Electromagnetic Theory

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Lecture No - 85
Numerical Solution of Laplace's Equation

In this short module, we will discuss numerical solution of Laplace's equation.

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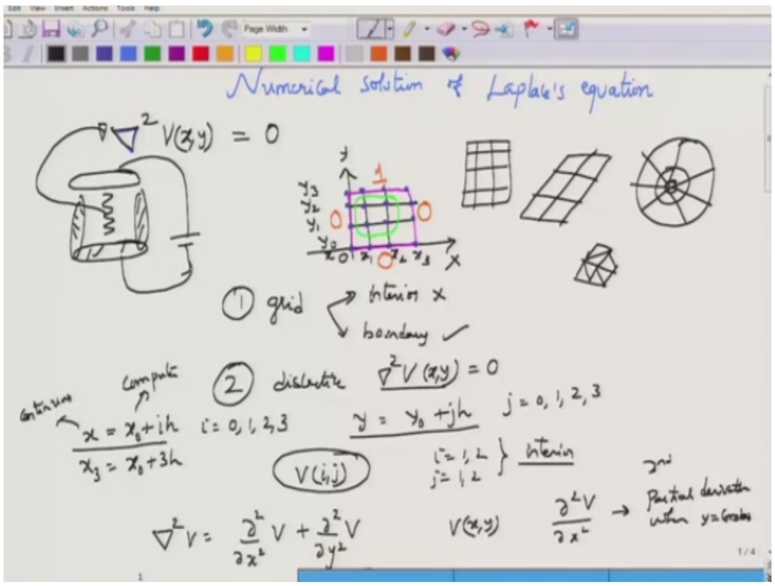

We have already seen analytical solution of Laplace's and Poisson's equation. This numerical solution can be it can actually supplement most analytical solutions. You can visualize them but in a more practical scenario you will be using lot of numerical methods to solve electromagnetic problems such as Laplace's equation, Poisson's equation, Wave equation. You know waves and the transmission line or wave guide.

These numerical solutions or numerical techniques are subject of deep interest because most problems in electromagnetics are not or cannot be solved simply by analytical means. There are a lot of numerical techniques that one can talk about but in this short module I will only discuss a simple numerical technique known as finite difference method for solving Laplace's equation in its most simplistic setting possible.

The two dimensional Laplace's equation is what we are going to solve numerically by using the fine difference method. But you should remember that there are lot of numerical techniques and these techniques are required because real electromagnetic problems are quite hard to be always amenable for an analytical solution. With that background for why numerical method is necessary let us jump right into the Laplace's equation, numerical solution of Laplace's equation.

By first looking at what Laplace's equation is of course we already know that for the wave which is sorry for the situation where you have a potential function V that need to be calculated. The Laplace's equation allows you to do so. So, delta square V of x gamma y, I am going to work in the rectangular coordinates. You can work of course Laplace's equation in any other coordinate systems. The formulation changes only slightly.

The formulas will change slightly the basic ideas would remain the same. So, let us say that I want to solve this Laplace's equation, delta square V of x y equal to zero. Obviously, this is zero but if it zero everywhere then there is a trivial solution for V. So, clearly this is zero only in the interior points on the boundary there are specific values of the potential V this is true because you will actually have physical electrodes placed.

And you are applying the potential differences that would create an electric field between the regions constrained by the electrodes. So, you have an electrodes let us say here you have one more electrode and you have one more electrode with a certain gap here. These electrodes can be electrodes or they could simply conductors and then you apply a certain potential here. So, may be this is time varying potential that you have applied.

This will create a time varying electric field inside this region which can be obtained by first knowing the potential function and then solving for this particular equation. So, once I know what is the potential function then I can go back and solve for the electric field. Of course, this is actually not time dependent I mean for this formulation we do not normally go for time dependent formulation in this manner what I am describing.

It is possible to solve this for time dependent but the formulation is slightly different. So, let us not complicate our life so I have this as a certain potential that I have applied. So, I will consider the simples possible situation. I will label this as a y axis. This as the x axis and let me define a certain you know let me define the problem consisting of this grid. A small gap is required physically but we will not really worry about that gap in the formulation.

So, this is the grid or you know if this is a tube, tubular structure you can think of this as a tube and then I am looking at the cross section of the tube. Let me also specify the values of the potentials at the boundary. Let us say these three values are zero and for simplicity let me put the value of 1 over here. Now how do I go about solving numerically? Well, computers cannot work with real variables x and real variables y.

