Electromagnetic Theory
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Lecture - 08
Cylindrical Co-ordinate system

How do I specify points? Go back to our idea of specifying points on a road on a plane. On a road, I show the particular co-ordinate axis. How this were lined up on to the right of my reference house or to the left of my reference house. When I graduated to two dimensions, how this scattered all over the plane and if I wanted to specify the location of a particular house, right?

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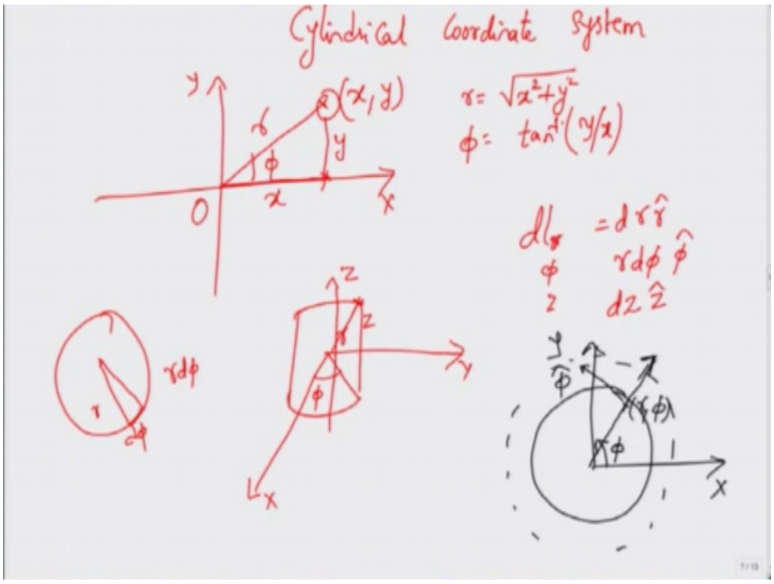

So if this is the plane, if I wanted to specify the location of this house, I had to specify them by giving the x and y co-ordinates. I had to tell you how to move along the x direction, what value you are going to move and proceed vertically until you reach this house, this is XY plane and this particular point was given the co-ordinates x and y indicating the amount that you had to move along x and the amount that you had to move along y.

So if you wanted to also specify the height, you had another option going to z co-ordinate system. This is the rectangular Cartesian co-ordinate system that we studied. Is there another way of specifying this location of the house, answer is yes. Instead of specifying the distances to move along X and Y, why not I specify the distance from the reference or the origin to the house and the angle of the road assuming that there is a road in that particular angle, what is the angled road that I have to take in order to reach that house.

So all I am saying is that you start from the origin and you want to reach this point you head towards this house at an angle of say phi as measured from the x axis because angle is some quantity that you will always have to measure with respect to two lines. So measure angle with respect to x axis, this is a conventional choice and what distance you have to move. That distance let us call as r.

Can I now find out what is r and phi? Here is a small point. In some of the textbooks you will find that this r is actually denoted by rho. I am not using rho for a very important reason, rho for us is the charge density. If I use rho there might be a case where I will be writing rho L rho hat and at least for me this kind of starts getting confused. Rather than that I know that r stands for radius, okay?

So I am going to use r as my variable over here and applying the law of right angle triangle if this is x and this y obviously r is equal to square root of x square plus y square. Now you see why I used r in the previous example preciously because I was anticipating that I am going to discuss cylindrical co-ordinates with you, that is why I used r is equal to square root of x square plus y square. How do I calculate phi? Well, use trigonometric relation.

Phi is nothing but Tan inverse or inverse tan of y by x. So these two will give me the equal end points on the cylindrical co-ordinate system just as the x and y points would give me the value of the point or the co-ordinates of that particular point. On three dimensional case you have to imagine that there is a cylinder of appropriate radius.

I am still going to use by xyz Cartesian co-ordinate system but now I am going to imagine a cylinder of some radius r and if I want to locate a point p here all I have to do is to give the radius of the cylinder and also give you the height of the point above the z is equal to zero plane. So I have this height which is z, okay? Then I also have to specify what is the angle.

So in order to do that when I have to drop down this particular like, I have to drop down this line on to the xy plane and then measure the angle of this line with respect to the x axis so as to give me the angle value phi. Okay, so probably a slightly easier way to show you that would be to take up this piece of paper, roll it up. Now I have a cylinder here. This is my z axs, imagine that this is my x and y plane.

