Principles of Communication- Part I
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Module No 6
Lecture 36
Spectrum of Sampled Signal, Aliasing and Nyquist Sampling Theorem

Hello welcome to another module in this massive open online course, so we are looking at the sampling of an analog signal to (con) to extract samples from it to convert it to a digital signal, alright. And we have said that the signal can be sampled by multiplying it by an by an impulse train, alright which is termed as ideal sampling, alright.

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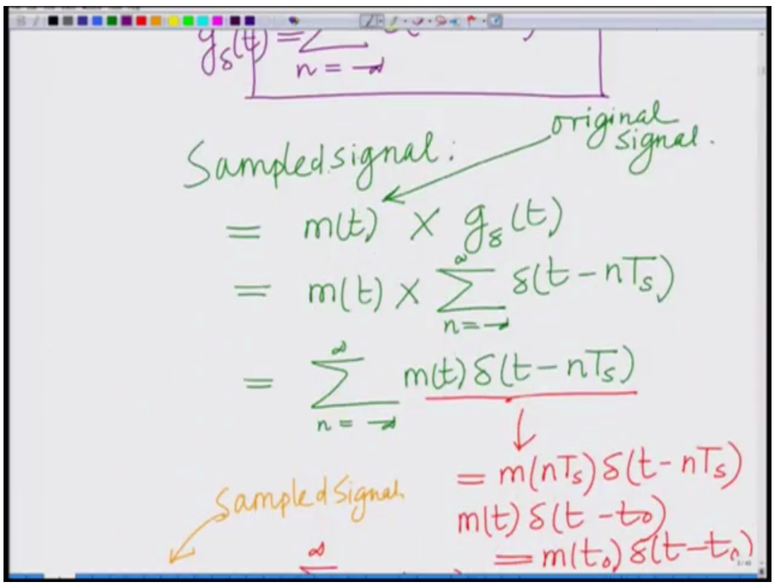

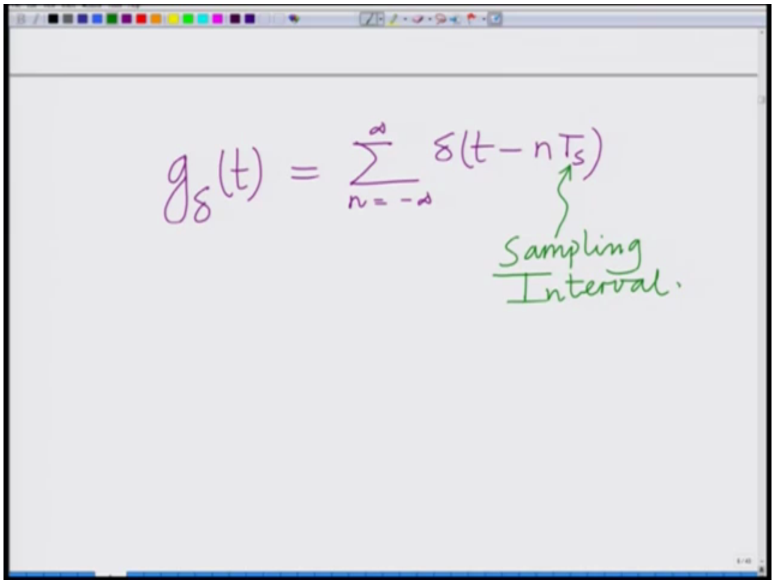

So let us say let us consider our impulse train which is given by g delta t equals summation summation this is your summation n equal to minus infinity that is summation n equal to minus infinity into infinity delta t minus nTs where Ts is the sampling interval, alright which is basically the time interval between the successive sampling instants, okay. Delta is of course the direct Delta function this is basically we have our sequence of impulses at train of impulses at every multiple of Time scale.

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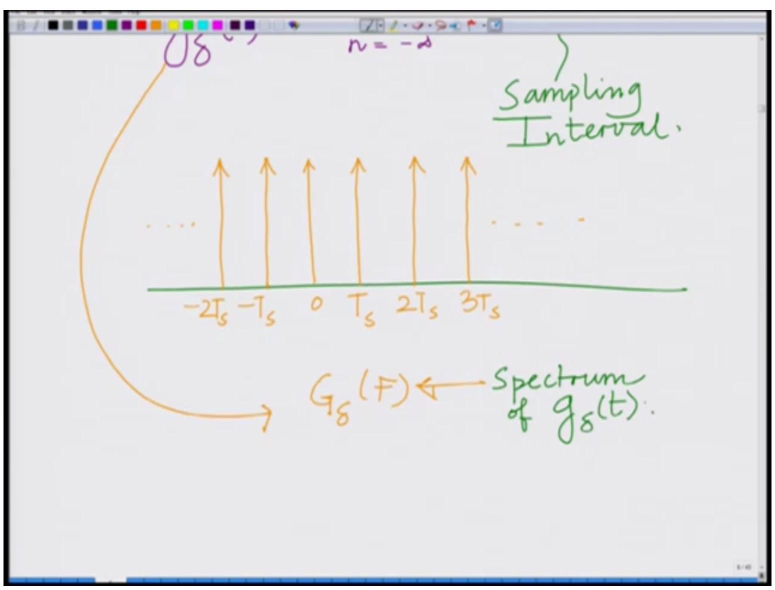

So we have impulse at Ts we have impulse so this is our impulse at Ts we have impulse at 2Ts, we have impulse at 3Ts and so on and of course we have impulse at 0 we have impulse we have an impulse at we have an impulse at minus Ts, minus 2Ts etc and now what we want to do is we want to find the spectrum g delta F that is the spectrum g delta F which is basically your spectrum of g delta t, okay. We want to start with that towards eventually find the spectrum of the spectrum of the sampled signal.

