Electromagnetic Theory
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Lecture No - 84
Group Velocity & Phase Velocity

In this module we will discuss concept known as phase velocity and another concept known as group velocity. We have not really talked about these two terms in the earlier lectures because we, I was waiting for waveguides part to be completed so that we may have a better understanding of what phase velocity means and what group velocity means.

Now we recall that plane wave which maybe polarized along say x or y direction and propagating along z direction would carry energy with it and the steady state solution, that is when all the transients have died down, the electric field for a x polarized wave can be written as,
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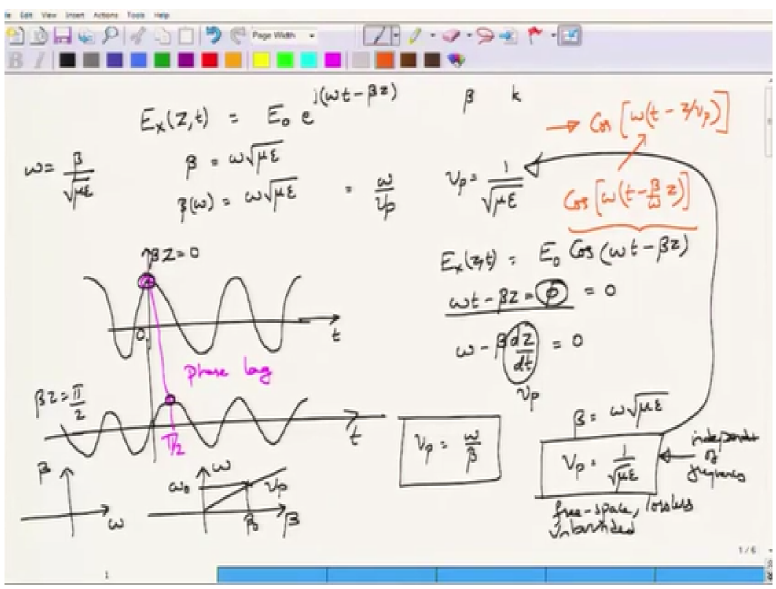

by giving it x component and propagating along z as you know of a particular frequency Omega zero as some constant E zero, e to the power j Omega t minus Beta z. If the medium is free space and lossless; unbounded. Free space is vacuum, so vacuum is we consider it to be lossless and unbounded. That is, it has no boundaries then for corresponding to every frequency Omega, there is a corresponding value of Beta.

And this relationship that we have between Beta and Omega we already know how to get that relationship. Beta is actually Omega into square root Mu Epsilon. In maybe in the earlier classes I might have used instead of Beta I might have used k, both essentially mean the same thing. These are the propagation constant. So for every value of Beta you can even imagine this Beta being a function of Omega.

Because for every frequency of the plane wave that is propagating, Beta of Omega is given by Omega into square root of Mu Epsilon. And for the free space Mu and Epsilon are both constants and square root of Mu zero Epsilon zero in that case will be the velocity of wave in free space which is C. So this would be equal to Omega by V in general or let me call this as V p for reasons that will become very clear very shortly okay.

So from this equation that I have written it is very clear what V p should be. V p is this parameter one by square root Mu Epsilon. It can be applicable for a homogeneous dielectric medium which is unbounded or it could be applied for free space where in you substitute Epsilon is equal to Epsilon zero. Now this is the complex notation for the wave as you might remember. So this would be the Cosine solution that I have.

Of course this solution keeps propagating all the way from minus infinity to plus infinity so let us say this is the wave that would appear at Z equal to zero. So this would be the wave at Z equal zero this is the electric field, you know the units of electric field you can write down the corresponding unit for that one. Now let me arbitrarily assume that this is time equal zero reference okay.

Any point can be considered time equal to zero reference because this wave extent all the way from minus infinity plus infinity in time, so I will arbitrarily choose this as zero. Now what would be the wave at a later stage right, at later place? So let say at Z equal to in such a way that Beta z is equal to Pi by 2. Then what I have is Cos of Omega t minus Pi by 2 would be the wave that would exist at the location Beta z equal to Pi by 2.

