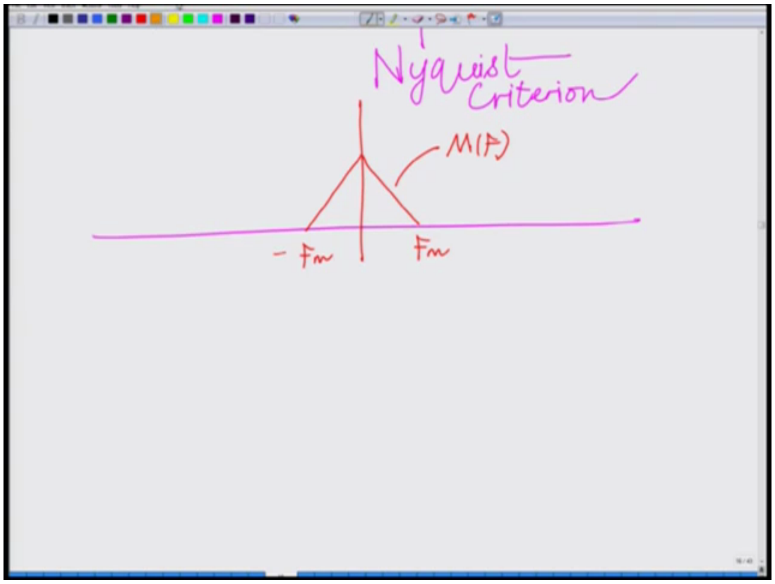

And remember when Fs is greater than Fm what is going to happen? This is your original spectrum let us say this is your original spectrum.

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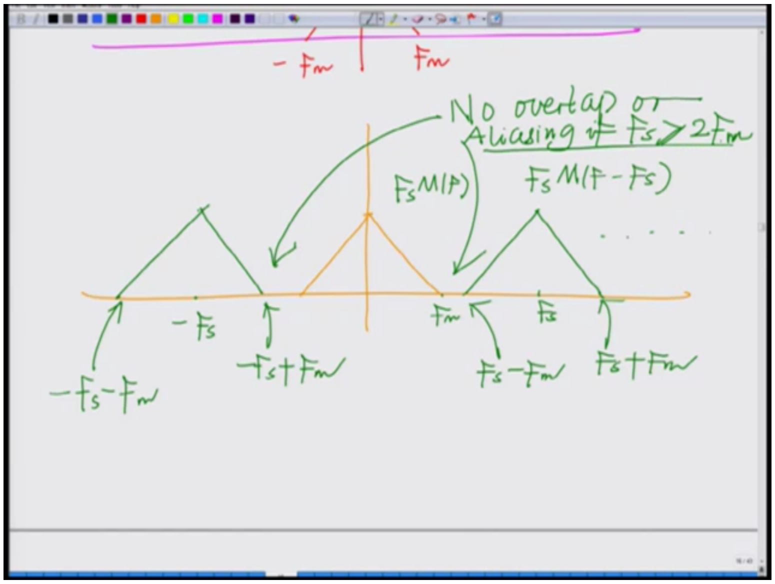

MF between minus Fm and Fm you are sampling at sampling frequency greater than Fs then you are going to have this original spectrum first scaled by well, scaled by Fs. So this is your copy of Fs into MF. Now if Fs is greater than FM you are going to have another copy over here. So this is Fs so this is Fm this point is Fs minus Fm, this point is Fs plus Fm similarly if you look at this over here so this is Fs MF minus Fs, similarly and then you will have various components FS MF minus 2Fs and so on.

Similarly here your have at minus Fs this is your minus Fs plus Fm this is your minus Fs minus Fm and now realize there is no overlap or no aliasing or aliasing if Fs greater than or equal to Fs is greater than or equal to 2Fm, so there is no overlap or aliasing is F Fs is greater than or equal to 2Fm and now there is no distortion, alright since the sampling frequency is greater than equal to twice the maximum frequency there is no distortion, now you can see that we can reconstruct the original message signal from the sampled spectrum and the way to do that is by filtering this copy the original the copy of the spectrum which is intact at the (seri) at the at the frequency 0 that is we have Fs times MF which is intact at in the baseband that is which is intact at which is centered around the 0 frequency I can extract that by low pass filtering this with an ideal low pass filter, okay.

