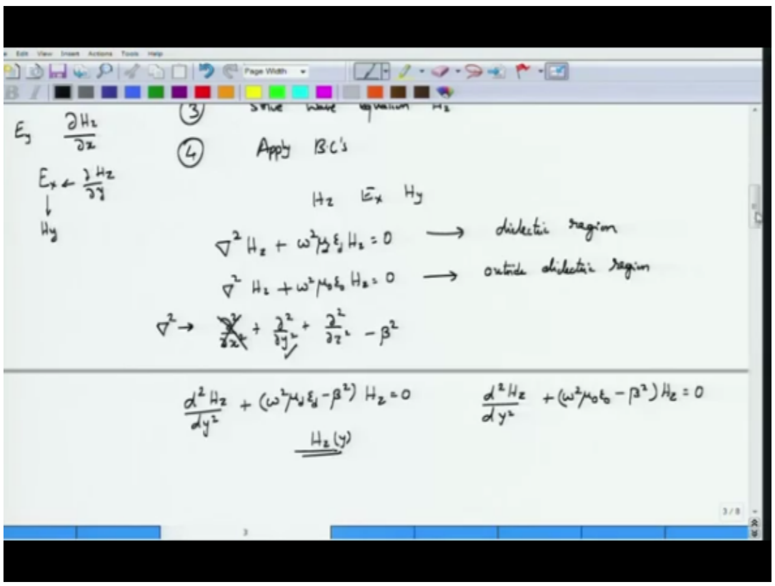
Electromagnetic Theory
Prof. Pradeep Kumar K
Department of Electrical Engineering
Indian Institute of Technology – Kanpur

Lecture No - 77
Dielectric Waveguide

So strategy seems to be alright okay. Everything seems to be fine. We just need to start with Wave equation for Hz. But we already know what the wave equation for Hz right.

(Refer Time Slide: 00:24)



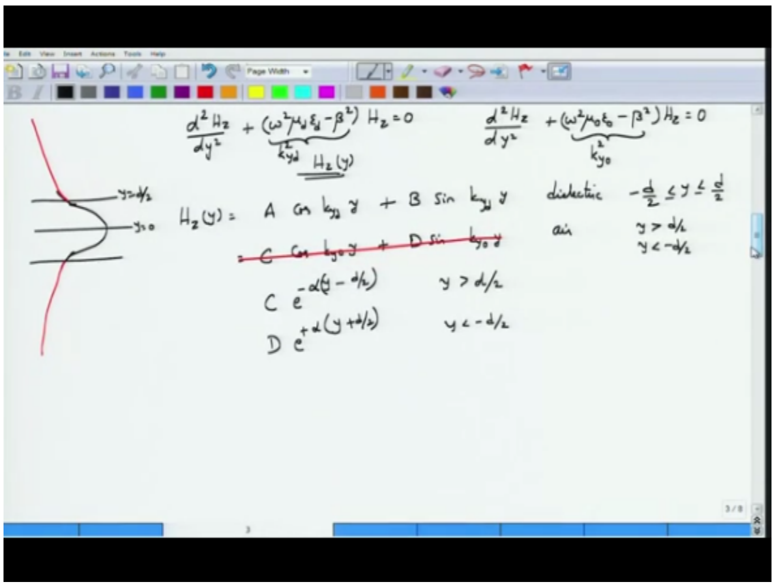
What is the wave equation for Hz? I have del square H z minus omega square mu d epsilon d, this d stands for dielectric okay, times Hz must be equal to 0. This of course, works only in the dielectric film region or dielectric region in this particular case. The same equation will hold, except you replace this mu d and epsilon d by mu 0 and epsilon 0 for the case of, this must be plus. So this is plus, for the case of outside the dielectric region okay. So these are my equations.

From them, I will have to write down, I mean, I have to solve Hz okay. Del square of course consists of del square by del x square plus del square by del y square plus del square by del z square. I have no dependence on x, therefore the first one vanishes. I do not know the dependence on why therefore the second term remains. I know the dependence on del square by del z square, which is basically minus beta square right. So this is what I have.