So this would be the case when Beta z equal to zero. This is actually the Z equal to zero plane and for the case where Beta z is equal to Pi by 2 which I have taken arbitrarily, then there is a Pi by 2 phase lag as we would see because the wave that would exist at Beta z equal to Pi by 2 plane; would I am drawing this on the same graph hopefully I am going to get the graph also right. Beta z is equal to Pi by 2.

So you go to maximum here and then you should go to a minimum here. So this is the Sin wave which I am trying to plot but I am slightly struggling over here. So let me actually draw it on the line that is below okay. So that would be easier for me to draw. So I have this line over here okay, the same time now instead of having the wave at this point, at t equal to zero, being equal to zero the value of this one will be when Beta z equal to Pi by 2 at t equal to zero Omega t is equal to zero.

So you have looking at the real part of it Cos of Pi by two which would be zero. So what you actually have Sin wave right and this Sin wave would be the one that would be propagating for all times. So if I were to hook up two oscilloscopes, one oscilloscope at Z equal to zero and other oscilloscope at Beta z equal to Pi by 2 or Z equal to Pi by 2 Beta. Then I will see that the wave that would be there at Beta equal to zero is slightly different from the wave at t.

The properties remain the same. The wave at Z equal to Pi by 2 Beta or Beta z equal to Pi by 2 will have the same polarization; same frequency everything, except there is a fixed phase relationship. The maxima that would be occurring at Z equal to zero is now shifted at Pi by 2 right so or wait Omega t equal to Pi by 2. So this shift is what we call as phase delay or phase lag.

So we say that the wave at z Beta z equal to Pi by 2 is lagging with respect to the phase of the wave or with lagging with respect to the wave at Beta z equal to zero. So this phase lag or phase delay, what would be the velocity of this phase lag? That is to say if I were to follow this maxima right and then if I were to look for the velocity of this maxima, I will actually follow along the wave and then see how quickly this maxima is travelling along the wave.

That would actually mean to take the argument of this E x of z t right, so let me write down this in terms of Cosine thing because we know that real electric field will be, I mean will have a Cos term actually because it will be real part of this one. Let us not use the complex notation. So if I look at this particular wave and if I want to follow the peak of the point, actually it can be any point that would be considered.

It could be the constant phase point that I am actually looking at, so if I were to follow this phase and how quickly so how much time did it actually take to go from this maxima here to maxima at a different point? So this is the velocity of this constant phase point and this constant phase point velocity is called as the phase velocity of the wave right. To obtain the phase velocity of the wave I am going to follow the maxima or I can actually follow at any particular phase point Phi then I am looking at how quickly Phi travel.

Which means that I am looking for d Phi by d t and if it is a constant phase point the argument should be such that, if I fixed z and time t increases, the relationship if both z and t increase they have to increase together such that Omega t minus Beta z is always equal to this phase Phi right. So this is the constant phase that we have considered and that phase should be the same value so that the value of the electric field will always be the same.

That is what essentially means by look at the phase velocity of the constant phase point right. So how would I be able to obtain? Sorry this is not d Phi by d t, the velocity is d z by d t how quickly this phase point is moving. So for simplicity I will take Phi equal to zero which means that I am following the maximum okay. So since I am following the maximum, I can then now find out the derivative of z with respect to t which will give me the phase velocity.

So differentiate this equation with respect to time you get Omega. Because Omega t differentiation with respect to time will be Omega minus Beta is a constant with respect to time so Beta comes out and then I have d z by d t. That would be equal to zero. But d z by d t precisely what I have called as the phase velocity of this constant phase point. So I seem to now have a relationship with says V p is equal to Omega by Beta okay.

And since Beta happens to be for free space, a parameter that would be independent of, I mean Beta would be linearly proportional to Omega with this proportional defector. So we have V p is equal to 1 by square root Mu Epsilon. So this is the reason why I called this constant V p by 1 by square root Mu Epsilon because I knew kind of beforehand that this would be the velocity of the constant phase point for a single frequency wave which is propagating along the z direction.