I am keeping on this board which is the xy plane and this is the z axis. To specify any point here I have to specify at what height I am and what is the radius of the cylinder. And if I want to find out the angle, I have to drop this line down to the xy plane and then connect the origin to that line in order to give me phi. So this is the z axis or this is the z value that I am going to get.

This is the radial value or this would be the radial value or if you take the origin here that would be the radius value here. This is the z axis, this is the r value and then you drop down this point on to the xy plane which would come out to be somewhere over here let us say. This axis would then become the, this angle would then be the value of phi. So quickly let me finish up by giving you the values, the lines along r along phi and along z.

Along r is fairly simple, this is dr, this is dr in the direction of r, in the increasing radial direction. Along phi interestingly will have to be r d phi in the direction of phi because phi itself is angle it is not distance, you need to measure that and you need to multiply with the appropriate radius that you are looking at. So if this is your radius r and this is the distance that you have moved d phi then the arc length here is r d phi and that is the line segment here and along z it is simply dz z hat no change from the rectangular co-ordinate system.

So how do we obtain the unique vectors r phi and z that I have written down for the cylindrical co-ordinates in the last lecture? Well, let us recap the cylindrical co-ordinate thing. Now this is a cylindrical co-ordinate system. I am writing everything in this two dimensional case, that is with only x and y here because for z it is essentially the same as rectangular co-ordinates.

So there is no change in the rectangular co-ordinates, line segment. So let us not get bothered about the 3-line segment for that one. So you have the x and y co-ordinate system over here. Any point here we have previously mentioned as x and y. Now we are representing this as a point r and phi. Okay? So where should the r and phi unit vectors be located. Well, remember what is r, r is the distance from the origin to the circle over here.

So it is increasing in this particular direction. So if I want to write down a circle of a different radius, I will be writing down a circle like this. So the circle is actually expanding. So r is increasing in this way. So the unit vector for r would also be along this particular direction. This point the r will be along this direction. In which direction will unit vector for phi be located, first of all we know that this is the angle phi measured with respect to the x axis.

So the unit vector must be in the direction of increasing phi. So phi must be along this direction. So you have r vector and phi vector. I can rewrite these vectors here in slightly better way.

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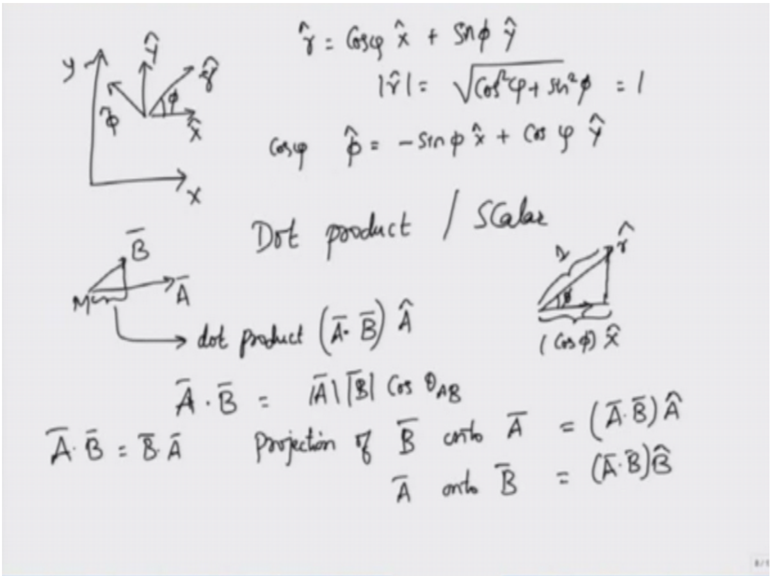

So I have r vector, unit vector and the phi vector here. And the same point, I also have the unit vector for x and y, with the idea that this angle is phi, this angle is 90 minus phi, this angle is phi. So the question is what is the relationship between the vectors r, x, y and phi. What is the relation? I know that any vector in this particular x, y plane can be expressed in terms of x and y unit vectors.

So I can write down, the r vector as something, times x vector plus something else times v vector. What is that something else? You can see that the relation between r and phi can be obtained by taking this r and then if you imagine that this is one and this angle is phi, you can find out what is the length of this particular component and this particular component length will be cos phi and this other length will be sin phi.

So this something times x hat plus something times y hat in these equation are expressed cos phi. So I have r is equal to x cos phi plus y sin phi. So I have sin phi. Okay? What is the magnitude of r? The magnitude of the r unit vector should be equal to one, and is this equal to one? Yes, because this will be equal to square root of cos square phi plus sin square phi, this is equal to one.