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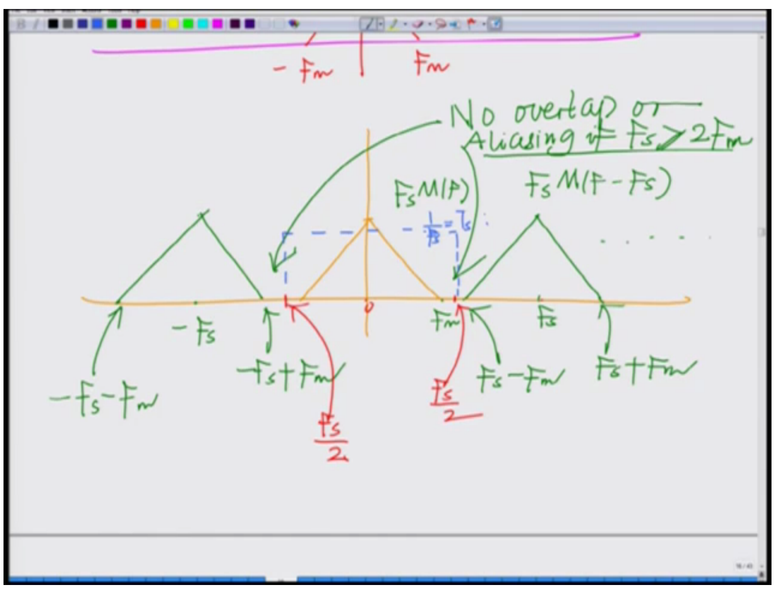

So now I can extract this now this is remember this is Fs this is 0 so this is your Fs by 2 Fs over 2 sampling frequency by 2 this is your sampling frequency by 2, now I can extract this by low pass filtering this by low pass filtering this you can clearly see if I have low pass filter with cut-off at Fs by 2 and there is a scaling factor of Fs, so this has to be 1 over Fs the height of this has to be 1 over Fs equal to Ts.

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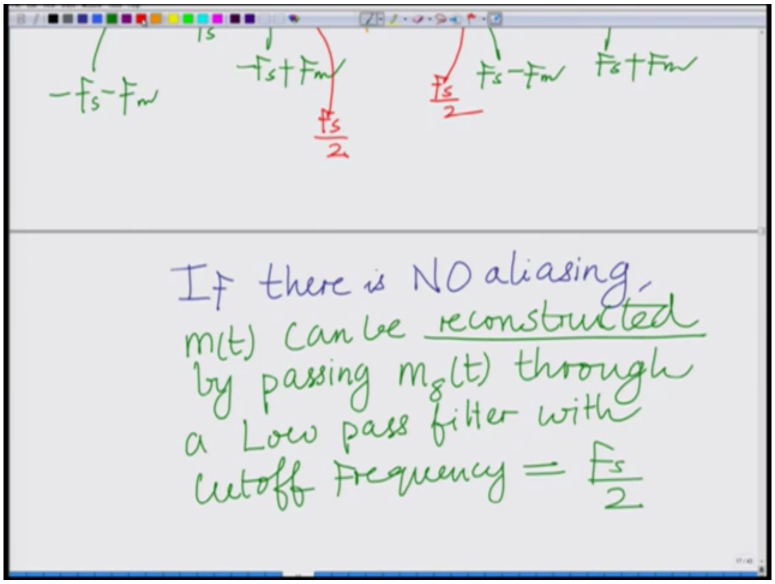

So now I can reconstruct the original so now we realize something interesting I reconstruct if there is no alias remember if there is no aliasing that is an important condition if there is no aliasing reconstruct or original message signal or m(t) can be reconstructed there is a ready signal can be reconstructed by passing m delta t through a low pass filter with cut-off frequency equals Fs by 2 the cut-off frequency of low pass filter is Fs by 2 that is it is one if F magnitude F is less than Fs by 2 and 0 otherwise and also there is a scaling of Fs. So to invert that you have to multiply by 1 over Fs therefore and so cut-off frequency is Fs by 2 and the height of the response is 1 over Fs that is Ts.

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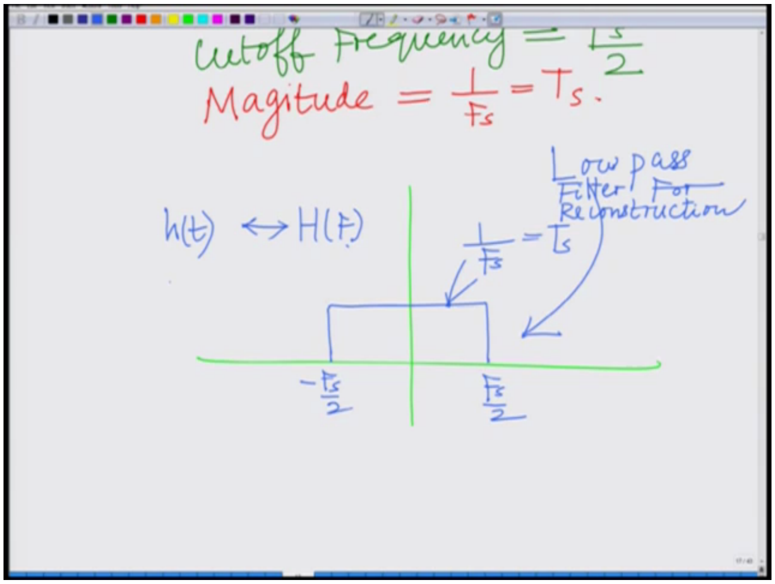