This is important. It is a single frequency wave having frequency of Omega. If you were to graph this Beta versus Omega, so let say Beta is here and Omega is here. In some cases, you will also see the axis interchange with Omega on the x axis and Beta on the y axis, both representation are equally used widely used. We will also be using this notation so that you would become very familiar with these two ways of visualizing Omega and Beta.

So for free space it is fairly simple. The relationship between Omega and Beta is obtained by inverting this. So what you have is Omega is equal to Beta divided by square root Mu Epsilon and clearly the ratio of Omega by Beta is the phase velocity which we have obtained. The relationship here is linear relationship and at any particular point which corresponds to frequency Omega zero and corresponding point of Beta zero.

If you were to look at the slope of this line or by you know by finding the ratio of Omega zero by Beta zero, the slope will give you the phase velocity V p okay. And it is important to note that for free space condition and lossless condition, free space; lossless; unbounded medium right for all these cases or for entire this case, the phase velocity is independent of frequency right independent of frequency.

You might appreciate why this independent thing would be important if you were to look at this Cos of Omega t minus Beta z argument and we have seen that there is a particular phase lag. So I can rewrite this Cos Omega t minus Beta z in a slightly different way. So let me rewrite this one by taking Omega out. Now I am just writing this part so I have Cos of, Omega is taken out, so t minus Beta by Omega into z right.

Now Beta by Omega, Omega by Beta is V p so Beta by Omega should be 1 by V p so this entire thing can be written as Cos of Omega t minus z by V p right. This clearly indicates a travelling wave form with respect to time but its phase velocity is V p. If you remember your physic classes, this would be the wave equation for the wave that is propagating having a frequency Omega having a velocity at that point it was not specified what velocity you were considering but it is actually the phase velocity.

So it is clear that the phase velocity happens to be independent of the frequency and therefore the phase lag that you would obtain would be the same for all frequency signal. So if you have if you change the frequency from Omega 2 to Omega zero or from Omega zero to Omega 1, the corresponding values of Beta also changes in a proportional manner such that both the phase velocity as well as this one by V p which would be the phase delay right phase delay per meter kind of the thing. So that would also be independent of frequency.

These are really important concepts. The reason why this would be important is seen in the next case where are we going to look at a situation where the phase velocity is not linearly proportional to or is not independent of frequency the reason being Beta is not linearly proportional to frequency.

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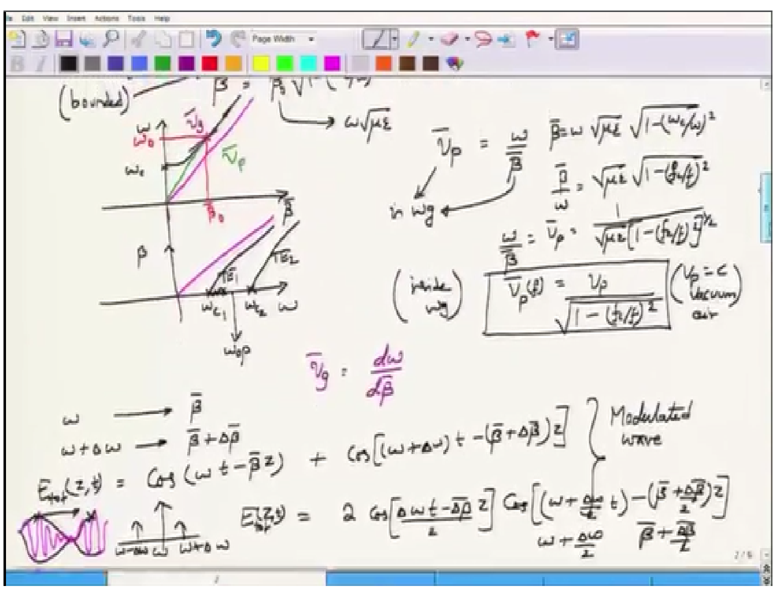

This is the situation where we have obtained; we have already solved this one for the waveguides right. So in waveguides we had this peculiar situation that Beta was related in a nonlinear manner so it was equal to Beta zero, so I will use a bar over bar to indicate that we are considering Beta in bounded medium right. Waveguide is an example of a bounded medium and I am continuing to assume that the medium is lossless.