Similarly, you can show that the unit vector for phi can be written as because this phi is in this particular direction. Sorry, this is phi. So phi along y will be cos, phi along x will be pointing in the negative direction. So phi hat vector or the unit vector along phi is given by minus sin phi x hat plus cos phi y hat. Okay? So I sort of talked about a vector along a particular direction.

So I have shown you that r vector can be written as cos times phi along x and sin phi y and phi vector can be written as minus sin phi x plus cos phi y. So I talked about this particular thing that r vector along x is cos phi times x hat, r along y is sin phi times y hat. What exactly did I mean when I said that one vector is along another vector. What did I mean by, let us say if I take two general vectors, let us call them as vector A and vector B, both defined in a particular common origin M.

So what do you I mean when I say that vector B along vector A. What do you I mean by that? What I mean is that in order to tell you what I mean I need to introduce you to dot product. Dot product is an example of vector algebra multiplication of the two vectors. We talked about two vectors and we added the two vectors, we subtracted the two vectors but we did not introduce how to multiply the two vectors and that is precisely what we are going to do now.

We are going to multiply two vectors and the result of this multiplication is going to be a scalar or a number, that is why sometimes dot product is also called as scalar product. Dot product is also called as scalar product. So what is this dot product. Take two vectors, A and B. Now I have two vectors over here. This stick represents a vector. This is a good way to show the vector and have another vector B.

Dot product is defined as the length of the component B along the vector A. That is imagine that if someone takes up another ruler or a scale and then drops a perpendicular from the tip of the vector B to the vector A, to the point on the vector A. So this tip would fall here. So when this falls, you can then look at what is the length from origin to this one. So this particular length gives me the dot product of the vector A and B.

So what is a dot product of a vector with itself, the length of the vector itself will be the dot product because vector dropping on to itself will be exactly equal to this particular length. In a general scenario I have this vector and a vector A and we now take a perpendicular and drop it over here you are going to get the length of the vector B along vector A. That is I dropped down a perpendicular from B to A.

And this length that I have is called the dot product of the two vectors A and B and this is indicated by A dot B. So there is actually a dot here. Sometimes it is very difficult to find the dot. So please look for the dot here. So here is a dot between the two vectors A and B and the result of the dot product will always be a scalar because it is only giving me the length. So this is dot product but can I associate a vector with this length. Yes, I can because I know what is the length.

If I multiply this length by the unit vector, so along the direction of A I will get the vector value or the vector component of B along A. So this is exactly what I meant when I said I have r hat vector which is the unit vector in the cylindrical co-ordinate system and then I have the x hat vector which is the unit vector in the Cartesian co-ordinate system but I know that both r can be expressed in terms of x and y.

We have already seen how to decompose any vector. So if I now drop a perpendicular from r to x, the length of this perpendicular will be exactly equal to, so if this is the angle phi, this length is cos phi, why? Because the length of this is equal to one. So if you look at this projection this will turn out to be cos phi. How do I associate a vector with this cos phi? I need to multiply this one by the unit vector along x direction, so I am going to get cos phi x hat.

So more specifically the dot product between two vectors A and B is given by the magnitude of the vector A times the magnitude of the vector B and the angle between the two vectors. There are two angles here. But we are looking for the smallest angle between A and B. So if the angle between the two vectors is theta AB then the dot product or the length of the component of B on A is given by magnitude of A times magnitude of B.

Sorry, these are vectors and cos theta AB and this will be a scalar quantity and you can of course multiply this one by the unit vector along A to get the vector component of B along A. We also call this as projection of B on to A, okay? Projection of B on to A is given by the dot product value which gives me the length of B along the vector A times the unit vector along direction A. Can you figure out what is the projection of A on to B.

Well, it might not come as a surprise, but the length of the component of B along A will exactly be equal to the length of A long B, right? So if B to A, that projection value is something, A to B projection value will also be the same. So this will be, you can either write this as B dot A or A dot B does not matter giving us another rule for dot product that dot product is commutative, A dot B is equal to B dot A.

So this will be A dot B times vector B, okay? So this is the projection of A onto B and this is the vector projection of A onto B. So I hope that in the spirit of this discussion, this relationships of r is equal to cos phi x hat plus sin phi y hat and phi hat is equal to minus sin phi x hat plus cos phi y hat are very clear to you now. Two of other things that we have not finished with the cylindrical coordinates.