And frequency and height and frequency response magnitude equals 1 over Fs equals Ts, so the filter looks like this basically I have my filter, okay. And now I have a low pass filter with cut-off equals Fs by 2. So this is Fs by 2 this is minus Fs by 2 the height of this is 1 over Fs equals Ts, okay. So this is my reconstruction low pass filter low pass filter for this is my low pass filter for reconstruction let us term this as h of t, so this is this is H of F which is the response of let us say the impulse responses h of t. So we have H of F the response is basically it is easy to write this down this is H of F which is equal to 1 or basically Ts if F magnitude of F is less than or equal to Fs by 2 where Fs is sampling frequency and this is 0 and this is 0 otherwise, okay. So this is basically the low pass filter that the LPF or the low pass filter that we want to employ for reconstruction.

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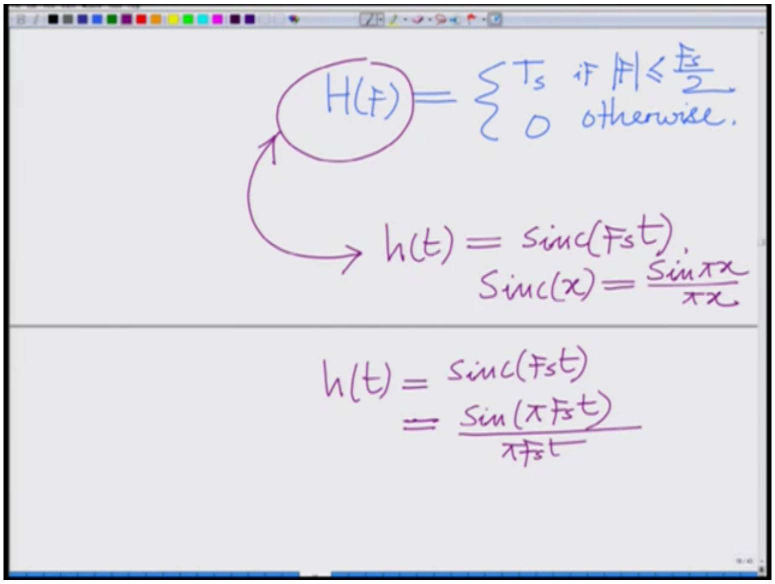

Okay, which is magnitude Ts if F is less than or equal to magnitude F is less than or equal to Fs by 2 0 otherwise, okay. And now this low pass filter you can (re) you can find out the impulse response of this that is the time domain impulse response and the impulse response of this we already know this is a pulse remember if it's if the signal is a pulse in the time domain then in the frequency domain it is (())(20:17) and if you apply duality if it is a pulse in the frequency domain naturally the impulse response of the response in the time domain must be a sinc function and the response of this in the sinc domain that can be seen as follows that is that can be seen as follows.

From duality this can be easily obtained as this is sinc of the corresponding h of t equals sinc of Fst, okay. So this is basically the sinc of Fst that is a sinc function in the time domain this is a where sinc of x remember sinc of x equals sin pi x by pi x, so sinc of Fst h of t equals sinc of Fst equals sin of pi Fst divided by pi Fst where Fs is the sampling frequency, okay.

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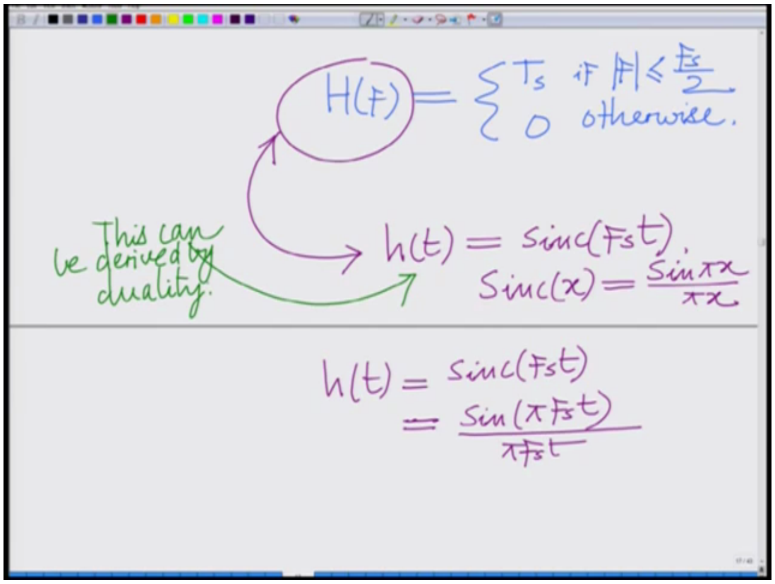

So this is the sinc function in the time domain and we can derive this and this can be derived by let us note that this can be derived by duality. And this can be derived by duality, okay. So this can be derived using this can be derived using duality, okay. And therefore what we have this can be derived by using the principle of duality we know that the principle of duality, okay. Now therefore we have the signal m(t), so what we have to do to reconstruct the signal for reconstruction.