But I am indicating this bounded medium or the waveguide propagation constant by writing a bar over here so Beta bar is equal to Beta zero into square root of 1 minus Omega c by Omega. Remember the Omega is the cutoff frequency and this equation of course is valid when Omega is greater than Omega c that is mode is actually propagating right. So this would be the equation for Beta, Beta bar.

And if you were to sketch this one on say Beta bar axis versus Omega axis right, you would see that until Omega is equal to. I mean as long as Omega is less than Omega c the corresponding Beta bar will actually be equal to zero. Since Omega is increasing horizontally this way I will consider this as the cutoff frequency Omega c until this Omega c the value of Beta will be equal to zero.

Thereafter it increases in this manner okay. For Omega being very large compared to Omega c right, for a case where the operating frequency is so high compared to the cutoff frequency then I can neglect this Omega c by Omega and this would actually approach, Beta bar would approach Beta zero because Omega c by Omega can be neglected. Its square certainly can be neglected and therefore the square root of 1 minus small quantity.

The small quantity can almost take to zero and this would be equal to Beta zero. So for very large frequency there is an approximation. This particular Beta versus Omega curve will approximate the Beta equal to Beta zero curve. So in another words the propagation constant becomes nearly the same as that of the free space propagation constant Beta zero which again is given by Omega into square root of Mu Epsilon.

Where Mu and Epsilon are the constitute and parameters of the medium that is consist, that is used to form the waveguide. So you have Beta bar equal to Beta zero which is actually Omega into square root Mu Epsilon and multiplied by a certain factor and this is the factor which is making the relationship between Omega and Beta as nonlinear. A slightly different way of putting the same thing would be to plot Omega on the x axis okay.

So you plot Omega on the x axis and you plot Beta on this axis and here you will see that until Omega is equal to Omega c, the cutoff frequency Beta will be equal to zero and then it would go like this. There is a associated asymptote as well, sorry I should have curled this one slightly better. So there is a corresponding approach to the asymptote as well. This would be the situation for the fundamental mode. What would be the Beta versus Omega relation for a different mode?

Well for a next mode that would be like this okay. So let us call this as Omega c 1, this as Omega c 2, these would correspond to the fundamental mode. For example, if this is parallel plate waveguide, this would be T E 1 mode, this would be T E 2 mode and so on. For rectangular waveguides similarly the modes would be t 1 zero, t 2 zero and so on okay. So if you are operating frequency happens to be anywhere here.

So if this is my operating frequency, then only one mode exists and for every value of frequency Omega there is one to one correspondence between Beta and Omega. Whereas if the operating frequency goes over here then for every value of Omega there are two values of Beta, these two values of Beta indicate that there are two modes which are propagating okay. So please keep these two in mind.

Now the primary problem that has happened here is that Beta is now a nonlinearly related to Omega. It is not exactly the linear relationship that we were looking at okay. Now what would be the phase velocity? If I go back to the same condition that the phase velocity can be defined as the ratio of Omega to Beta, I can still do that, so I will be defining this as Omega by Beta except that I am putting a bar over here indicating that this is phase velocity in waveguide.

Similarly, Beta is the propagation constant in waveguide okay and if you now look at this, what is this relationship between Beta bar given by Omega into square root of Mu Epsilon so maybe I can just try to find this Omega by Beta. So when I do that one what will I get here is square root of Mu Epsilon into square root of 1 minus Omega c by Omega hold square right. So this is Beta is equal to Omega or Beta bar equal to Omega into square root Mu Epsilon into this particular factor.

So if I take the ratio of Beta to Omega, so I will be pushing this Beta bar by Omega equal square root Mu Epsilon square root of 1 minus f c by f hold square because Omega c is 2 Phi of c; Omega is 2 Phi f, so for the corresponding frequency f this would be the relation. So Beta by Omega is the phase delay so Omega by Beta bar which is the phase velocity inside of a waveguide will be given by 1 by square root Mu Epsilon into 1 minus f c by f hold square under root.