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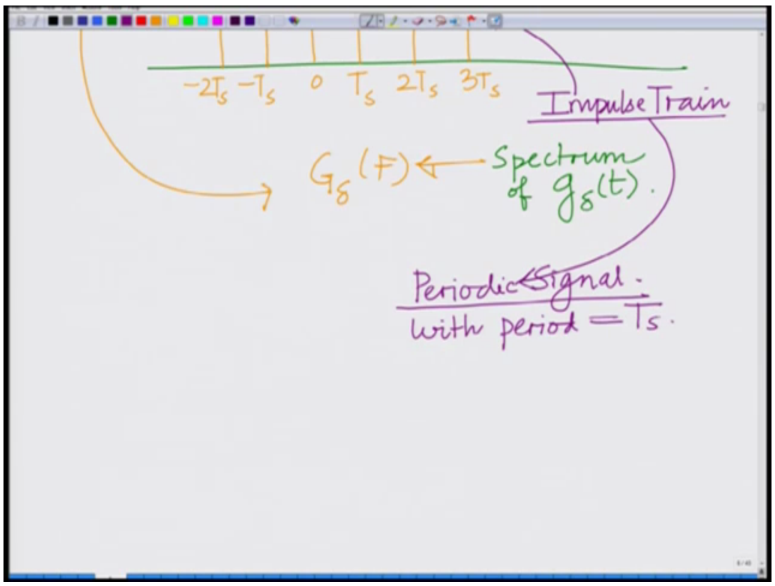

Now you will see that this signal if you can if you look at this signal this impulse train, correct? This impulse train this impulse train you will realize this is a periodic signal, correct? The impulse train is a periodic signals is there is one impulse every Ts. So we have impulse at Ts impulse at 0 impulse at Ts, 2Ts, minus Ts, minus 2Ts and so on, alright. So this is a periodic signal with period Ts, okay. So this is a periodic signal with period equal to Ts, okay.

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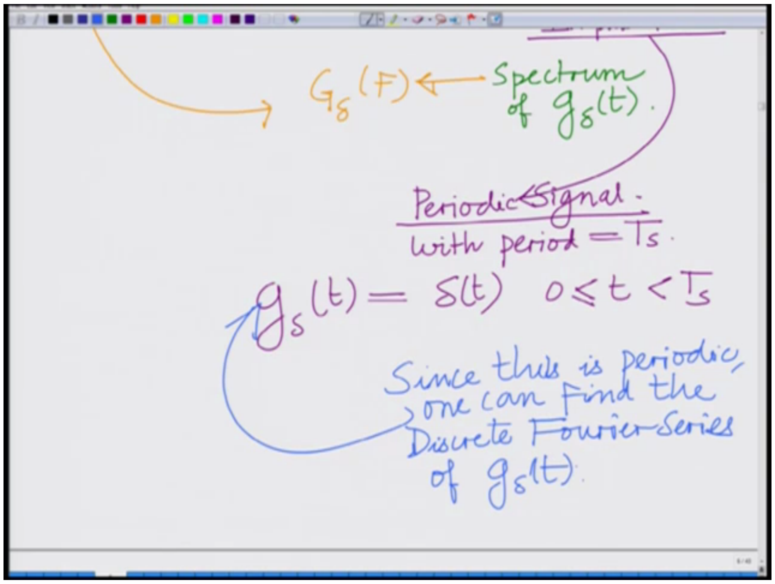

So in one period let us say this is basically your x(t) or your g delta t equals basically Delta t 0 less than equal to t less than Ts, alright this is the signal and therefore since this is periodic weekend find the Fourier discrete Fourier (se) series representation, since this is periodic since this is periodic one can find the one find the discrete Fourier series of g delta t, okay.

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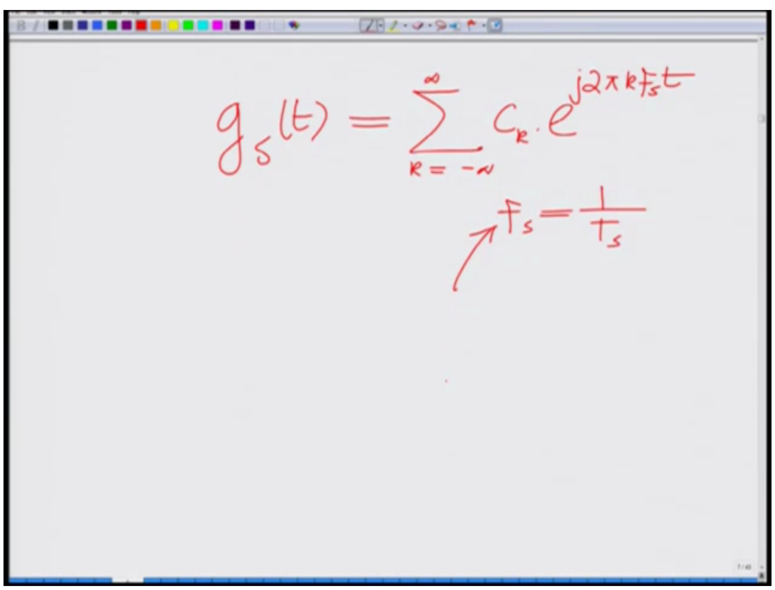

So let us express g delta t so g delta t can be expressed as summation well, k equals minus infinity to infinity Ck e to the power of j 2pi kFst where Fs equals 1 over Ts the sampling frequency is the fundamental frequency, alright. Fundamental frequency is 1 over time period, correct? The fundamental frequency is 1 over the time period the time period is nothing but the sampling interval, right? Sampling interval that is Ts therefore the sampling frequency Fs itself is the fundamental frequency, okay.

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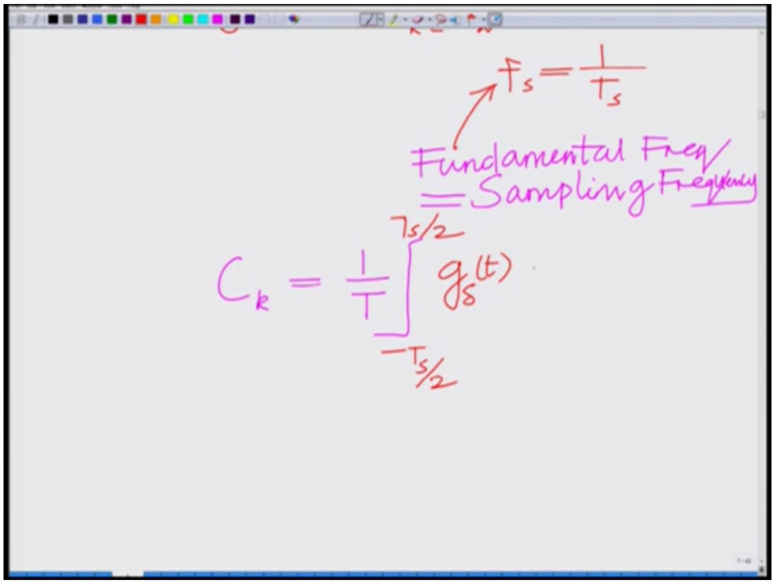