So, this is minus beta square and you can substitute this into the expression to obtain Hz both in the dielectric and outside the dielectric region and because there is only one dependence on y, I can change this partial derivative of del by del y into the total derivative d by d y. I have d square Hz by d y square plus omega square mu d epsilon d minus beta square into Hz equal to 0. This is for the dielectric region.

The same equation will be d square Hz by d y square plus omega square mu 0 epsilon 0 minus beta square times Hz is equal to 0. In these equations, Hz is a function of y.

(Refer Time Slide: 02:32)



We can have a short hand notation, so call this omega square mu d and epsilon d as k y d square, and call this mu omega square mu 0 epsilon 0 minus beta square as k y 0 square okay, that 0 stands for outside the dielectric region. Now, how do I solve these equations? Well, these are second order ordinary differential equations. The solution for these equations is also known. Hz of y will be some A cos k y d y or it could be sin k y d y. Now, this is for the dielectric region correct.

And for the A region or outside the dielectric, the same equation will be some C cos k y 0 y, sorry I made a small mistake here, so it must be k y d, not d coming in the numerator, it is just the subscript for k y plus d sin k y 0 y. this is for the solution in air. So, we have this solutions, of course the dielectric region goes from minus d by 2 less than or equal to y less than or equal to d by 2. This would be y greater than d by 2 or y less than minus d by 2 right. So these are the dielectric and air regions that we have written down over here.

I have now four unknowns A, B, C, D. I will have to apply boundary conditions to mach them and see whether I am actually able to get anything over here. Now, before we proceed, let us actually understand whether this equations, the way we have written down are reasonably alright or not okay. If you look at the top one, you wanted a propagating kind of a solution right. So you wanted a cos k y y and a sin k y y type of a solution.

However, in the air region, we seem to have made a mistake. What is the mistake that we have made? We do not want a way which is propagating along y. In fact, what we want is a way which is decaying along y right. So, if you graphically look at this, cos k y y assuming that, that is what I am looking at. So if this is my y equal to 0 axis. So, cos k y y would be a way which would go like that. If it was metallic wall, it would have gone down to 0 at both ends.

But, since this is not a metallic wall, there would be a small penetration of the wave here, outside the dielectric region. Now, once outside you are there, you do not want further, you know a sin or a cos kind of a solution. Although in some cases you will not have any control and then you will have to accept this propagating solution or sometimes you desire this okay. In cases, where you want to couple light from one wave guide to another wave guide, you want a cosine kind of situation right, outside the dielectric region of the first wave guide as well.

But, that is not our intention, we are not coupling anything from one to another. So for us, this inside the dielectric is perfectly fine, but outside the dielectric, we want the wave to basically decay to 0 okay. So, we want the wave to essentially go down to 0 outside the dielectric region. Which means that, I cannot choose them to be propagating type of solutions, rather I have to choose them to be decaying kind of solution.