This 1 by square root Mu Epsilon is nothing but the phase velocity of a wave. If the medium would have been a free space right so this would be the free space phase velocity V p divided by 1 minus f c by f hold square under root. What has happened here is that although there is a one to one relationship between V p bar that is the phase velocity and the frequency as long as you are working in the single mode condition.

Let say that would be one to one relation but the problem is that V p bar depends on the frequency f. So if my operating frequency is here; if my operating frequency is here; if my operating frequency is here so for all these three different cases that corresponding value of Beta is different in such a way that the phase velocity will also be different. So I should more clearly label this as the frequency dependent phase velocity.

And the frequency dependent phase velocity is the one would cause a lot of problems okay. What would be the slope kind of relationship? Let say I am operating at some point over here, I know that corresponding to this there would be a corresponding Beta bar zero and this is the operating frequency Omega zero. If I were to draw a line okay, so this red line would correspond to that.

If I were to draw a line and take the slope of this line which would be Omega zero by Beta zero that would be the phase velocity V p. So what I have obtained is phase velocity V p bar. However, the actual relation between Omega by Beta if I look at the local slope here that would be different and that local slope if I were to find out by drawing a short line here and then taking the slope of this one that will give me what is called as the group velocity V g.

So group velocity V g bar is given by d Omega by d Beta bar okay. Why is that relationship? Let us actually look at this case okay. So let us consider that we have two waves okay one having a frequency of Omega the other one has a frequency of Omega plus Delta Omega, both are propagating along z direction. Corresponding to this Omega you have Beta bar; corresponding to this one you have Beta bar plus Delta Beta bar, that is we assuming that for given Omega there is corresponding one to one Beta bar right.

So that is clear because this relationship is still one to one relationship. Whereas for Omega plus Delta Omega where Delta Omega is a small displacement from Omega Beta bar would also change only slightly right. We will later find out what Delta Beta bar is. So for this particular case that two waves can be assuming that both have the same amplitude and the amplitude is equal to one, I can write the waves as Cos Omega t minus Beta bar z and because the total

electric field must be the sum of both individual frequencies so I have Cos of Omega t minus Beta bar z plus Cos of Omega plus Delta Omega into t minus Beta bar plus Delta Beta bar into z okay. So this is clear that the total electric field is a sum of one electric field component and the other electric field component. These two components have slightly different frequencies of Omega and Delta Omega.

You would actually recognize if I were to put one more term here and say, if I put this term as Beta bar minus Delta Beta bar okay into z if I have this electric field then you would clearly recognize that this is the result of modulating this carrier and producing these two side bands. This would be the upper side band and this would be the lower side band and then this electric field total would be called as the modulated wave.

And this is true you can actually call this as the modulated wave and this modulated waves are the one which actually allow you to convey information because this side band would then carry the information. If you were to just have the carrier, then there is no information because if you fix Omega Beta bar is fixed and amplitude is not usually considered here so there is actually no information being transmitted.

If someone knows what is Omega Beta bar and this one or you can very easily measured, there is really no information being conveyed with this one. Because if you look at the frequency domain picture the corresponding frequency here would be that of whatever the frequency Omega that we have to considered here right, so there is actually no spread around this Omega indicating that there is no temporal changes happening.

This would simply a boring Cosine wave propagating all the way from minus infinity to plus infinity. When you have this situation then there are side bands Omega plus Delta Omega and Omega minus Delta Omega and the corresponding time domain wave form would be different right. Now you will have in addition to the envelope right, there would be carrier. So you have this carrier and there is corresponding envelope.

And it is this envelope which is changing that conveys information to you. In more general case the spread will not be like this sinusoidal one but we will not consider that at this point okay. And for simplicity that is for mathematical simplicity I will assume there is only carrier and it is upper side band. Now I can use some well-known trigonometry identity to simplify this expression.

And write down for the total electric field as 2 Cos Omega Delta Omega t minus Delta Beta bar z. So please verify this one its coming from Cos a plus b formula, plus sorry this is multiplied by Cos of Cos of Omega plus Delta Omega by 2 into t minus Beta bar plus Delta Beta bar by 2 into z okay. So this would be the total electric field. So let us call we have called this as E total of z t and now you can see two terms right, on the - on this term; on the right hand the right most term you have the frequency of Omega plus Delta Omega by 2.