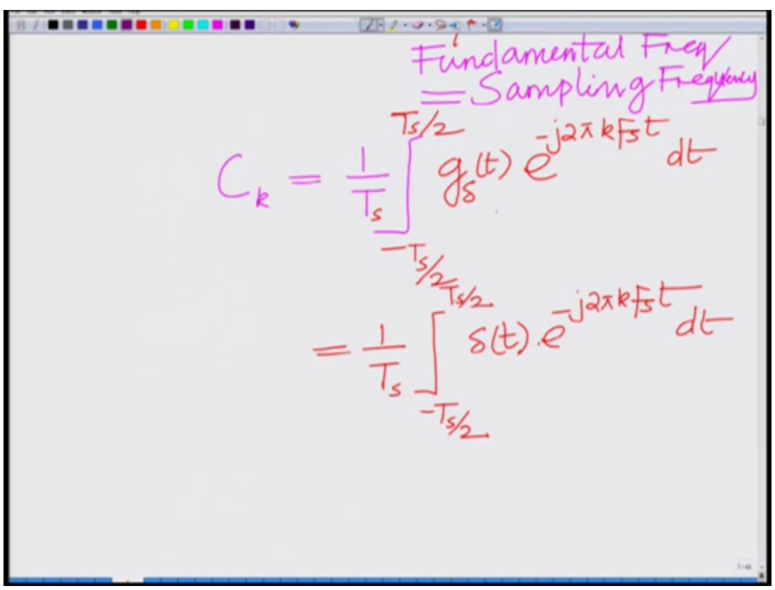

So the fundamental frequency equals the equals the sampling frequency, alright. And further now we have to find these coefficients Ck and you can see that this coefficient each Ck equals well, 1 over Ts we know the expression for Ck 1 over Ts integral 0 to Ts, correct? G delta t or we can make it from minus Ts by 2 to Ts by 2 g delta t dt which is equal to 1 over Ts minus Ts by 2 to Ts by 2 delta t sorry g delta t e to the power of minus j 2pi kFst e to the power of e to the power of minus j 2pi kFst dt, alright there we have replaced g delta t by delta t that interval minus T s by 2 to s by 2.

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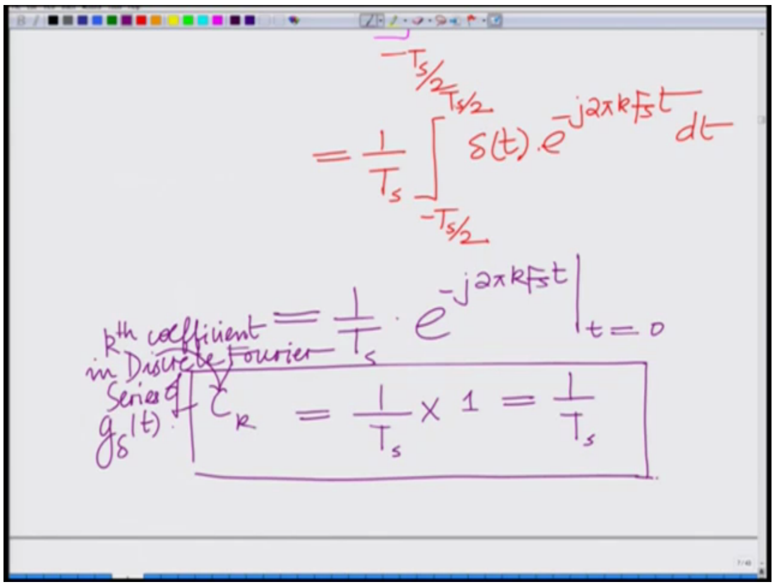

Now the integration multiplying by delta t is very simple that is when I multiply a function by delta t and integrate that extracts the value of the function at t equal to 0. So this is simply e to the power of minus j 2 j 2pi kFst evaluated at t equal to 0 that is simply 1 over that is simply 1 so this quantity is simply 1 over Ts into 1, so 1 over Ts so each Ck has a very simple expression each Ck this is the kth coefficient in the discrete Fourier series kth coefficient in the discrete Fourier series of g delta t that is your that is basically your that is basically your Ck, so we have evaluated each coefficient Ck as 1 over Ts.

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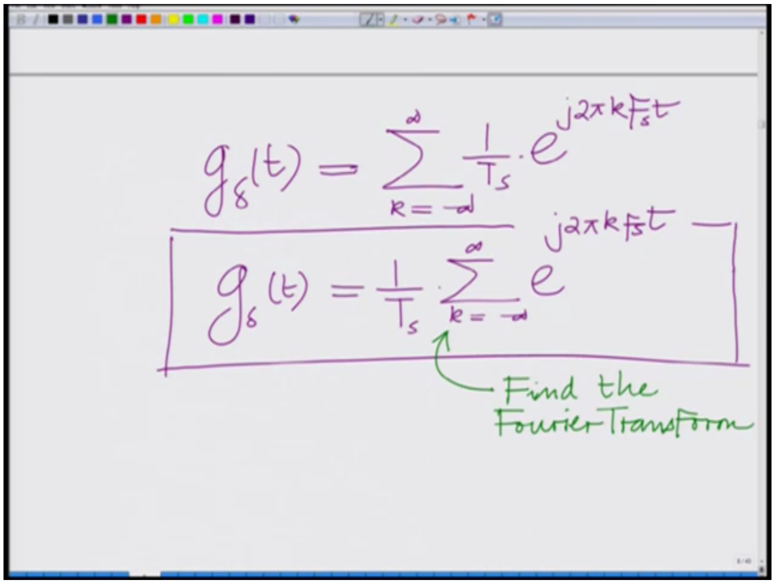

And therefore now the discrete Fourier series representation of this impulse train is basically given as I have g delta t equals summation k equals minus infinity to infinity 1 over Ts e to the power of j 2pi kFs e to the power of j 2pi kFst which is also basically you can write it since 1 over Ts is common so you can take it outside of the summation 1 over Ts k equal to minus infinity to infinity e to the power of j 2pi kFst this is your discrete Fourier series. This is basically your discrete Fourier series this is basically the discrete Fourier series representation okay, correct?