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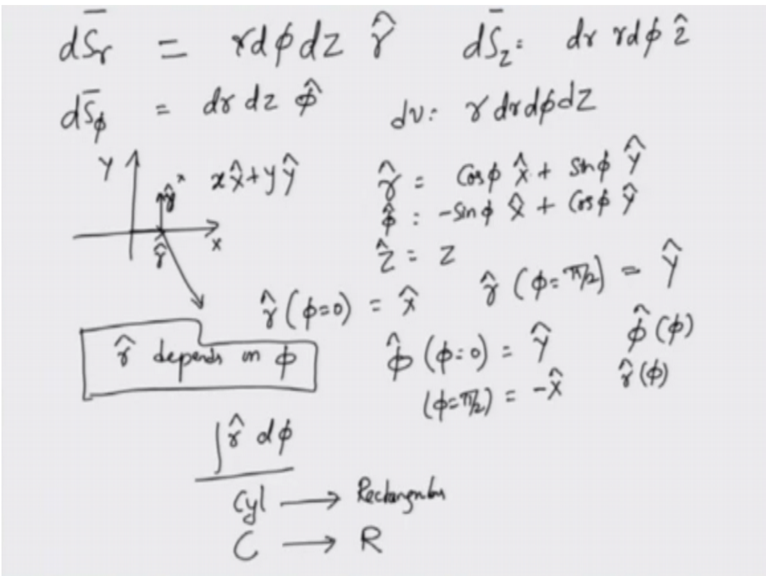

How do I write down the vector surface areas? First consider the vector surface area along r direction. That is - I need to hold r constant and vary the other two variables. So how do I hold r constant, well you go back to this illustration over here, so now imagine that I have this particular cylinder of constant radius r, now I have to hold r constant and I have to move along two directions along phi and along z.

So I move along phi, moving along phi means rotating or circulating this particular this one. So let us say I move along phi by an amount of d phi. How is this an amount of d phi? Remember from this point you can drop a line to the origin, origin would be somewhere at this particular point. So you can actually prick some holes here. This will be my y axis and this will be my z axis and then there will be another one which will be x axis.

So if you now look at this point, you can draw a line from this point to the x axis and anther point at which point I have ended you can draw another line. The total angle you have moved will be d phi and the length that you have moved will be r times d phi, because length, arc length is not just the angle d phi, it is r times d phi, so that length will be r d phi. How do you move along z?

Let us assume that I am going to move along z in this direction, so I am going to move upwards in the z direction, how much I have to move. I would be moving a length of dz. So r d phi, dz and I can complete this complete square or the particular surface element assuming all of this to be very small, then the total area that I have generated which would be pointing along the constant r direction or in the radial direction will be given by r d phi multiplied by dz, okay?

Now suppose I want to find out the surface area along the phi direction, so along phi direction means I have to hold this particular phi to be constant. So this is my phi, so you can imagine that something is coming out like this, so I am holding phi constant over here. The paper is little thick, therefore I am having some amount of trouble, but this is a particular phi. You can see there is some amount of phi over here which is constant.

Now in what two directions I can move. I can move along r and I can move along z. So I can move along r which means that this is going to be little tricky for me, but to move along r from a particular point would be to move like this and then move along z would be to move upwards. I can complete the square again or the parallelogram again, so that I get the component of this vector, this is the component of the vector surface area, along the phi direction.

So this pencil or pen indicates the direction of phi and this vector area will be equal to dr dz and it will be pointing in the phi direction. There is no phi here. It is just dr, dr gives me the length along r, dz gives me the length along z. How am I going to get the surface area along z? Well I need to move along phi and I need to move along r, right? So this particular lock or this particular plane is the z is equal to constant plane.

If I move along r, I am moving along this and if I am moving along phi, I am moving, rotating around here on a constant r values, so I am moving r, that is I am moving dr and I am moving r d phi. So I have r d phi dr along z direction. So these are the 3 surface, vector surface elements that you will be meeting later and which you need to keep in mind.

So hopefully my very crude experimental or graphical way of showing you how the surface elements have worked and you can now very well understand whatever the surface areas. The vector surface area along the constant r is move along phi, you are going to move r d phi and then move along z you are going to move dz attach the unit vector along r. You can also move along constant z; you are going to move along r.

You are going to move along phi and attach the z unit vector, moving along the constant phi plane will be equal to move along r and move along z and attach phi. What is the volume element? Well, volume element is how much you move along r, how much you move along phi and how much you move along z, that could be r dr, d phi, dz okay? So hopefully this you will be familiar. Now we said something about the vectors in cylindrical co-ordinate system.