There is a finite range of numbers which is determined by the word size of the computer. It could be a 32 bite computer or a 64 bite computer depending on that there is a certain range of numbers. This range is quite large but it is not enough for us to consider that the entire real number line can be implemented on a computer. So, computers do not take in continuous values. The values that you are using must be discretized.

So, the first step in solving Laplace's equations numerically would be to define a grid or a stencil. Sometimes there words are used interchangeably I prefer to use the word grid. So, what is this grid? It basically consists of considering certain points alone forming a grid. So, these are formed by the points. Let me try to use the different color here. So this is one point, this is another, three, four. So, these are the other points that I have.

So, please note the points that I am calculating here. This is a grid that we have plotted which means that on a computer since I do not have all the values of x so between these two grid points I mean I do not have the ability or the computer does not have the ability to represent all these points. So the computer recognizes only the points that I have shown on this grid. Now, you can see that this grid points can be separated nicely into two groups.

On this line where you know, at the edge of the grid or at the boundary the points all lie on the boundary. So, the outer grid points or all the boundary points with a specific values of the boundary values that are given in the problem itself. So, these are grid points. These are called as the end point or the boundary grid points or sometimes called the exterior points and there are certain grid points sitting inside.

So, there grid points are called as the interior points. These are called as interior points or these are the interior grid points and this is how you actually create a grid. This is not the only way to create a grid you can actually create a grid that would look triangular that could look skewed situation. So, let me try and put the skewed grid here. So, this is a skewed grid. A rectangular grid is the one that we have constructed. So, this is my rectangular grid.

You can also have circular grids. So you can have circular grids cut at different angles. So, this is one circular grid you can have triangular grid, you know this is a slight example of a triangular grid where in you are defining the grid points in a triangular way. So, these different grids are used for approximating the continuous range. So, in this case it’s easy for me to consider the rectangular grid.

If I were to do the circular grid I will be missing out some of the regions of interest. So, depending on what the shape of your problem is you might have to consider different grids and the performance of the numerical technique depends on the grid size the type of grid that you have used and the number of grid points that you actually take which is again corrected to the size of the grid.

The general thing is that try large numbers of grid points to get better or accurate results. You have to take this advice also with a pinch of salt because there are situations where you can increase the grid size but you do not actually improved accuracy in the solution. So, first step is to create grid. We have assumed a grid. This grid will now have two points the interior and the boundary points.

On the boundary, I already know the values of the potential because that is the problem statement. If I do not know the boundary values, then well this would be a boundary value problem. So, I know the boundary values or the values of the grid points at the edge or the boundary. I do not know the interior grid point values and this is what I want to calculate from numerically solving.

So, what would be the second step? Well, the second step would be to discretize delta square V of x y. This equation I need to discretize. Again, before going to this discretization I need to know how this v of x y itself can be represented. These are the grid points as I have told you let us label these grid points. So, I have a x which is going from so I have cut four plains, right along x. This is one plane, second plane, third plane and forth plane.

Let us say the values of x r 0 and so let us label this as x 0 x 1 x 2 and x 3 these are the four points that I have created. This could be for example x equal to zero and x 3 equal to 1, if that was you case and these are now four points along that. Similarly, for y I will have four points y 0, y 1, y 2 and y 3. Now, every point on the grid can be represented by a two dimensional array. I mean the entire grid point can be presented in a two dimensional array where x takes on values of x 0 + I into h, h being the separation between the grid points.

So, h is x 1 minus x 0 or x 2 minus x 1, or x 3 minus x 2. We will assume uniform grid spacing which means that h is constant for all this and i will go from 0, i will take on values i equal to zero, 1, 2 and 3. So, I have generated all the four points with x 3 being given by x 0 + 3 into h. So, this was h, 2h, 3h. So, 3h + x 0 will give me x 3. Similarly, y variables are actually on the computer. So, this is the continuous and these are the computer representation of x values.

So, y will be y 0 + j h and j again takes on values of 0, 1, 2 and 3 in this particular example, okay, depending on the number of repels this will be m. This would be n. So the total grid size would be m cross n array. Any point inside that can be represented by v of i, j where i would represent the i-th point. So, it would be i equal to zero, i equal to 3 would be the last point. This is kind of the C notation that we are looking at.