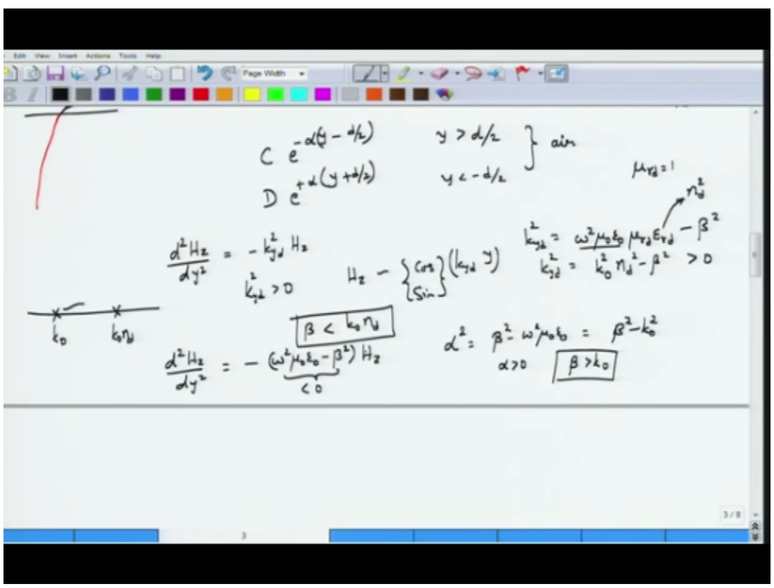
So, the solution here in the region above y equal to d by 2, we can choose this as C exponential, we want them to be decaying exponentially and that would happen when beta square is actually greater than omega square mu 0 epsilon 0. But when beta is less than omega square mu d epsilon d, then you will have the propagation inside and decay outside, we will see that one in a moment. So C e to the power minus, call this as alpha and y okay.

So, I just need a decaying solution, I do not want an expanding solution, therefore, I will take this as minus alpha y. But I know that, at this point I do not want the expression for the decaying wave to be equal to C, you know I want this value to be valid at d by 2. So I just shift the origin from y equal to 0 to y equal to d by 2 and call this as e to the power minus alpha y minus d by 2, which would be valid for the region above the dielectric.

Why because, you put this y equal to d by 2, you get this expression, exponential will be one and the value here will be equal to C okay. Similarly, for y less than minus d by 2, I shift the origin okay, I take this as e power plus alpha because, now the value of y is negative. So, I take this as y plus d by 2 okay. So, now I have complete solutions for Hz, except for the constants C and D, which I will now have to evaluate by applying boundary condition.

These two are the solutions in air, one above the dielectric, one below the dielectric. This is the solution inside the dielectric region. Let us write down also what is this alpha.

(Refer Time Slide: 07:38)



See, I have this d square Hz by d y square, the way I would solve this second order partial differential equation would be to push this k y square d outside. So, I will have minus k y square d Hz right. As long as k y d is greater than 0 or k y d square is greater than 0, the solution of this one will be Hz as cos of k y d y or sin of k y d y okay. So, this would be the solution of Hz, as long as k y d square is greater than 0.

But, what is k y d? k y d is nothing but omega square mu 0 epsilon 0 mu r d epsilon d right minus beta square. This is my k y d square, the propagation constant inside the dielectric region. But, I can write down this mu square mu 0 epsilon 0 as some k square ok inside air. This is the same thing that would be there in the air as well. So, I can as well have a short hand notation for this, and if I assume that mu r d is equal to one okay.

That is, the dielectric medium does not, is in fact non magnetic, then I can eliminate this mu r d from the discussion and I also know that epsilon r d would be the refractive index of the dielectric region. Therefore, I can consider this as n d square okay, n d standing for refractive index of the medium okay. So, I have k 0 square n d square minus beta square. This is your propagation constant k y d inside the dielectric region and this has to be greater than 0, which means that beta must be less than k 0 n d, this is the first condition.

Now, to the same equation outside the dielectric, let us say above the dielectric, this should be equal to minus omega square mu 0 epsilon 0 minus beta square times Hz. Again, if omega square mu 0 epsilon 0 which can be written as k 0 square is greater than beta square. The solution will be of the form of cos and sin, which is what I do not want. So, what I really want is that this k square, this fellow should actually be less than 0 right.

So, if this is less than 0, then this would be negative okay, so that, I can have a exponential solution, then I will have, actual solution will be in the form of beta square minus omega square mu 0 epsilon 0 times Hz and then that would be the solution. So, what I will do is, I will define alpha square as beta square minus omega square mu 0 epsilon 0 or beta square minus k 0 square okay. So, if this is my alpha or rather we can define alpha as minus beta square minus k 0 square.

