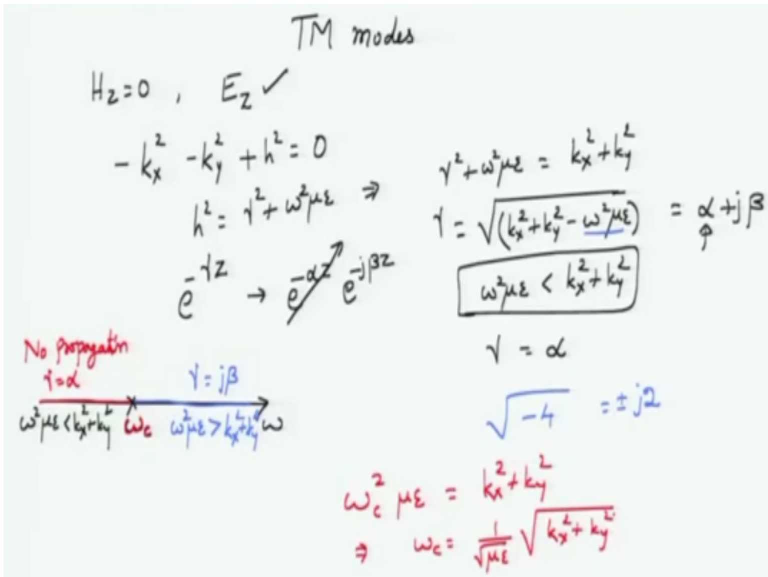
Electromagnetic Theory
Prof. Pradeep Kumar K
Department of Electrical Engineering
Indian Institute of Technology – Kanpur

Lecture - 71
TM modes in Waveguide

(Refer Slide Time: 00:15)



In this module, we will continue to discuss TM modes first and then we will go to TE modes. As we discussed in last module, TM modes actually have no 'H z' component. This makes our analysis slightly simple because we just have to solve for 'E z'. And then all the other quantities are expressed already in terms of 'E z'. So, you can find out all the other components of the electric and magnetic fields.

We wrote down a Helmholtz equation for 'E z' and then simplified it to a certain extent. So please refer to the previous module to find out, where we have left it. So, I will be actually continuing from the point, where we left. So we had written down 'E z' as product of 'x' and 'y' and then substituted that. And that is where we had left. So, we had introduced two constants 'minus k x square' and 'minus k y square', such that this plus 'h square' was equal to zero.

But 'h square' itself, so if you rearrange this equation, 'h square' is 'gamma square plus omega square mu epsilon'. So, this implies that 'gamma square plus omega square mu epsilon' is equal to ‘k x square plus k y square’. So, gamma is equal to ‘square root of k x square plus k y square minus omega square mu epsilon’. Now, before continuing with our solution, let us look at whether we would actually have a solution.

You know for what conditions for ‘k x’ and ‘k y’ or whether we will not have a solution. First of all, gamma is equal to ‘alpha plus j beta’. If you want to have a solution, you do not want to have an alpha there. Why? Because if alpha is non-zero, that would indicate away, which is attenuating because electric fields are all assumed to be going as ‘e power minus gamma z’ which implies that we have ‘e power minus alpha z’ as the attenuation factor.

And ‘e power minus j beta’ as the propagation factor. If I somehow ensure that alpha is equal to zero, then this ‘e power minus alpha z’ term can be removed and then I will have a pure propagation of the waves. Well, what condition would give me alpha equal to zero? Look at this expression for gamma. So, if ‘omega square mu epsilon’ happens to less than ‘k x square plus k y square’, we have not in fact told you how to calculate ‘k x’ and ‘k y’.

We will be doing that shortly. But if for some frequency, mu and epsilon are constant, so if for some frequency omega, it so happens that ‘omega square mu epsilon’ is less than the sum ‘k x square plus k y square’, then the quantity under the square root will be greater than zero and gamma will be equal to only alpha. There is no possibility of any wave propagating, as long as this ‘omega square mu epsilon’ is less than ‘k x square plus k y square’.

Assuming that ‘k x square’ and ‘k y square’ are some constants, I mean which are constants, so as you gradually increase the frequency. So you have, let us say omega axis, as I gradually increase the frequency until certain point until a certain frequency, which is known as the critical frequency or the cut off frequency, the product here, would continue to be less than ‘k x square plus k y square’.

And therefore, gamma will be equal to alpha here. So, gamma is equal to alpha, there is essentially no propagation, only attenuation of the waves. So, waves just get attenuated to create a launch at ten hertz signal into a wave guide that signal would not propagate for too long before getting attenuated and dying out. However, at this critical frequency, what happens is that, this ‘omega square mu epsilon’ just becomes equal to ‘k x square plus k y square’.