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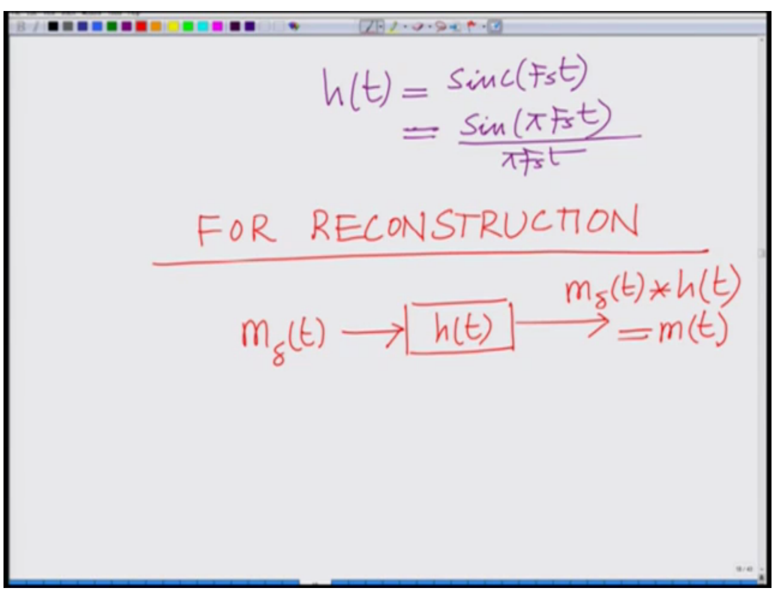

Now let us highlight that procedure for reconstruction we take our sampled signal m delta t pass it through the system with impulse response ht and that gives this, so m delta t convolved with h of t that gives back our original signal m(t) when there is no aliasing, right? Because we are a low pass filtering to extract the component at 0 frequency that is the component between minus Fm and Fm which is simply a replica of the spectrum of the original (sa) signal m(t) scaled by Fs.

So we are multiplying it by a low pass filter which is response which has which has a cut off frequency Fs by 2 and obviously to invite for that scaling to account for that scaling, we are scaling this by 1 over Fs or basically Ts, okay. That gives us back the original spectrum it gives us back the original spectrum in the frequency domain in the time domain naturally in the frequency domain we are multiplying so in the time domain it is basically convolution, okay.

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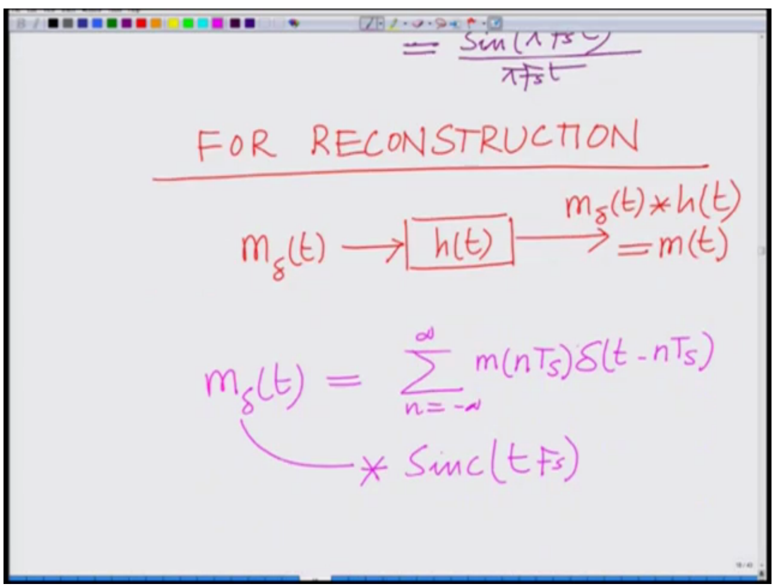

And that can be expressed as follows, now if you look at that our m delta t the sampled signal is summation n equal to minus infinity to infinity m of nTs delta minus nTs, now I have to take m delta t and I have to convolved with sinc of tFs that is sinc of Fst, remember this is our reconstruction filter sinc of tFs or basically synchronous Fst and therefore what we can see is m of t equals m delta t convolved with sinc of Fst which is equal to basically, now we have our m delta t also I can write this as convolution s commutative, so I can write this as sinc of Fst convolved I can interchange m delta and sinc.