For the interior case, i is equal to 1 and 2, j is equal to 1 and 2. So, this would be the interior situation. The exterior points are all with i equal to zero or j equal to zero, I mean these are the outer layer points that we are considering when i and j are having at least one of those elements will be equal to zero. So, the first step is to grid and once you have grid you have defined a grid x and y will now become discrete values or discrete variables taking on these values which I have shown here with x 0 y 0 being the origin point.

And you can refer to the value of the potential at any point on the grid either it is boundary or in the interior by writing this as v of i j where i and j can take on the values of zero to 3. This is symmetric grid that we have chosen. Now, the second step is to discretize this delta square v of x gamma y equal to zero. Now what I mean by delta square v let us expand that delta square v is actually delta square/ delta x square of v plus delta square/ delta y square of v.

Now, this seems to indicate that if v of x y is the potential function then delta x square v/ delta x square is the partial derivative. Is actually the second partial derivative when y is taken to be a constant. In terms of upgrade what it means is that if I take the y equal to constant plane. Let us say y is equal to y zero plane how would my v of x y vary along this grid when y is held constant is given by delta square v/ delta x square.

Similarly, if I take an x equal to constant plane let us say x equal to x, x 2 plane I take then x 2 value is fixed whatever the value. x 2 is equal to x 0 + 2 h that is fixed. But then how does v of x y change or with respect to y is given by delta square v/ delta y square. So, clearly I need to represent this delta square v/delta x square and delta square v/ delta y square which are both continuous quantities on a computer. How do I do that?

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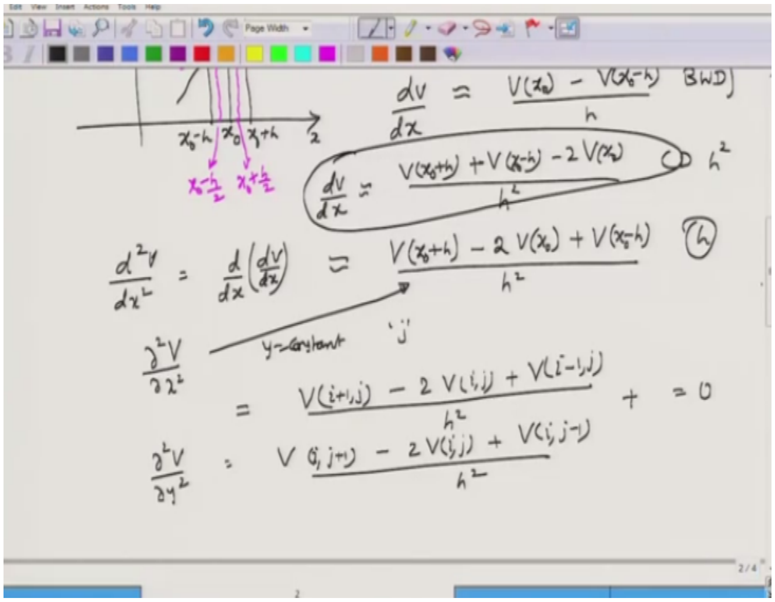

Let us take the simplest case of the single variable way or fixed. Let us assume that v of x is some function then if this function has a certain form which is continuous function let us say I pick a point x 0. If I pick a point x 0 and an additional point x 0 + h and one point to the left have x 0 minus h then what I am trying to find by v of dv/dx is that dv/dx is actually the slope of this one at the point x 0.

This slop can be obtained in two different ways so d v/ d x can be written as v of x 0 + h minus V of x 0 divided by h. This is called as the forward difference formula. This can also be written as V of x 0 minus V of x 0 minus h divided by h. In the classes on mathematics you would have seen that this limit of h must be tending to zero but on a computer I cannot tend h to zero without really blowing up the problem in a very complicated ways. So, I have h which is 6. It is small but it is not really going to zero.

So, this is called as a backward difference formula both formulas appear with an error of h that is the order of h. So, improve h to reduce the error you need to take (()) (13:57) h smaller. These are fairly bad ways of evaluating the derivative. A slightly different way which give you and error of the orders h square is known as the central difference formula which is obtained by appropriately adding these two terms. So, this is d v/ d x. This is also d v/ d x.