The propagation constant is Beta bar plus Delta Beta bar by 2 but we

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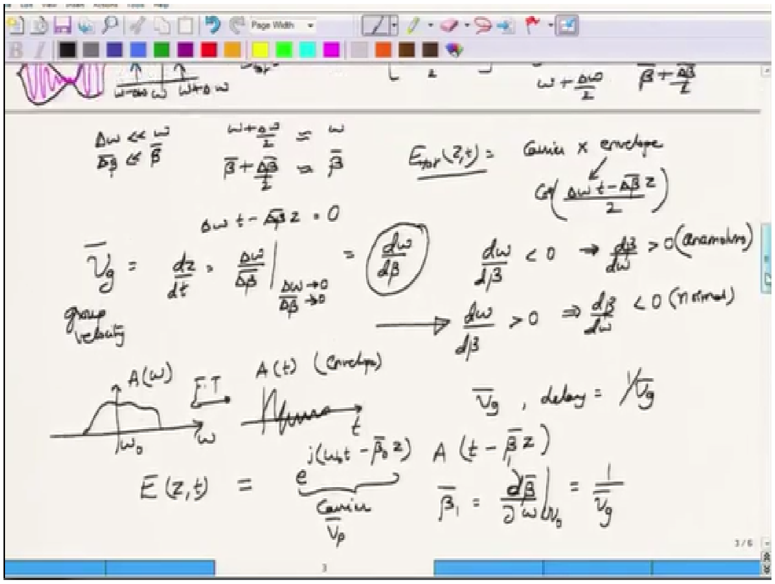

have assumed that this Delta Omega is so small compared to Omega and similarly Delta Beta bar will then for will be so small compared to Beta bar. That I can write down Omega plus Delta Omega by 2 as Omega itself approximately and Beta bar plus Delta Beta bar by 2 approximately as Beta bar itself okay. So this was really giving me the carrier so I will now have the total electric field component being carrier. I am not writing what the carrier is.

It is very clear what the carrier here if you make approximation times there is some envelope out here. What is this envelope? This envelope is Cos of Delta Omega t minus Delta Beta bar z divided by 2. If I now look for how quickly the phase of this envelope varies, that phase velocity of the envelope can be obtained by setting again Delta Omega t minus Delta bar z to particular phase value which I am going to consider as zero and then differentiate this equation with respect to time.

So I get Delta Omega by Delta Beta bar d z by d t which is now the phase velocity of the envelope. Please remember this is very different phase velocity that we are considering. One hand you have a phase velocity of the carrier itself that is given by Omega by Beta bar. Whereas you are now considering the phase velocity of the, so this is the velocity with which the constant phase point on the envelope modes okay and this velocity of the envelope is called as the group velocity okay.

So this is called as the group velocity and group velocity is given by Delta Omega by Delta Beta bar and in the case where Delta Omega tents to zero or Delta Beta bar tents to zero equivalently this relationship is given by d Omega by d Beta which is precisely the slope that I had talked about okay. So these are the two velocity that you need to consider and you need to understand that for the cases that we have considered free space velocity V p is different.

This would be equal to z c for vacuum or air okay, however the velocity the phase velocity inside a waveguide is different and it depends on the frequency. There is also corresponding group velocity V g bar which is obviously a function of the frequency. Why? Because d Omega by d Beta is the definition for the group velocity and this particular case also shows that V g is dependent on frequency Omega.

In most cases you have d Omega by d Beta to be less than zero or you will have the situation where d Omega by d Beta is greater than zero, if this happens this indicates that d Beta by d Omega is greater than zero right and this indicates d Beta by d Omega is less than zero. So this particular case is called as the anomalous dispersion case and this is called as normal dispersion case.

And the case for group velocity normally defined very well for this situation that is for normal situation of normal dispersion is where we can define very clearly what the group velocities are. The situation for the anomalous dispersion is very complicated. So we will not pursue this anomalous dispersion case here okay. So before leaving this group velocity point I want to emphasis one thing which might normally cause confusion.