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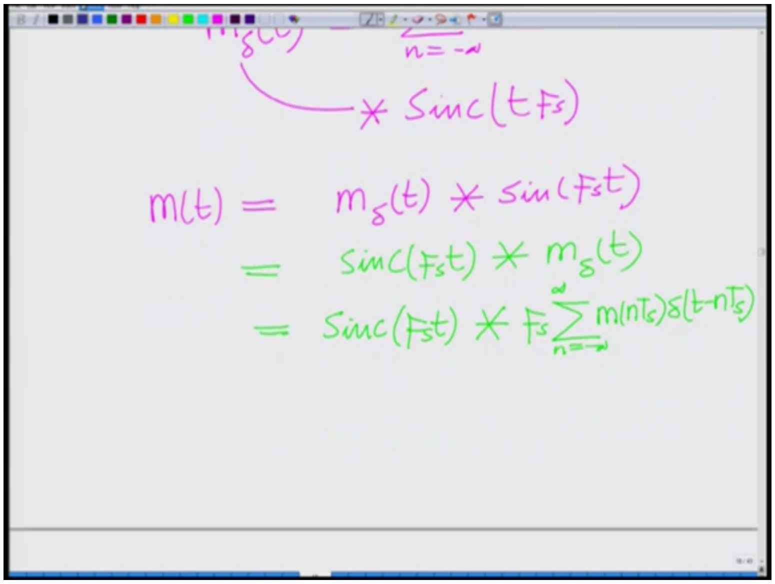

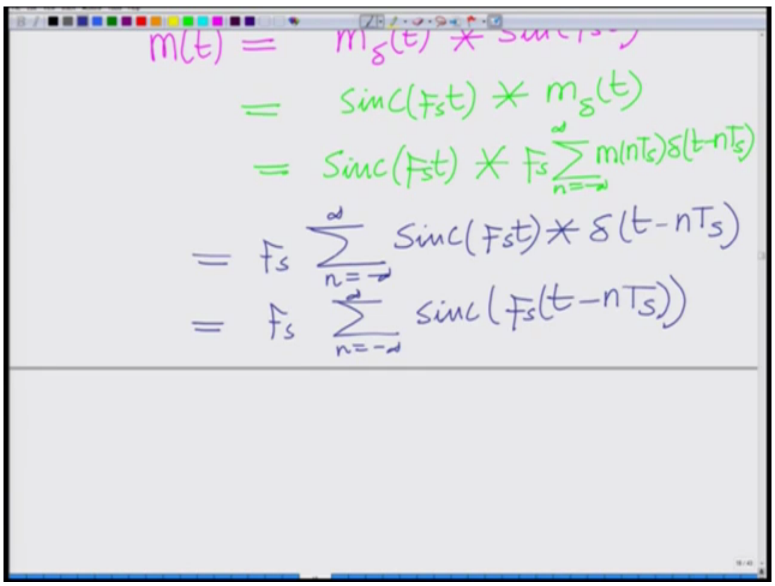

So this is sinc of Fst convolved with will m delta t we have already seen that is n equal to minus infinity to infinity m of n Ts into delta t minus nTs which is basically equal to which equals well, Fs taking the sinc inside that is n equal to minus infinity to infinity sinc of Fst convolved with delta t minus nTs. Now a function convolved with delta t minus nTs the function is simply shifted to nTs so that is Fs shifted by nTs, so that is n equal to minus infinity to infinity sinc of Fs of t minus nTs where we are once again using the property that is function x(t) convolved with

Delta t minus t not is simply x of t minus t not that is x of t shifted by t not, so sinc of Fst convolved by delta t minus nTs is simply sinc of Fs t minus nTs.

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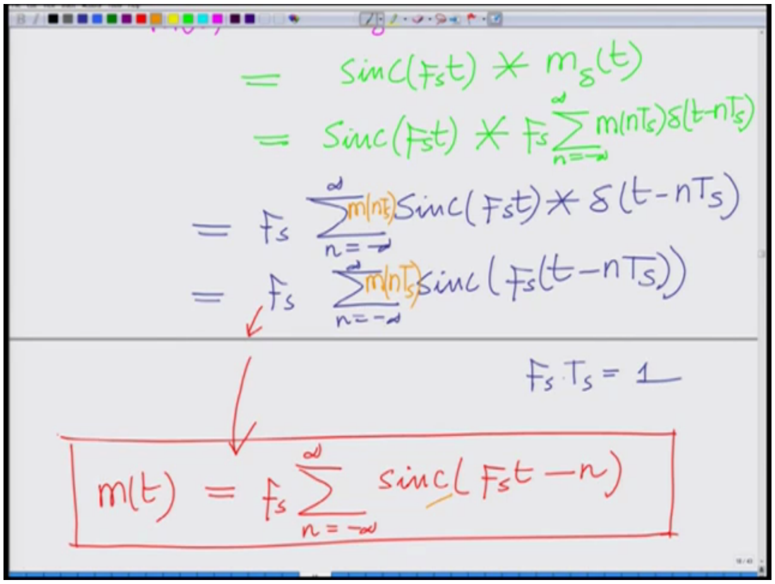

And now you will realize that Fs into nTs is one because Fs is a sampling frequency Fs into Ts because Ts is 1 over Fs, so Fs into Ts equals 1, so if I use this principle over here what I get is summation n equal to minus infinity to infinity Fs sinc of Fst minus Fs into nTs that is simply n and that is the original reconstructed signal that is how we can reconstruct the original signal by passing it through the sinc filter or this is basically a filter whose response is basically this will be your I am sorry I am missing the the mnts there has to be an mnts m of nTs and this is also there will be m of nTs.