You still do not get a propagation because the quantity under square root has become zero and gamma is equal to zero would imply alpha is equal to ‘minus j beta’. So you still have an attenuation and waves are still not propagating. But beyond this frequency, what would happen is that, the product ‘omega square mu epsilon’ becomes greater than ‘k x square plus k y square’. The moment this ‘omega square mu epsilon’ becomes greater than ‘k x square plus k y square’.

Gamma will be equal to pure ‘j beta’. Why pure imaginary, why? Because the quantity under the square root will become negative and you are now looking for something like ‘square root of minus four’, for example, which will have ‘j two’ as your solution. Of course you will have ‘plus or minus j two’, indicating that you will have propagation along plus ‘z’ direction as well as propagation along minus ‘z’ direction.

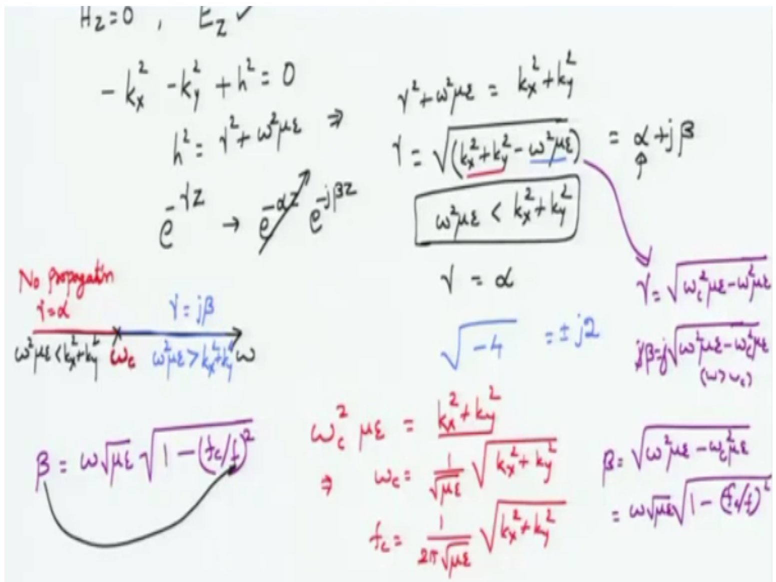
We will choose one of them because we know in which direction we are launching the waves. So for us, let us say, we have chosen only the plus sign because we already have ‘e power minus j beta z’ as the wave, which is propagating along plus z direction. So you understand the critical frequency concept or the cut off frequency concept. And this is one of the most interesting formulas that you will come across in waveguides.

By the way, let me tell you here, waveguides will have lot of formulas, you know, it will be very difficult for you to remember them. The easiest way to go about remembering them is to practice deriving all these equations and understand them how these equations are derived and then sit one day on a piece of A4 size paper, you start writing down the formulas. Keep using that formula, as long as you want in this particular course.

There is, once you understand the concept, there is nothing much to the formula. The formulas are only going to be complicated. You can always look up the formula. So do not worry, you can look up the formula. You do not have to memorize them. But you understand how these formulas are derived. So one of the first formulas, we have figured out is the expression for cut of frequency ‘omega c’.

So ‘omega c square mu epsilon’ must be equal to ‘k x square plus k y square’. This implies that ‘omega c’ itself is equal to ‘one by square root mu epsilon, square root of k x square plus k y square’.

(Refer Slide Time: 06:19



Sometimes, instead of writing this as ‘omega c’, some people like to split this in terms of ‘two pi f c’. If you do that, then ‘f c’ will be equal to ‘one by two pi square root mu epsilon, k x square plus k y square under root’. So this is an important equation that you can remember. But now, look at what we have done. ‘k x square plus k y square’ is equal to ‘omega c square mu epsilon’ that is the critical frequency.

Therefore, if I substitute for this expression for gamma, I can actually write down gamma as, from this expression I can write down gamma as ‘square root of omega c square mu epsilon minus omega square mu epsilon’. And I know that propagation happens only when omega is greater than ‘omega c square’. I can write this as ‘j beta’ is equal to ‘square root of omega square mu epsilon minus omega c square mu epsilon’, but with the j outside.