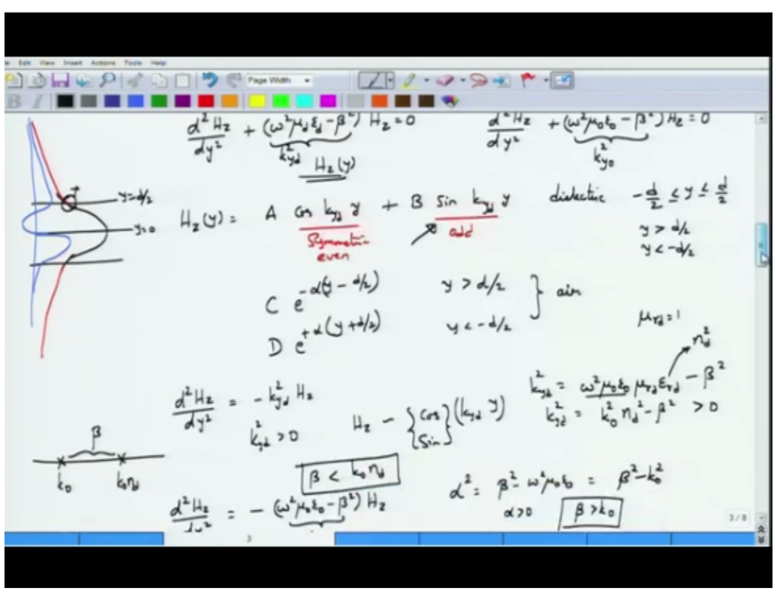
So, if I define this way, then this is k 0 square minus beta square, so, this would actually be minus alpha square. So everything is alright. So, if I define alpha square as beta square minus k 0 square and then demand that alpha should be a positive quantity, the way it can happen is, when beta is greater than k 0. So, I have two conditions for propagation, that beta must be greater than 0, but beta must be less than k 0 n d.

So, if I were to plot the numbers, this is where the lower limit for beta would arise, that would be k 0. This would be the higher limit k 0 n d and somewhere here, will be the value of beta. You can actually combine these two equations and eliminate beta, but then you will be expressing everything in terms of alpha okay. So, we now have the solutions for Hz, we know where the solutions would be and how the solutions formats are, and all that we are going to do with boundary condition is to stitch them together at this point okay.

So, we apply boundary condition to stitch them together. What sort of boundary condition should we apply? Can I apply or you know tangential Hz is going to 0 at the two boundaries. Unfortunately, I cannot, this is a dielectric and a dielectric boundary right. Well in this particular case I can, but I f you go to e z I will not be able to do so. So, I can actually try and apply the continuity condition.

Moreover, the form of the solution indicates that, this itself can be split up in to two types, one is cos k y d and the other is sin k y d of y.

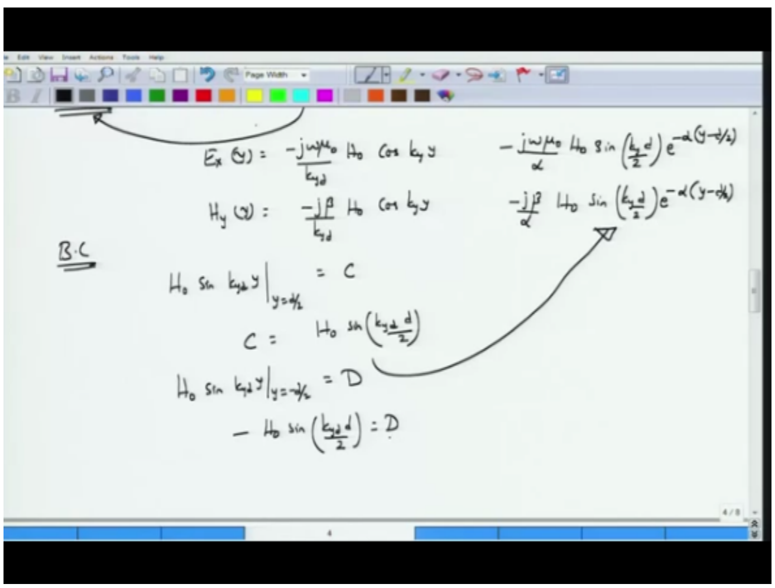
(Refer Time Slide: 12:08)



This cos will be a symmetric wave guide with respect to the center okay. So, this would be symmetric or we can call this a s even mode, because the mode shape is essentially even okay. For the odd mode, which would be in the form of a sin wave right. So, it would be a sin, which means that, if this is my axis, then this should be a sin, so it should actually go something like this.