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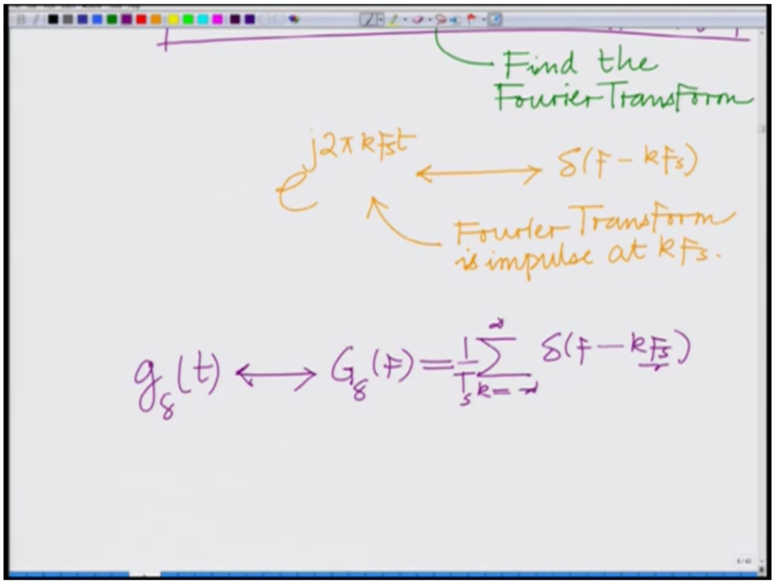

And now from this discrete Fourier series representation we can find the Fourier transform so from this find the Ft or the Fourier transform and that is simple one can extract the Fourier transform because e to the power of j 2pi kFst if you look at e to the power of j 2pi kFst e to the power of j that is nothing but an impulse this Fourier transform of this is an impulse at kFs, correct? Fourier transform is an impulse at kFs.

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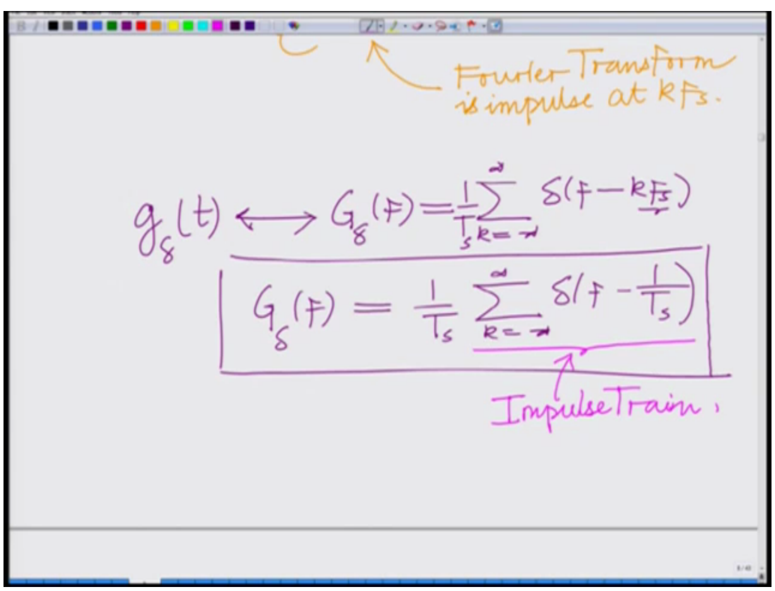

The Fourier transform or basically spectrum is an impulse at kFs which means now the Fourier transform of this is basically g delta t is basically each Fourier transform of each is basically an impulse Delta F minus kFs therefore the Fourier transform g delta F which is basically the Fourier transform of g delta t is basically, so if you look at g delta t and let us say we consider it's Fourier transform capital G Delta F that is simply summation k equal to minus infinity to infinity 1 over well, there is a 1 over Ts each e to the power of j 2pi F kFts kF e to the power of j 2pi kFst has a spectrum delta F minus kFs, okay.

And Fs remember Fs is basically 1 over Ts, so this is basically nothing but 1 over Ts summation k equal to minus infinity to infinity delta F minus 1 over Ts, okay. And therefore you can see something very interesting which is basically if you look at this summation if you look at this summation 1 over Ts summation k equal to minus infinity to infinity Delta F minus 1 over Ts this is basically this is basically another impulse train.

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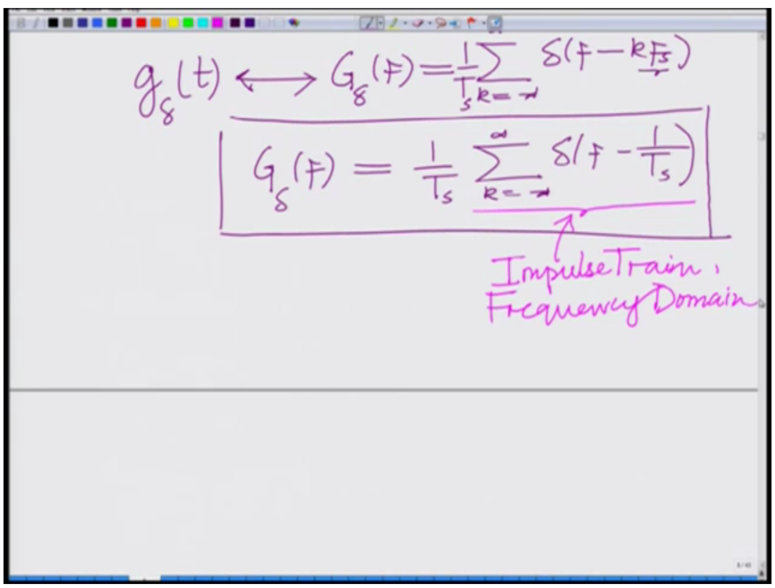

Now you can see that this except for the scaling factor of 1 over Ts this is basically another impulse train and this is a train of impulses in the frequency domain, alright. So we have started with the train of impulses, impulse train in the time domain and what we are seeing is its Fourier transform or spectrum is basically an impulse train in the frequency domain impulse train in the frequency domain is an impulse train in the frequency domain.

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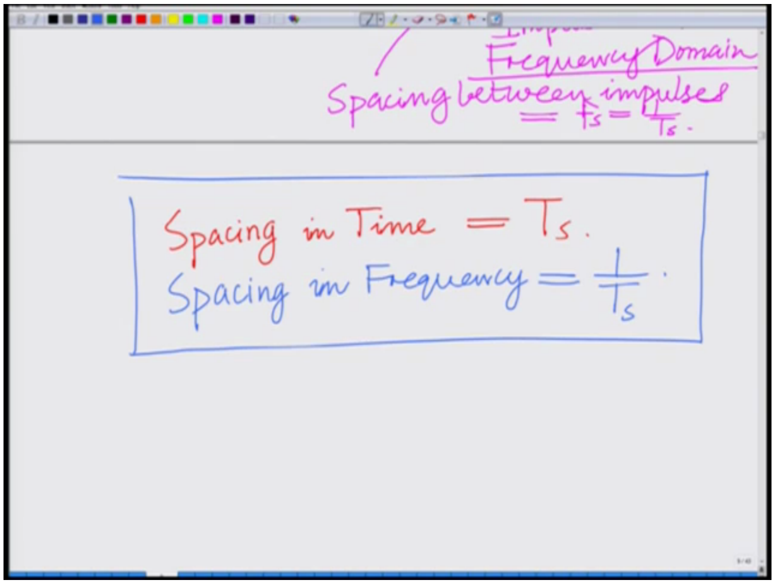

Further the spacing between the impulses that is each impulse I am missing a factor of k over here this is F minus k over Ts, so spacing between impulses equals Fs equals 1 over Ts remember spacing in time it was Ts the impulses are spaced at because Ts is a sampling interval, alright. So the impulses are spaced Ts apart but in frequency impulses are spaced, so spacing in frequency equals spacing in frequency equals this equals 1 over Ts, alright.