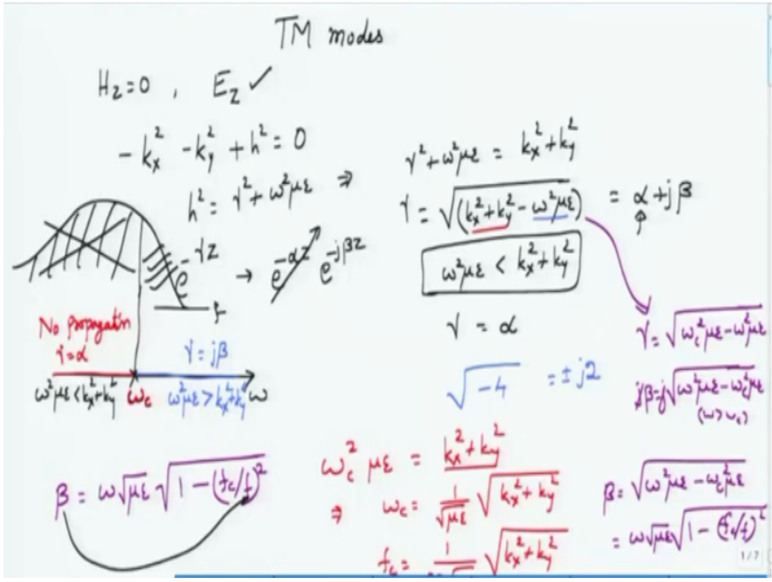
You remember this omega is greater than ‘omega c’ in this expression. So with this, I can cancel out ‘j’ on both sides. And therefore obtain an expression for beta, the propagation constant given by ‘omega square mu epsilon minus omega c square mu epsilon under root’. This is again sometimes written by factoring this ‘omega square mu epsilon’ out and then you write this as ‘omega square root mu epsilon’.

‘One minus mu epsilon’ will cancel with each other. ‘Omega c square’ can be written as ‘two pi fc square’ or ‘two pi fc whole square’. ‘Omega square’ can be written as ‘two p f whole square’. So this becomes ‘one minus fc by f whole square under root’. So let me rewrite that same expression over here. Beta is ‘omega square root mu epsilon’, which would remind you of a plane wave solution.

So for a plane wave, we just had this propagation constant as ‘omega square root mu epsilon’. But now there is a nice factor, sorry, there is an extra factor, which will change the value of beta. And most critically, although we have not explored that, we will explore that, beta is now a function of frequency. This will lead to lot of problems later. And we need to introduce what is called as dispersion and group velocity.

We will do that. So, it is interesting that we did not even solve for ‘k x’ and ‘k y’. But we already learned so much about this waveguide. So, TM modes and as TE modes would be possible to propagate at or propagate only when the frequency becomes greater than ‘omega c’. So in this context, sometimes this is called as the bypass nature of the waveguide.

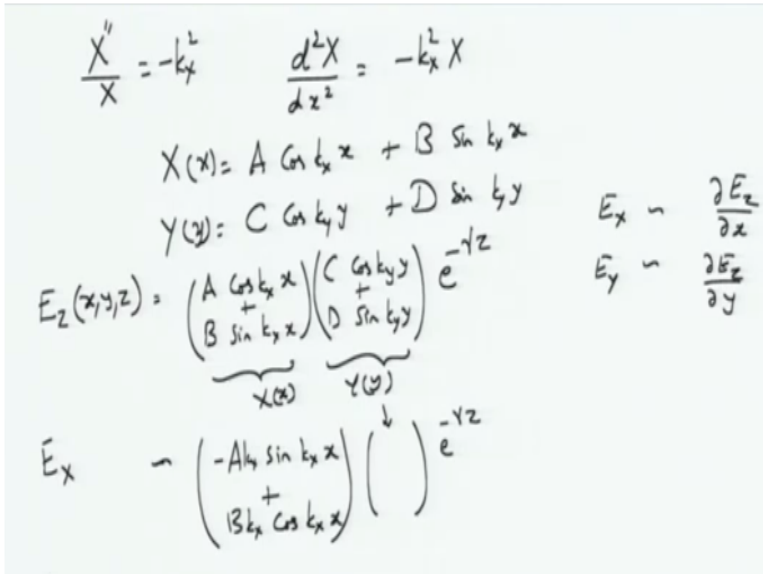
(Refer Slide Time: 08:57)



It means that if you have a frequency spectrum, so far some way you have a pulse, which is to be propagating inside, then if the frequency components are all lying, you know some frequency components lying below 'omega c', they would all be filtered and cancelled out or they would be attenuated out. Only the components, which are greater than 'omega c' will be allowed to propagate.

Obviously, this would induce distortion because if you lose some frequency components, there is no way you can actually obtain a undistorted wave form from reconstructing the remaining sinusoids. Something that you must have been very familiar with Fourier transform theory. Now let us get back to finding the expression for electrical magnetic fields.

(Refer Slide Time: 09:41)



We have seen that ‘x of x’, you know we had two equations. ‘X double prime of X’ is equal to ‘minus k X square’. So let us solve this. The other one will easily follow from this. And go back to the differential notation. So this is actually ‘d square X by d small x square’, which is equal to ‘minus k x square into X’. There is too many ‘x’s involved. But please excuse the notation. This is quite standard in variable separable method of solution.