So, this is how the sin wave should look and they are actually odd, in the sense that, at the y equal to 0 plane, they actually carry opposite signs, whereas the cos will have the same sign and therefore that would look like a symmetric or even wave form. So, let us look at odd wave forms first okay. So, which means that my solution inside the dielectric will be of the form of sin.

(Refer Time Slide: 13:06)



So, I can write down Hz y for the odd, odd TE case. I will leave the even TE case as an exercise for you. So for the odd TE case, Hz of y is equal to B sin k y d. Rather than calling this as B, let us simply call this as H 0, that 0 nicely indicating that this is an odd TE mode that we are considering. So you have H 0 sin k y d y, for this one, solution is valid for y less than mode by less than d by 2 that is inside the film.

We can also find out what are the x and y components for this particular mode. So E x of y will be equal to minus j omega mu 0 divided by k y or k y d times H 0 cos k y y okay, after applying the boundary conditions and then appropriately differentiating this Hz because E x will be proportional to del Hz by del x multiplied by some constants over here. Similarly, you can show that H y of y will be equal to minus j beta divided by k y d H 0 cos k y y okay.

So, these are the solutions for E x and H y. We also know the solutions for Hz. Now, we can actually apply the appropriate boundary conditions. We can write the equations for air region as well. So, first consider y greater than d by 2 for which E x of y becomes minus j omega mu 0 but, the differential of Hz, Hz is now e power minus alpha. Therefore, it would be alpha that is coming out in the denominator. So, that would be alpha H 0 sin k y d by 2.

That would be at the boundary the continuity condition e power minus alpha y minus d by 2 ok and then, you will also have the solution for H y, which is minus j beta divided by alpha H 0 sin k y d by 2 e to the power minus alpha y minus d by 2. If you are slightly confused at this point we can actually go back and apply the boundary condition. I assume that you could apply the boundary condition very nicely. So, let us write down this as H 0 sin. So, this is how you apply the boundary condition.

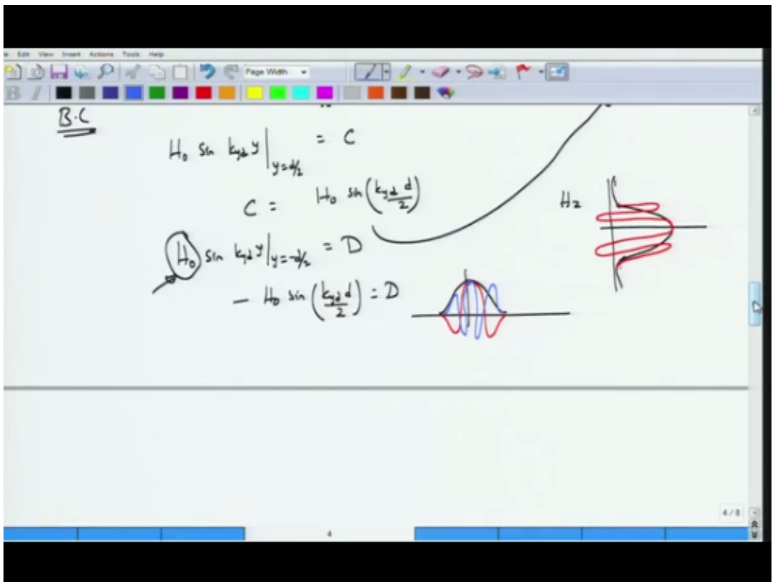
I am going to just motivate the boundary condition to you. I will not go into details for this one okay. So, this is H 0 right, sin k y d y this was the solution as you would be there in the dielectric region. This at y equal to d by 2, must be equal to the condition for the wave at air right. So, at y equal to d by 2, if you approach from the top, the solution would be in the form of C e to the power minus alpha y minus d by 2. Therefore, at y equal to d by 2, this would be equal to c.