So spacing in for x as the (spaco) spacing in time is Ts, spacing in frequency is 1 over Ts so basically as the sampling as the sampling interval Ts increases the spacing in time increases the spacing in frequency that is 1 over Ts decreases because 1 over Ts is the sampling frequency as the sampling interval increases Fs that is a sampling frequency decreases, so spacing in frequency is the sampling frequency alright a sampling time sampling interval is increasing the sampling frequency is decreasing therefore spacing in the frequency domain is decreasing.

And conversely as the spacing in the and vice versa, right? As the spacing in the time decreases there is your sampling at finer at a (fi) at a very fine at finer and finer interval, right? that is Tsis decreasing the sampling interval Ts is decreasing which means the sampling (increa) the sampling frequency Fs is increasing the spacing in frequency increases, alright. So this is spacing in frequency is Ts equals 1 over Ts equals to Fs.

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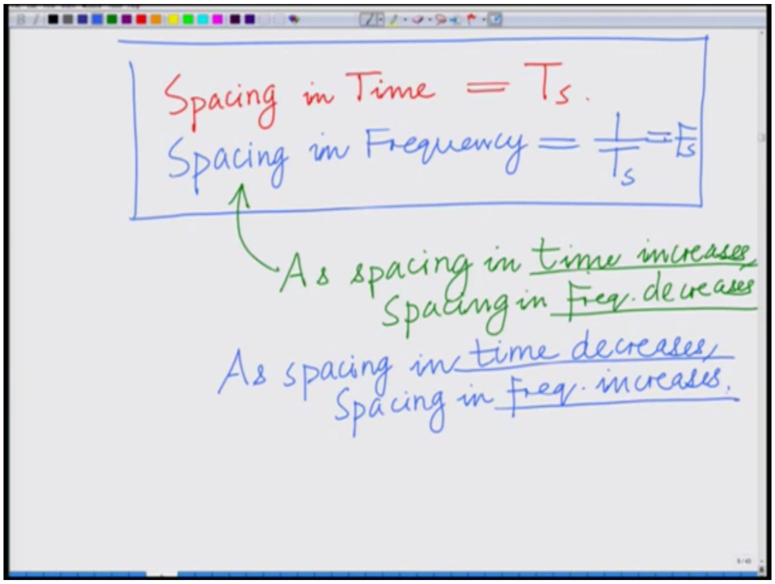

So you can note that as spacing in time (decrea) increases spacing in the frequency domain decreases, okay. As spacing in time increases spacing in frequency domain increases and vice versa. As spacing in time decreases as spacing in time decreases spacing in the frequency that basically increases, alright. So that is basically it means that as your sampling finer and finer in time that is Ts is decreasing the spacing in frequency that is the sampling frequency Fs is increasing therefore the spacing between these successive impulses of the impulse train in the frequency domain is increasing.

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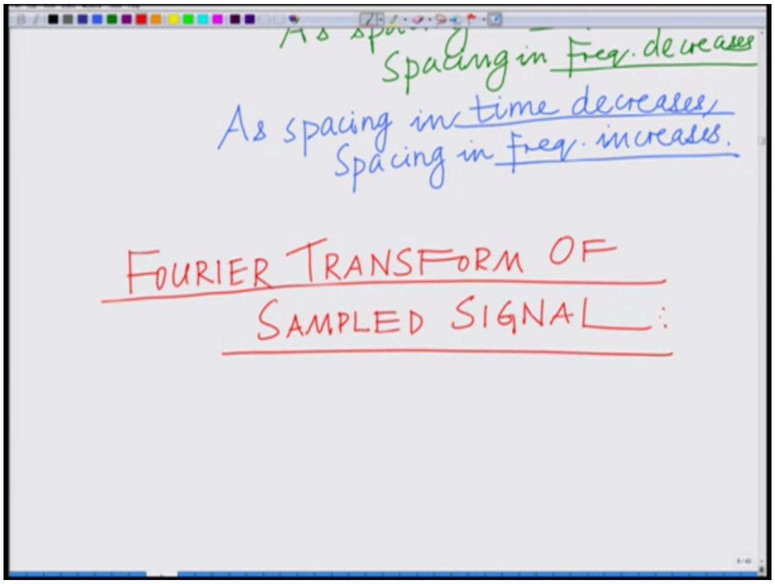

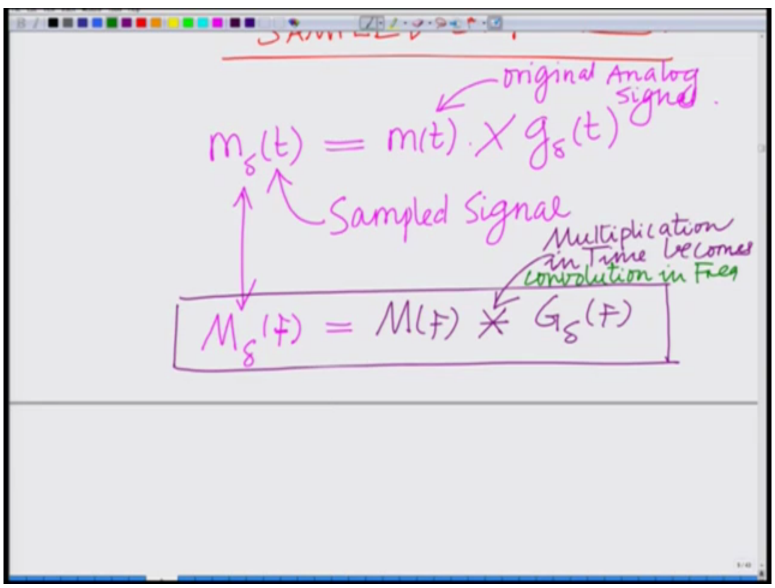

Now let us look at what is the Fourier transform of the sampled function, alright. So now we are in a position to look at the Fourier transform of the sampled functions, so we want to look we have found the Fourier transform of the of the impulse train now Fourier transform of the Fourier transform of the sampled analog signal, now you can see that will be given as follows remember m delta t or sampled analog signal is simply m of t times g delta t this is your sampled analog signal.