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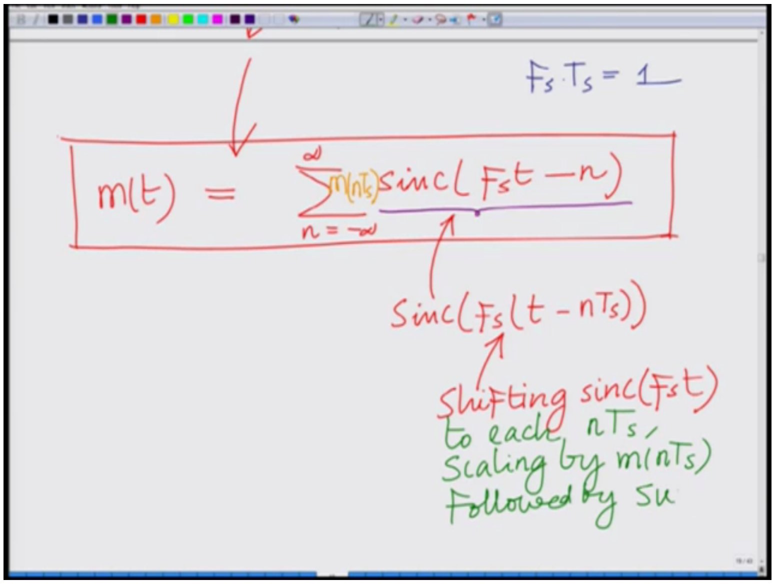

So what we are doing is basically we are shifting, correct? we are shifting of course there is no Fs there is no Fs because this is simply your mnts into Delta t minus Ts, so this is simply m of nTs so anyway what we are doing is basically we are taking each sinc Fs shifting it to nTs where as shifting sinc of Fst to each nTs to each nTs scaling by m mnts followed by the.

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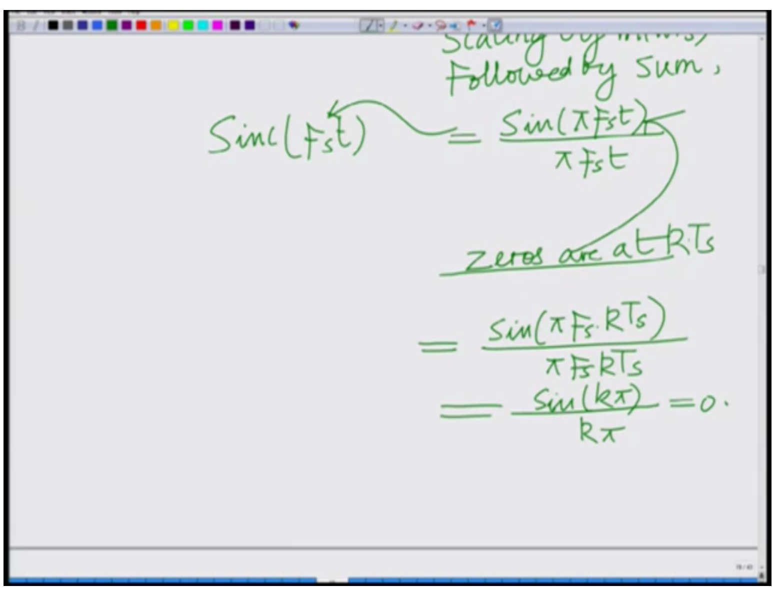

So what we have essentially is basically and remember sinc of Fnts sinc of Fst the zeros this is equal to basically (())(28:33) sin of pi Fst divided by pi Fst. So this is zeros are at the zeros of this are at integer multiples that is at k times Ts because this is at every k times Ts this is sin of pi Fs pi Fs k times Ts divided by pi Fs k times Ts Fs into Ts is one, so this is sin of k pi divided by k pi which is 0. So the zeros are at that is at every integer multiples at integer multiples of Time scale.