So, this would be equal to C, so, C is equal to H 0 sin k y d d by 2 and this is precisely what this factor is coming in over here okay. So, this is the boundary condition and this is the amplitude of the wave outside the dielectric region. Similarly, you have H 0 sin k y d y at y equal to minus d by 2 must be equal to d, which is the amplitude at y equal to d by 2 right. So, y equal to minus d by 2.

But if you substitute for this, sin becomes sin of k y d into minus d by 2 becomes minus H 0 sin k y d d by 2 is equal to D ok and this is precisely what you get. If you were to write down the expressions again for the region y less than or equal to minus d by 2 okay. So, once we have determined everything and you can now clearly understand where H 0 must be coming from? H 0 comes from the total power normalization. What is the power that is actually carried by the wave and that integral will tell you how to fix the value of H 0.

So, we have figured out everything, we know how Hz looks like, we know how E x looks like, you know how H y looks like okay.

(Refer Time Slide: 17:22)



We know that Hz has a function of y is a cosine wave and depending on the order, so you can have m is equal to, the solutions will be of the form of a different orders right. Because you have a cos k y d by 2 and you can approach the same value in two different waves. So, you can approach it by one half cycle, you can approach it by two half cycles okay. So, let me try to draw two half cycles over here and this is how the wave would decay outside okay.

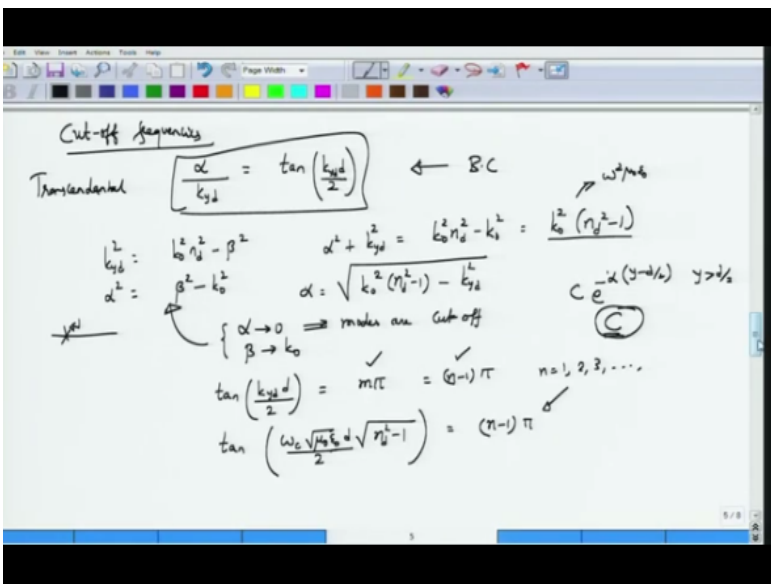
So, I need to approach it with a different or two half cycles, so let me try that. This is the second cos k y d by 2 or rather if you are able to plot it in this way, this is my fundamental mode, the next mode will be this one right. So, it would actually approach in this way. So, this is the next mode and now I can draw the higher order modes if you want. This is the next higher order mode again as I said the drawing is slightly confusing for me.

But, these different modes correspond to the different orders. Again, there is no specific greatness about this mode orders. They are just indicated the possible solutions as the different mode shapes contained by the electric and magnetic field combined. They are just come in different flavors in which a sustainable E and H patterns for the particular solution would exist. So, these are called as modes. Let me close this discussion on dielectric wave guides.