This is your sampled signal and m of t is the original analog signal, your original this is your original analog signal and now if I look at the Fourier transform of this, correct? That is very simple m delta F is the Fourier transform of m delta t that is the sampled signal this is equal to now you can see m mt let us say has a Fourier transform M of F and g delta t we know has a Fourier transform g delta F this will be the convolution between the Fourier transform M delta F MF and g delta F because multiplication in the time domain is convolution in the frequency domain. So multiplication in time becomes convolution multiplication in time becomes convolution in the frequency domain.

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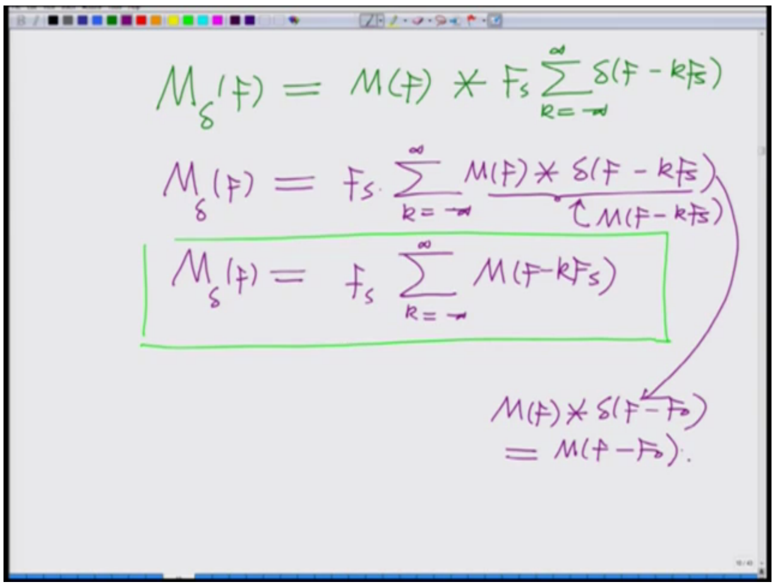

Therefore M delta F that is a spectrum of the sampled signal equals M of F convolved with g delta F g delta F we have seen is 1 over Ts summation that is 1 over Ts summation Delta F minus k kFs and 1 over Ts is also equal to remember 1 over Ts is also equal to Fs. So this is basically your convolution with Fs times summation k equals minus infinity to infinity, correct?

K equals minus infinity to infinity Fs Delta F minus kFs where Fs is the sampling frequency and this is equal to well, Fs summation k equal to minus infinity to infinity MF convolved with Delta kFs which is Fs now k equal to minus infinity to infinity MF convolved with Delta F minus kFs remember this is nothing but MF minus kFs, so this is MF minus kFs that is your M that is your M delta F M delta F is basically K equal to minus infinity to infinity Ms M delta F is Fs summation k equal to minus infinity to infinity MF minus MF minus kFs MF minus kFs MF minus kFs that is what you are doing here is something very interesting, alright.

First let me elaborate, the property that we have used here again is something that should be very familiar to you that is MF convolved with delta F minus F not (shim) simply shifts MF to F not, so MF convolved with each delta F minus kFs simply shifts each MF to kFs and therefore now what you are doing is you are shifting the spectrum MF to each kFs, alright. So for at each kFs that is k times multiple of the sampling frequency Fs, right?

You have a copy, alright of the original spectrum MF so now you are shifting the spectrum MF to each kFs and you are summing all these copies, so the spectrum of the sampled signal will be the sum of the copies of the original spectrum MM MF MF shifted to every multiple of the sampling frequency Fs, so what you are seeing here is something very interesting, what you have is?

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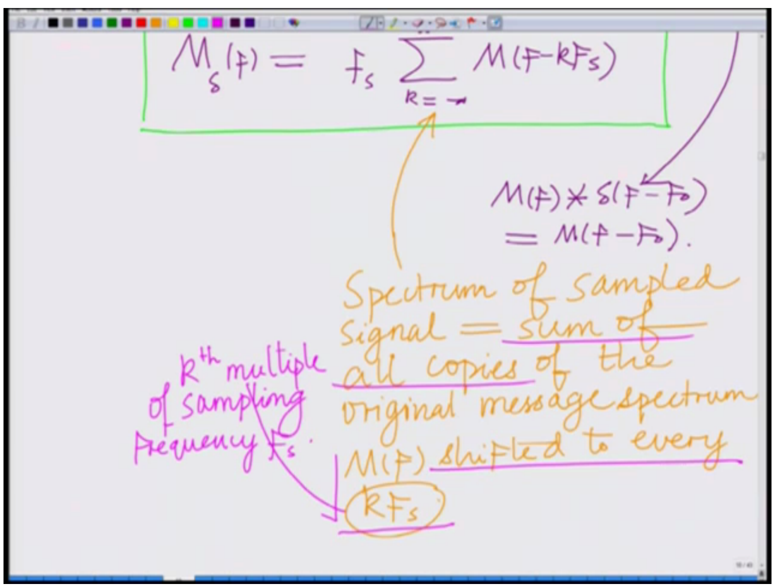

This is the (orig) this is the spectrum of sampled signal equals sum of all copies all copies of the original message spectrum MF shifted to every kFs where kFs remember kFs is kth basically this is the kth multiple of, this is the kth multiple of the sampling, so this is the sum of all copies of the message spectrum shifted to every kFs which is very interesting, so you have the original spectrum MF now we are shifting this to each kFs, right?

Each multiple of the sampling (freq) each integer multiple of the sampling frequency Fs therefore you are making an in finite number of copies of the original spectrum MF and then you are adding all these copies, now let us see in the frequency domain what does what this corresponds to, so you have your original spectrum MF, okay. So let us say you have your original spectrum this can lead to something very interesting in the frequency domain you have your original spectrum MF, correct?