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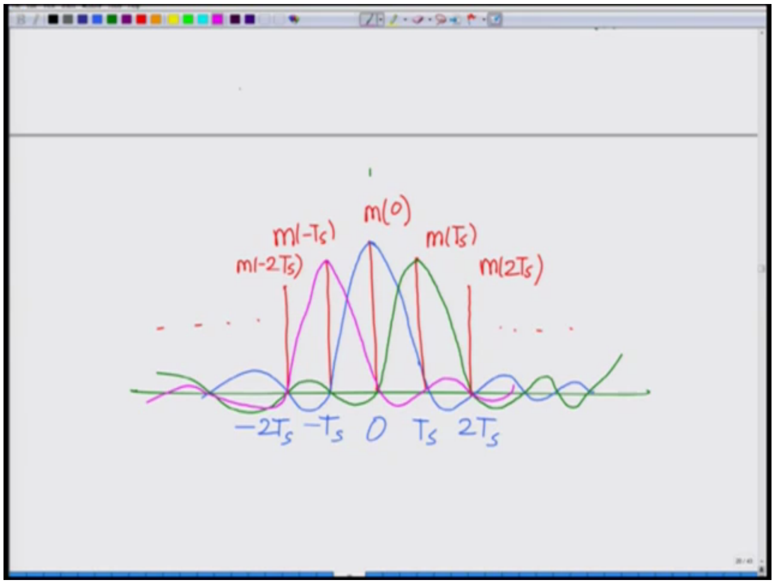

So therefore now what you are doing is essentially if you can look at this you have your you have your sampled signal m delta t which consist of impulses, correct? So this is your m of this is your sample which is basically your m of nTs, so let us say n equal to 0 so this is your sample which is m of Ts this is your sample which is m of 2Ts and so on and this is your sample which is m of minus Ts this is your sample which is m of m of minus 2Ts and so on. Now what you are doing you are shifting the sinc to Ts scaling it by m of nTs or m of 0 and then you are super imposing this sinc, okay.

So you are taking the sinc scaling it by m of 0 sinc amplitude at 0 is one so you are scaling it by m of zero so now and remember that is one sinc whose zeros are at remember this, so this is Ts, this is 0, this is 2Ts, okay. This at 0 so this is 0 Ts, 2Ts, minus Ts, minus 2Ts and now when you scale it by m 0 remember the zeros are still at Ts, 2Ts and so on. So what you are getting is the sinc which are which is like this that is the zeros are at basically the various samples.

And now then you are taking a second sinc scaling by m of Ts and the zeros of this sinc will again be at this is an important property of the **the** zeros of this which is basically scaled by 2Ts are going to be at scaled at Ts is going to be a 2Ts 0 and so on and then you are taking another sinc shifting it to minus Ts and the zeros of this are going to be at these are going to be at well, minus Ts is going to have zeros and so on and what you are observing is basically from this figure, what you can observe? is basically the reconstruction is nothing but a some of shifted sincs.

You are taking a each sinc function sinc of Fst shifting it to n times Ts and scaling it by the value of the sample that is m of n times Ts, okay. Scaling it by the sample m of n times Ts and then basically summing of all summing of all these shifted scaled sinc copies and the interesting properties are the zeros that is wherever your scaling it by n of Ts wherever you are the shifting the sinc scaling it by m of nTs shifting it to n of n n of Ts and scaling it by m of nTs the zeros of this sinc coinside with basically the other samples, so it is not affecting the other samples so the reconstruction at the samples exactly is an affected and that is true because if you have to reconstruct the original signal the original signal back then we know the values at the samples they are simply m of 0 m of Ts m of nTs and so on, alright.

So the reconstruction is not affecting the samples at the locations, alright. And this is an interpolation, alright. What we are doing essentially we have this samples and we are taking we are interpolating these samples to create the original signal original signal m(t) and this is interpolation using the sinc filter.

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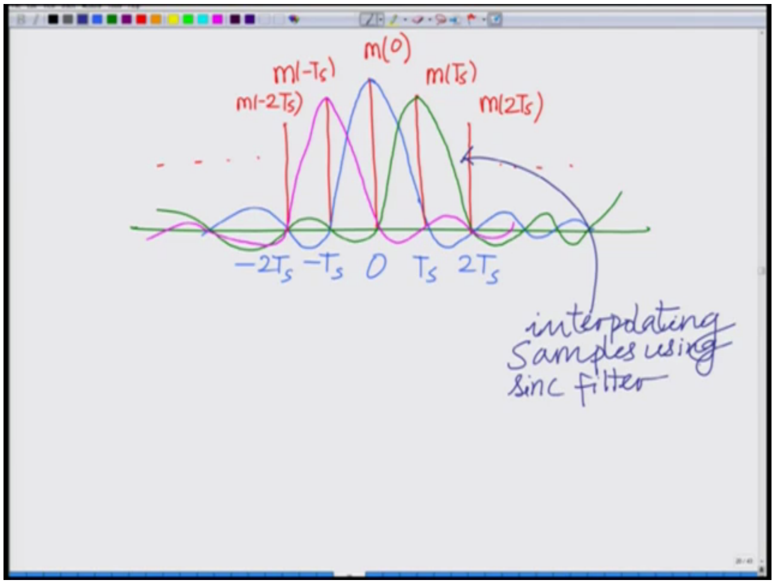

So what we are doing is literally, basically we are interpolating using the sinc filter this process is basically we are interpolating the samples. We are interpolating the samples using the sinc filter and what is our reconstruction?