If you look at the expression for the phase velocity V p bar inside of a waveguide right so this is inside the wave guide, I am emphasizing this again, you will see that there is term like V p divided by 1 minus f c by f hold square. Now when f is greater than f c, f c by f is a number which is smaller than one and a number smaller than one square is still smaller than one; one minus of numbers of smaller than one is number that is less than one and square root of number that is less than one still less than one.

But the implication seemed to be that V p bar of f is greater than V p and if the medium of the waveguide or something is that actually happens to be air or vacuum right, then it seems that V p bar for a given particular frequency f will be greater than c. Now how is it possible that something can have a velocity that is greater than c? We should remember that this is not really the velocity with which information data is actually is transmitted.

It is simply a mathematical velocity or the velocity of a hypothetical phase point and this hypothetical phase point can move at whatever velocity it wants. However, the actual information content is moving at a velocity given by the group velocity. You can see here right. So inside here the carrier is fluctuating quite a bit which means the phase velocity of the carrier would be higher compared to the group velocity which would be the velocity with which the envelope moves.

It will take a lot of time for the maximum of the envelope to move compared to the time where the carrier is fluctuating. So it does not violate the principal of relativity which states that no information can travel faster than light and this is clearly situation okay. It just because you get V p bar of f greater than V p it does not mean that the information is being transmitted. It just the hypothetical phase point that is moving along with the given velocity.

The information is carried by group velocity and as I said group velocity is a concept that can be nicely defined when you have normal dispersion. Let me just highlight couple of points before I leave here. In general, the information content will not be like a two side band thing that I have shown you but it would essentially be some sort of a spectrum which is spread around the carrier frequency Omega zero okay which has a certain band width of Delta Omega and this represents the modulation.

So if this would be the envelope and I call this envelope as A and this would be A of Omega, then this envelope A of Omega would be centered around Omega zero which is where the carrier is and the carrier is modulated to get this spectrum and the time domain response A of t would be something okay. So this let us say this is the A of t corresponding to this one, it is not but I am just considering an example here.

So this variation in time of the envelope is the one which carries information so these two are related of course by the courier transform and if you were to look for what would happen if instead of sending just a sinusoidal waves, what would happen if I were to send this type of envelope or equivalently this type of a spectrum. I will not go into the details of here, you can actually see that in the attached note that I will be putting in here okay.

So in the attached note you will see that derivation of this one and you will see that the electric field component, this is only envelope okay so the envelope has to be multiplied by the carrier to become electric field as you can see here right. So just the envelope of this A of t, so you will see that the total electric field actually can be written as e power j Omega zero t minus Beta zero bar into z which would be the carrier moving.

And the velocity of the carrier is the phase velocity V p bar okay times the envelope will be delayed version, assuming that this is the lossless medium the envelope will be the delayed version which is given by t minus Beta one bar into z. Now what is Beta one bar? Beta one bar is short hand notation for Del Beta which in price that Beta can be of, Beta can be of function of Omega as well as other parameters.

So this would be Del Beta by Del Omega evaluated at V zero, sorry Omega zero so this would be evaluated Omega zero and this is equal to 1 by V g bar. Remember V g bar is d Omega by d Beta so d Beta by d Omega is 1 by V g bar and this is the delay or the velocity of this one would be the group velocity and the delay will be the group delay. Delay is 1 by V g bar okay. This would be the group delay and the envelope arrive at a later time which is determined by the group delay.

As I said these concepts are very well defined for normal dispersion case and for the case where you are propagation constant happens to be linear, I mean one to one function. That is if the operating frequency is taken to be such that there is waveguide is essentially single moded then all this concept that we have discussed will be sufficient to understand the phase and group velocity.

These concepts need to be defined in a much more rigorous way when you go to anomalous dispersion or you go to the situation where you have multiple operating frequencies okay. So this is the phase velocity and group velocity. Let me just remind you that anomalous dispersion case, the wave actually the information in the form of a pulse is used to compress the pulse. So the anomalous dispersion used pulse compression, whereas normal dispersion is leads to pulse expansion.

More rigorous way and much more details of this one can be found in the literature or in the attached note. Please take a look at the note. Thank you.