If I subtract these two so this is d v/ d x which is evaluated this point. If I have the same d v/ d x which is v of x 0 minus v of x 0 minus h. Actually, by manipulating these two let me not go into the details of the manipulation if you do that then d v/ d x can be approximated as v of x 0 plus h plus v of x 0 minus h minus 2 into V of x 0 divided by h square. To do this you need to use terra series and then expand this appropriately.

Let us not do that one at this point. So I am not going to do this thing but you guys can actually verify this. There is a 2 in the denominator, I am not, really not sure but that's the really immaterial at this point. So, I have d v/ d x as this part but this is not what I want. What I want is d square v/ d x square. How do I obtain d square v/ d square? Well, I remember that this in nothing but d/ dx of dv/ dx itself. Now, since I need an evaluation with respect to point h.

So, I will consider x 0 minus h/2 and x 0 plus h/2. I will find out what is d v/ d x at these two points and then take the difference. Here I won't do the central difference I will do a single difference and when I do that I will be able to obtain the expression for, d square v/ d x square and that expression will be given by so this d square v/ d x square is given by –in the numerical way is given by v of x 0 + h minus 2 v of x 0 plus v of x minus sorry x 0 minus h divided by h square.

Well, this seems to be in the same vain that we have obtained and that is right. So, this you need to remember that the central difference formula. But this is actually the formula for, d square v/ d x square as well. This has an error of h. So, let us get back here. So, this is your d square v/ d square which we have obtained numerically but again you might say that is not what I want. What I want is delta square v/ delta x square.

Well, the solution is that in this expression I have assumed y is equal to constant which means I have held j as a constant value and this is really the changes in i that I am looking for that is if I hold the y equal to zero plane which would be horizontal and I am looking at how v changes with respect to x that is what I have obtained. So, this delta square v/ delta x square is given by v of i + 1, j.

So, now notice that I am actually referring to the i + .1 on the j equal to constant value minus 2 v of i, j plus v of i minus 1, j right. This would be going along the y equal to constant plane divided by h square similarly, delta square v/ delta y square will be equal to V of i, j plus 1 minus 2 v of i, j plus V of i, j minus 1 divided by h square if I add these two and then take the result equal to zero that would be the numerical approximation or discretize of the Laplace's equation.

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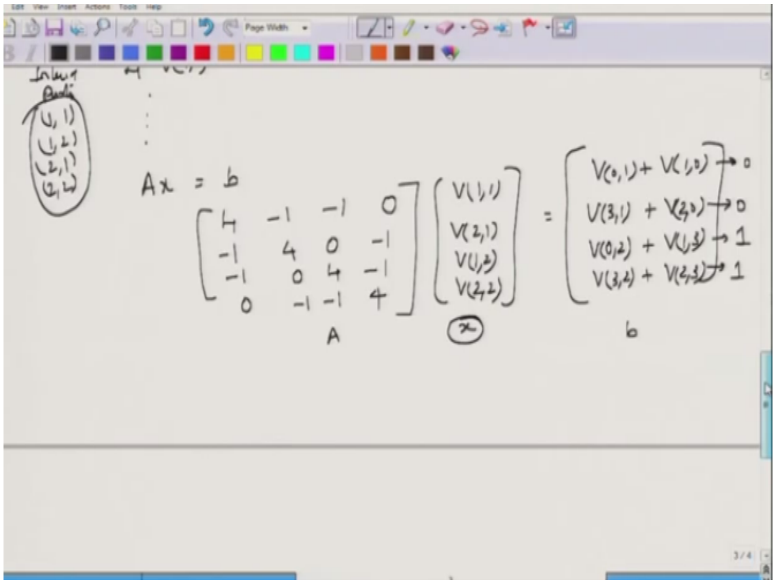

So Laplace's equation delta square v equal to zero becomes on a computer V of i + 1, j minus 2 V of i, j plus V I minus 1, j which would tell me how V is changing with respect to x then I have the corresponding expression for i which would be V of i, j +1 minus 2 V of i, j plus V of i, j minus 1 divided by h square is equal to zero. This is your second step. The third step is to assemble this equation in metrics form.

You might ask why we need to assemble equations in metric form? And the reason is because once you put this equation in a metric form then you can use very powerful metric solvers for solving the system of equations. So, you can find not only the value but you will also find the Igon Values which are sometimes required for this Laplace's equation or for other equations such as wave equations.