I mean, I have not written down the expressions here for y less than minus d by 2, I will leave that as an exercise for you okay. So, you write down the expressions for y less than minus d by 2. The solution for even TE would not change okay. Sorry I wrote down the expressions as even TE but I was talking in terms of odd TE. So, for the solutions for even TE there is no problem, instead of sin k y d y, you will write down this as cos k y and then you would simply continue applying the appropriate boundary condition.

The crucial point here is that, Hz was not equal to 0 but then, the tangential Hz was simply continuous. You do the analysis for Tm waves and I will leave that as an exercise for you.

(Refer Time Slide: 19:52)



Let us discuss one aspect which is cut of frequencies for these modes. So, how do we define the cut off frequencies? Well, when does we have the propagation right. So, you start with alpha, you also know what is k y d right. So, alpha is basically given by writing down the boundary condition, you will be able to express the relationship between alpha and k y d, because I have, in one case I have H 0 sin k y d by 2 and the other one will be a cos k y d by 2.

So, the ratio of alpha to k y can be written down as alpha by k y d is equal to tan k y d. So, tan k y d d by 2, this can be obtained again from the boundary condition. This relationship tells you the different values of k y d and alpha, how do you choose them? Of course, solving this equation is kind of very difficult because, k y d is present on both left hand side as well as on the right hand side. These equations are known as transcendental equation and the solution of this has to be done by a numerical analysis.

We will not go into those details in this particular case okay. So, you can obtain this equation as I said from the boundary conditions, we will not pursue this one, but you can actually add alpha and alpha square and k y square okay. So, I know what is k y d square. K y d square is nothing but k 0 square n d square minus beta square. I also know what is alpha square. Alpha square is nothing but, beta square minus k 0 square.

So, if I add these 2 together, I get alpha square plus k y d square equal to k 0 square n d square minus k 0 square or taking k 0 square as a common factor, I get n d square minus one okay. Now, I can express what alpha is? Alpha in terms of k y d, that would be equal to k 0 square n d square minus one minus k y d square. Remember, whenever you have k y d value, which is non 0 there will be a mode which is propagating okay. So, this is your expression for alpha.

What would happen to the phase constant beta right? See, as alpha goes to 0, beta would simply approach k 0, that is from this expression very clear right. So, if alpha goes to 0, the way it can go to 0 is when beta approaches k 0. So, for a given k 0, alpha will go to 0, because beta is approaching k 0. So with that, let us actually, so, when that happens with alpha is equal to 0, the waves are no longer cut off right.

The waves are no longer bound to the dielectric region, and therefore the waves are cut off in the sense that they would not be confined inside the wave guide. Why? Because if the wave, if alpha goes to 0, then alpha times y minus d by 2 was the solution outside the dielectric region at the air for y greater than d by 2, but with alpha going to 0, the solution would not decay, but rather it would remain a constant.

There was a C times e power minus alpha and we also saw how to obtain C, but the point is that, as alpha goes to 0, the C times e power minus alpha y minus d by 2 does not go to 0 as you increase y, rather it remains fixed at a given value of C and therefore energy is carried away from the mode outside the air region. Similarly, if you go to the bottom surface or the substrate region, you can see the same condition. So, alpha going to 0 implies modes are being cut off okay.

So, this is the situation, where the waves are cutting off and the solution for that would be to look at, since I know the relationship between alpha and k y d as alpha goes to 0, the solution can only happen when k y d d by 2 would go to 0 or you know the integer multiples right. So, this should go to either 0 for tan to go to 0 or it should go to, I will just write down this as multiple of pi m standing for different orders right.