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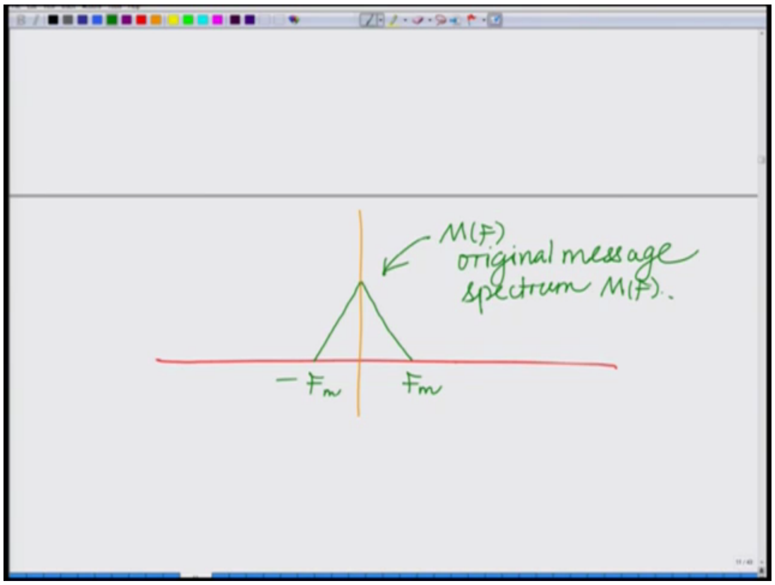

You have the original spectrum MF, okay. Let us say the maximum frequency is Fm between minus Fm, so this is your original message original message spectrum My family consists of.

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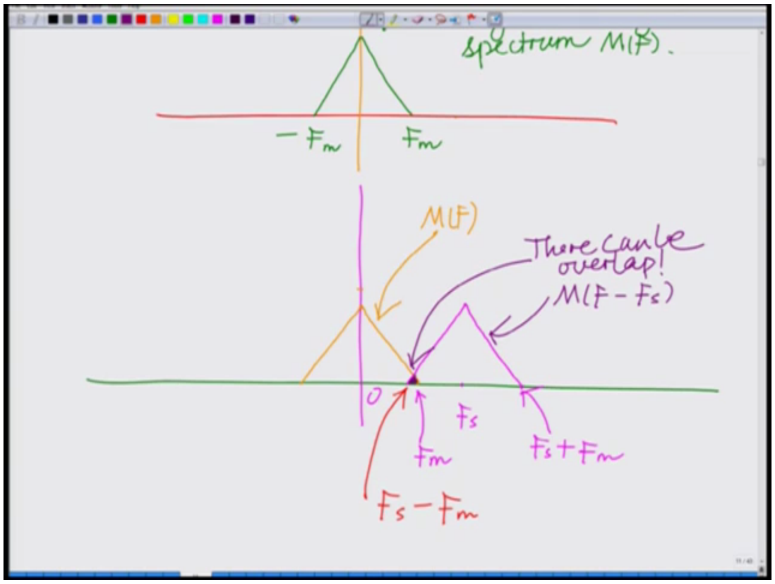

Now let us draw here the sampled spectrum so we are going to have well, we are going to have the original message spectrum intact that is going to be intact, so we are going to have the original message spectrum that is going to be intact and so that is going to be this is going to be MF. In addition at Fs you will have one copy at Fs, correct? So this is the copy at 0 in addition you will have one copy which is shifted to Fs, correct? So that will be that copy will be, so this point is your Fm now that is a copy at Fs so this point naturally is Fs plus Fm and this point is interesting if you look at this, this is Fs minus Fm. Now that can potentially be an overlap in this region this to copies so this is your MF this is your MF minus MF.

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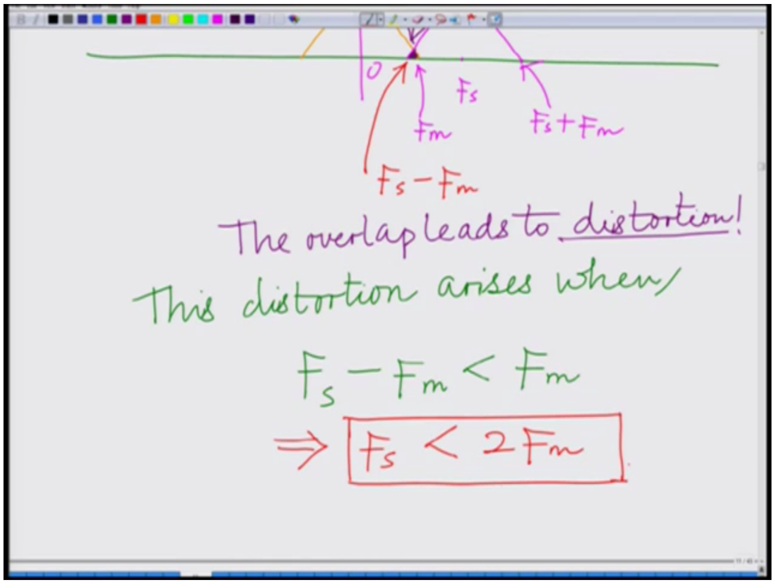

So there can be there can be an overlap between these 2 copies, so MF now if you can see this is interesting so there can be an overlap and the overlap the overlap leads to distortion, so you have a copy of the original spectrum that is another you have the original spectrum its 0 there is another copy shifted by Fs when you take the sum these 2 copies an overlap and therefore once you sum that can lead to distortion, alright.

And this can cause distortion of the sampled that is the (si) the spectrum of the sampled signal is not similar to the (spect) the spectrum of the sampled signal that is m delta t that is m delta F is not similar to the spectrum MF of the original signal it is a distortion version of the (spect) distorted version of the spectrum of the original signal MF, alright. And when does this arise this distortion arises if there is an overlap, now realize that this distortion arises when you can see when will there be an overlap? There will be an overlap when Fs minus Fm is less than Fm Fs minus Fm is less than Fm which implies Fs is less than twice Fm.

So these spectral the spectral overlap so this is a very important and very interesting condition the spectral overlap, alright. The spectral overlap between these different copies of the spectrum shifted by the various integer multiples of Fs will occur if the sampling frequency Fs is less than twice the maximum frequency Fm of the original analog signal, alright and therefore at this distortion which occurs because of the spectral overlap is termed as Aliasing, okay.