So, you can actually find solutions to lot more interesting situations where when you assemble the equations in the metric form. Now, you can do that for the case where i equal to 0, 1, 2, 3 and j equal to 0, 1, 2, 3. I will write down only one line of that equation and if you add these two there will be V of i j coming in with the factor of minus 2, minus 2 so that will become minus 4 and then I can push all these h square equal to zero on to that side.

So, if I write down one equation so that would be –and then change the signs of this one so that will be 4 V of 1, 1 I am writing this in the interior points only. I am writing this in the interior points which can be 1, 1 1, 2 2,1 and 2,2. So, these are the four interior point variables that I am looking at and corresponding to four interior points. I am writing the equations in here. So, this would be minus V of 0,1 minus V of 2, 1, minus V of 2, 1 minus V of 1, 0 minus V of 1, 2 is equal to zero.

So, now there are 1, 2, 3, 4, 5. So, you can continue to write the equation for the other three interior points. I will not write it here but you can and you should write this one and when you write everything and put them in a proper matric form you can write this as some A x equal to b where A will be this point 4 minus 1 minus 0 minus 1 4 0 minus 1 minus 1 0 4 minus 1 minus 1 minus 1 and 4.

So, you can write down the equation in this fashion and then say this is V of 1, 1. V of 2, 1. You have V of 1 , 2 and then you have V of 2, 2. This would be your x. So, this is x this is A, on to the right hand side you will have V of 0, 1 plus V of 1, 0 then V of 3, 1 plus V of 2, 0. V of 0, 2 plus V of 1, 3 and then finally you have V of 3, 2 plus V of 2, 3. So, you can substitute for the expressions here.

These two values will turn out to be zero so this is your b vector. So this will turn out to be zero. This will turn out to be zero and what is this V of 0, 2. V of 0, 2 point comes if you go back to the boundary this is zero and 2. So, on 1and y equal to 2 so the value here is zero but then there is one more term sitting here which is V of 1, 3 which is this interior point V of 1 and 3. So, this is 0, 1, 2, 3 which the point here. This is V of 1, 3 which is 1.

So accordingly these last two values will be equal to 1 when you solve this system of equations you will end up with the solution for the x vector and that would be the solution for v of x y and how it would be there in the interior points. On a Matlab or some other package you can then connect how we would be –and obtain surface plot or if you want you can obtain multiple line plots.

So, to summarize there are three main steps for numerically solving Laplace's equation using finite difference method.

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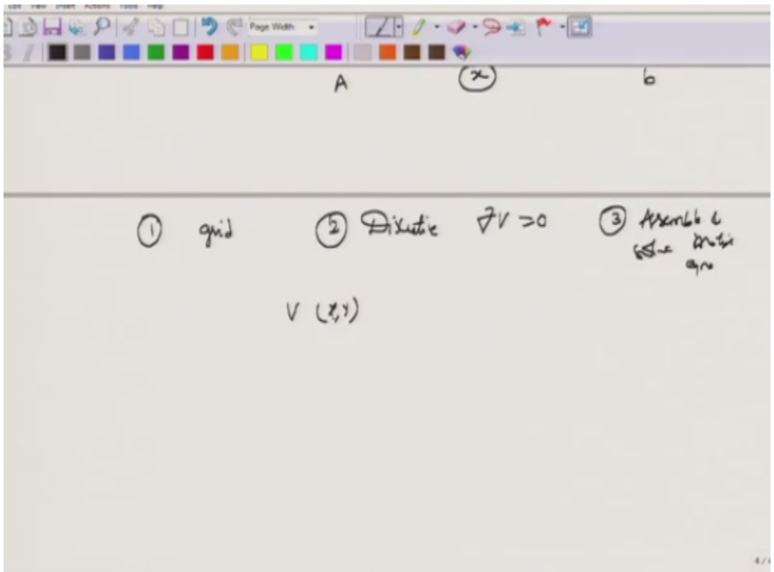

First is to define a grid second is to discretize delta square v equal to zero expression and finally you will assemble and solve the resulting metric equations to obtain V of x y. I would suggest that you understand this module and verify each statement that I have made and fill out this. So, I will leave this as an exercise to you and you will to obtain the solution. I will put up the solution in the attach note.

You can see that for more details and I hope that you will try out this simplest numerical technique to better appreciate the kind of problems that you can solve. Thank you.