Remember in the Cartesian co-ordinate system at any location of the vector, the corresponding vector was also very easy to find out. The corresponding vector was x, x hat plus y hat. The direction of x and y did not matter where I was situated, right, on this two dimensional case a plane, I do not care where I am situated because the x and y direction would always point.

For example, if this is my two dimensional plane, you have to imagine that my hand fills out the entire area, the direction of x will be along this, okay? My thumb is pointing along y. So at this point the direction of x is pointing here. The direction of y is pointing upwards. If I go at this point, the direction of x is pointing here, the direction of y is pointing here. So the unit vectors of x, y and in fact also on z, do not change when I move at different points.

Is that the same case with the radial vectors or the vectors in the cylindrical co-ordinate system? Well, we will try, okay? So how do I specify the vector in cylindrical co-ordinate, well I go back and relate them to the Cartesian co-ordinates. R hat is given by cos phi x hat plus sin phi y hat and z will be equal to z itself. Phi hat is equal to minus sin phi x hat plus cos phi y hat.

Now consider two cases, first phi is equal to zero, which means that I am actually along x axis. There is no angle change on there. So with phi is equal to zero, what will happen to r hat and phi hat vectors. Z will not change. Z will still be pointing upwards. It would be pointing along z. What happens to r cos phi, with phi is equal to zero will be one. sin phi with phi is equal to zero will be zero. So r hat at phi is equal to zero is pointing entirely along the x axis.

So if this is my x axis and this is my y axis r hat is pointing along this direction, interesting. What happens with r at phi is equal to phi by 2, 90 degrees. Go back and substitute over here. R at phi is equal to phi by 2 will be cos phi by 2 which is zero, sin phi by 2 which is one, so you are going to get this fellow along y hat. So the vector at phi is equal to pi by 2, the r vector will be pointing along y direction. Sorry, this is r.

So at any other point it would be pointing along a different direction, depending on the value of phi. What it the lesson here? The direction of the vector r depends on phi. This is a very very important lesson which is not found in the Cartesian co-ordinate system. So you would be not really thinking about this one when you try a different co-ordinate system. But it is very very important to note that, except for Cartesian co-ordinate system in general in other co-ordinate systems.

The direction of the unit vectors do change as you go at different points in space. Okay, this is very very critical that you remember this. Similarly, what will happen to the phi vector, phi vector at phi is equal to zero will be pointing along y direction, right, because cos phi will be equal to one at that point. Same phi vector at phi is equal to phi by 2, it is interesting that the same vector will be now pointing along at phi is equal to phi by 2 it would be pointing along minus x hat direction.

So again the phi vector, also depends on phi. So does it mean that I have to all the time specify r as a function of phi, phi hat as a function of phi, well, no. Because it will simply clutter up our equations. We are not going to write down this functional dependence on phi every time but we understand that in cylindrical co-ordinate system and the next co-ordinate system that we are going to discuss called spherical co-ordinate system, the direction vectors, the direction unit vectors are going to be different at different locations in space.

You might ask where is this important? Well this becomes very very important when you have vectors inside the integrals. Remember in the last example, there was a vector inside the integral but the vector was actually constant. I would move up and down along z axis but the direction of x or y and decide does not depend up on where I was located. If you try doing that same with integral of d5 and you have r vector in the cylindrical co-ordinate system.

This r will depend on phi. So different values it will depend on phi. So I do I solve these integrals. Turns out that these integrals can be solved provided you convert cylindrical co-ordinate system vectors into rectangular co-ordinate systems. So call this as c, call this as r, you need to convert from cylindrical to rectangular co-ordinate systems. And we will look at such conversions for a minute now, okay?

And in order to find these conversion methods we are going to use the concept of dot product that we have introduced. So go back to this equation r hat is equal to cos phi, x hat plus sin phi y hat. In this equation if you forget about y for a minute and look at this equation, r hat is equal to something times unit vector along x. Remember the definition for the dot product, A dot B is equal to some length times the vector component of B along A was equal to dot product value times the unit vector along x, correct?

So that was precisely what we discussed in terms of dot product. So in this case you can clearly see that this cos phi would represent the dot product of the unit vector r hat with the unit vector x hat. Similarly, sin phi would represent the dot product of unit vector r hat and the unit vector along y direction and similarly minus sin phi will be the dot product value along x.

And cos phi will be the dot product value along y, sorry, dot product will be sin phi and cos phi for phi hat and x and phi hat and y. The minus sign is associated with the x direction. That it could be pointing in the x direction. So with this knowledge we will now be able to transform one vector in the cylindrical co-ordinate system into vector in the rectangular co-ordinate system.