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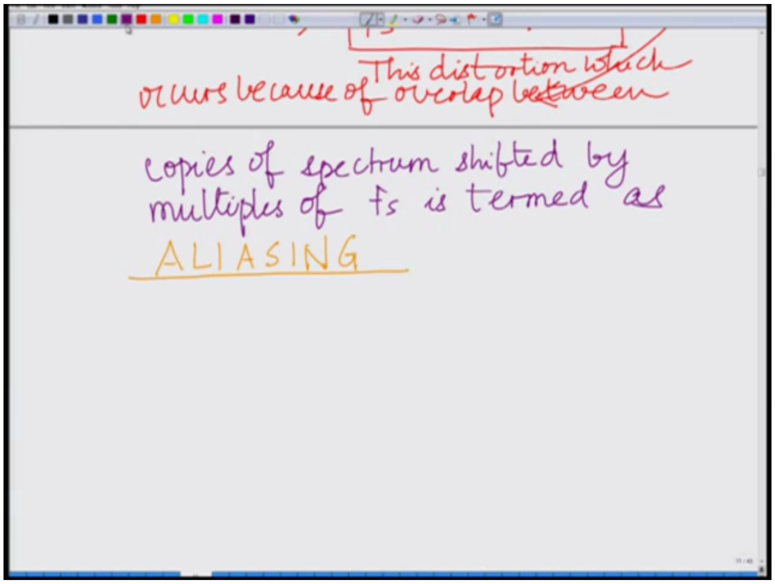

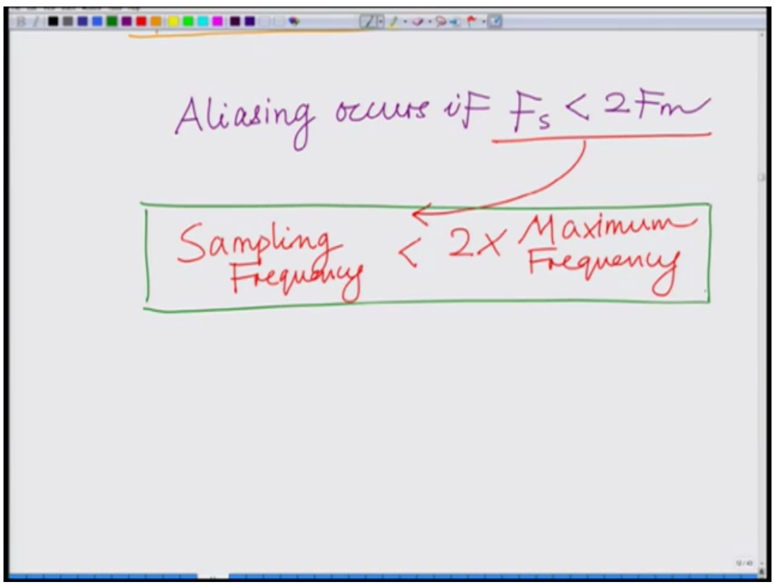

So this distortion this distortion which occurs because of overlap between copies of spectrum shifted by multiples of Fs is termed as is termed as Aliasing, this is termed as Aliasing. So this distortion which occurs because of this overlap this distortion is termed as Aliasing and aliasing occurs if Fs less than twice FM that is aliasing occurs this is a very important condition aliasing occurs if Fs less than twice Fm that is another way of stating this is basically sampling frequency is less than twice the frequency of the your sampling frequency is less than twice the maximum, sampling frequency is less than twice the maximum frequency of the signal being sampled. That is twice the maximum frequency of your m(t).

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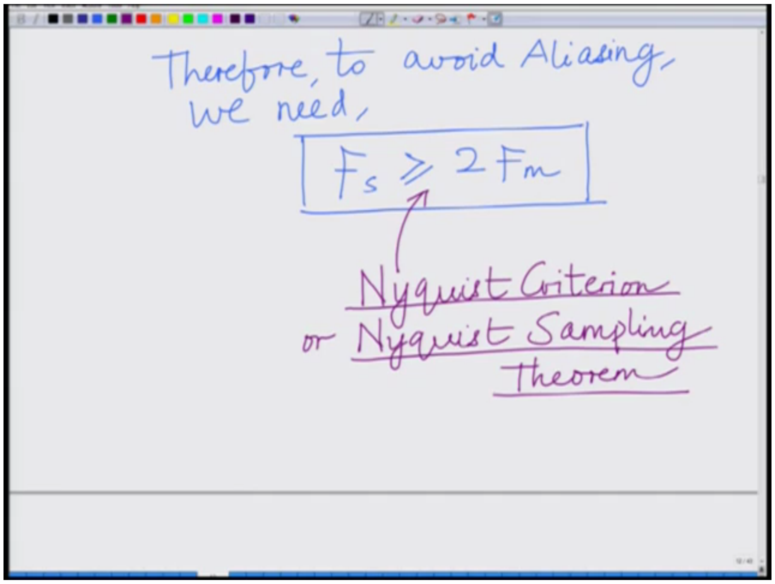

Therefore to avoid aliasing we need Fs greater than AM therefore to avoid aliasing to avoid aliasing or aliasing distortion we need Fs greater than or equal to twice Fm and this is termed as a this is termed as a Nyquist this is termed as a Nyquist criterion for sampling to avoid aliasing this is termed as the this has many names this is termed as the Nyquist criterion this is termed as a Nyquist criterion or your Nyquist, this is termed as a Nyquist criterion or the Nyquist sampling theorem which basically says that for to avoid aliasing which arises from the overlap of the spectral copies of the shifted spectral copies at the different multiples of the sampling frequency Fs one has to sample at a sampling frequency Fs greater than twice Fm where Fm is the maximum message frequency, alright.

Maximum message frequency Fm you can think of it also as the bandwidth of the original signal you have to sample at great sampling frequency greater than twice the bandwidth this is termed as the Nyquist criterion or the Nyquist sampling theorem which states that to avoid aliasing one has to sample at Fs greater than twice Fm, alright. And therefore what we have seen is something very interesting that the original sample the original the original sampled signal can be a distorted version of the original signal and that distortion occurs because these different spectral copies that is the spectrum of the sampled signal which comprises of the sum of the spectral copies which are shifted by different multiples of the sampling frequency various shifted by all possible multiples of the all possible integer multiples of the sampling frequency Fs, alright.

And because that can be possible overlap between these spectral copies that results in distortion which is termed as aliasing distortion occurs if Fs is less than twice Fm therefore to avoid this distortion we need sampling frequency greater than or equal to twice Fm this is termed as the this condition, alright this result is basically timed termed as a Nyquist (cri) this criterion or this condition required that is the minimum sampling frequency required to avoid aliasing is termed as a Nyquist criterion and this result is termed as a Nyquist sampling theorem, alright. So we will stop here and look at other aspects in the subsequent modules, thank you.