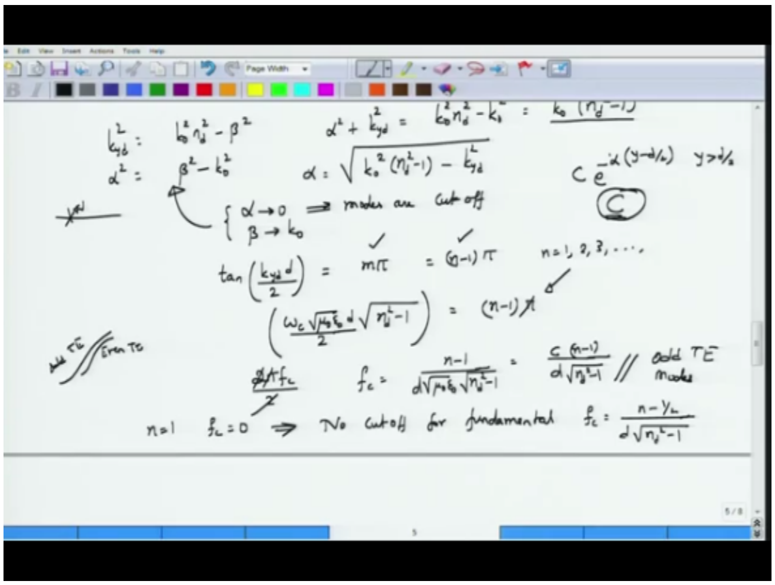
So, with m is equal to 0, 1, 2 or 3 you will have different modes coming in, if you want you can replace m pi by n minus 1 into pi and set n is equal to 1, 2 and 3 and so on. This is sometimes used. This m notation or n notation is used in literature both are used. So, either you, the first mode will be TE 0 mode or TE 1 mode. TE 1 mode would indicate a half cycle, TE 0 mode would indicate no minima in the region okay.

So, these are just kind of two different notations but the essential idea is that, these mode designations actually follow because they are getting multiples of pi and making this tan of this expression equal to 0 okay. But I know what is k y d right. I know what is k y d, I can write down the condition at which this happens by specifically writing this as omega c d by 2 square root of epsilon r d, which would be the expression for this one, minus 1.

Why minus 1? Because epsilon r of air is equal to 1 and this epsilon r d under root is nothing but n d square okay. So, this is the condition where we would have this one. So, k 0 square of course is omega square mu 0 epsilon 0. So, I am basically using this expression for k y d square, when alpha is equal to 0, k y d square will be equal to omega square mu 0 epsilon 0 n d square minus 1 or k y d will be equal to omega square root mu 0 epsilon 0 and under root of n d square minus 1.

So, let us write down that. So, it would actually be omega c, which is the cut off frequency square root mu 0 epsilon 0 divided by 2, that is also a d here, then there is square root of n d square minus 1 okay. So, this must be equal to n minus 1 into pi, if you are going to this notation for the waves to be cutting off or rather this expression must be equal to n minus 1 into pi.

(Refer Time Slide: 26:19)



You can replace omega c by 2 pi f c right and divide by 2, 2 will cancel, pi will cancel from left hand and the right hand side. So, f c can be written as n minus 1, don't confuse n with refractive index, n minus 1 is the mode designation divided by square root of mu 0 epsilon 0 d n d square minus 1 under root. But I know 1 by square root of mu 0 epsilon 0 is c. So, I have c n minus 1 divided by d square root of n d square minus 1.

So, these are the cut off frequencies for higher order mode. So with n is equal to 1, clearly f c is equal to 0, indicating that no cut off for the fundamental mode okay. There will always be this particular mode, that would exist. So, there is no cut off for fundamental mode. So, for the odd modes, the cut off condition is this one, this is for the condition for odd TE modes okay. What would be the cut off condition for the even TE modes? You can actually show that, instead of tan it would be cot, so, I will not derive this one.

So, the cut off condition for the even frequency modes will be equal to n minus half divided by d into square root of n d square minus 1 okay. This minus half simply indicates a pi by 2 shift in the argument that needs to be used. But the important point is when n is equal to 1, f c is not 0, which means that first the odd TE mode will propagate the fundamental, then the even TE mode will propagate.