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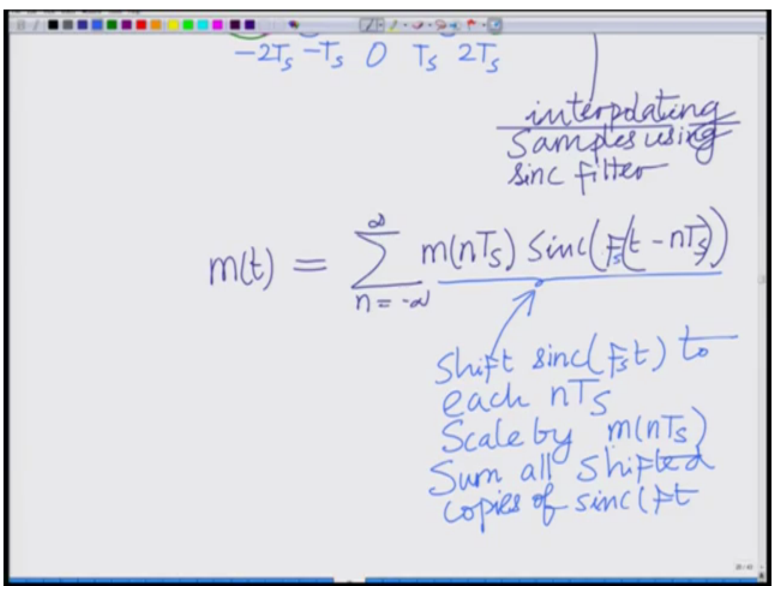

If you look at this relation over here that is m of t equals summation n equal to minus infinity to infinity m of nTs sinc of F Ts minus nTs that is this is basically shift that is basically shifts sinc FTs to each nTs scaled by mnTs and sum all shifted copies of this function sinc of sorry this is sinc of Fst sinc of Fst sinc of Fst minus Fst minus nTs, correct? Sinc of Fst minus nTs so you are shifting this sinc of Fst scaling by m of nTs and sum of all shifted copies of the function sinc of Fst and that is what you are doing.

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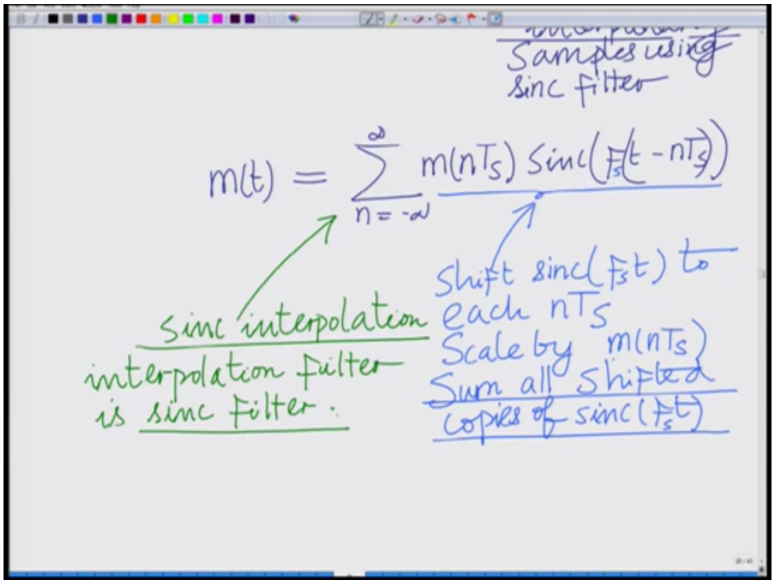

And this is basically your sinc interpolation or basically interpolation sinc is the interpolation filter, so interpolation filter is sinc function has a sinc response basically in the time domain. This is the interpolation filter the sinc function sinc of Fst is nothing but the response of the interpolation filter, alright. So basically what we have illustrated is how to reconstruct the original signal, which is very simple when we do not have aliasing when Fs is greater than 2 Fm then we can low pass filter it with cut-off frequency Fs by 2 and to invert the scaling we can have the height of the filter the magnitude response of the filter is 1 over Fs that is Ts, alright.

The corresponding response in the time domain is sinc Fst, so you are passing the sampled signal through this filter with response sinc of Fst we are able to interpolate the samples, alright. Interpolate the samples of the sampled signal and basically reconstruct the original message signal m(t) without any distortion since there is no aliasing, alright. So that is that concludes the reconstruction aspect of the original signal, alright. So will stop here and look at other aspects in the subsequent modules, thank you.