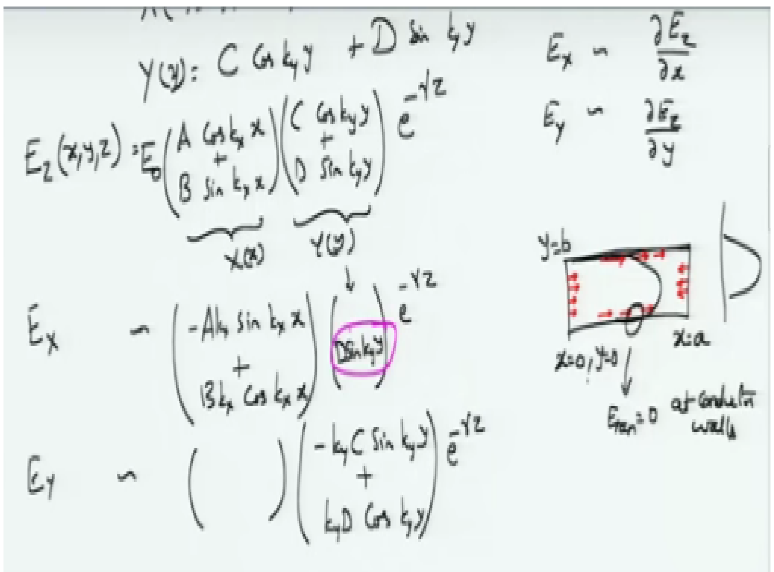
How do I solve this equation? Well, this is second order differential equation and the constant term is ‘minus k x square’. Therefore, the solution for this ‘x’ will be ‘some constant A cos k X x plus some other constant B sin k X x’. Similarly, I can find out ‘Y of y’ will be equal to ‘some other constant C cos k y Y plus another constant D sin k y Y’. So, I have these two expressions for ‘x’ and ‘y’.

So the total expression for ‘z’, which will be function of (x, y and z), will be given by ‘A cos k x X and B sin k x X times, C cos k y Y D sin k y Y’, I am trying out a new notation over here. I need to put a plus sign here because there are four terms, ‘A cos k x X into D sin k y Y, A cos k x X into C cos k y Y’. So, it is four times I am writing this, in terms of these brackets. So this would be ‘X of x’, this would be ‘Y of y’.

All of these terms would be propagating as ‘e power minus gamma z’. So, this is your electric field component ‘E z’. Of course one needs to also find out the components for ‘E x’. You also need to find out the components for ‘E y’. So, to do that one, let us first find out what is ‘E x’ and ‘E y’, how do they vary with respect to ‘E z’. We know that they would vary with respect to ‘E z’ in terms of, ‘E x’ would vary as ‘del E z by del x’, it could be proportional to ‘del z by del x’.

And ‘E y’ will be proportional to ‘del z by del y’. There are some constants involved there, but you do not have to worry about those constants. All that is required is the proportionality constants, I mean proportionality relations. I have ‘E z’, I have expressions for ‘E x’ and ‘E y’. So if you actually do that expression substitution. ‘E x’ will be something like ‘minus Ak x sin k x X plus Bk x cos k x X. The other things would not change. So I just write down as it is.

(Refer Slide Time: 12:43)



Whereas for ‘E y’, what would change is, ‘E y’ would have, so ‘E y’ will be proportional to, there could be some other constant over here, we can put one more constant E zero, just to indicate what is the power, but is not really important at this point, form of the solutions is important. So, ‘E y’ because it is dependent on ‘del by del y’, this term will not change, whereas in this term you will have ‘minus k x C sin’, sorry this is k y.

So this is ‘minus k y C sin k y Y plus k y into D cos k y Y. of course this will also propagate as ‘E power minus gamma z’. We have obtained three components now. Have you obtained completely? We still do not know what is A, B, C and D. We still then do not know what is ‘k x’ and ‘k y’. How do we go about solving them? Well, here is where we need to apply the boundary conditions.

So what are the boundary conditions? Look at the cross section of the waveguide. Along ‘x’, I have two boundaries ‘x equal to zero’ and ‘x equal to A’. Along ‘y’, I have two more boundaries, ‘y equal to zero’ and ‘y equal to B’. So, this entire top plate at this cross section will be having y equal to B. Whether at two ways you can actually write down the solutions for ‘E x’ and ‘E y’, we know what to retain and what not to retain.

A simple way would be to, depending on which one you would call it simple, one way would be to just substitute the values. So, ‘E x’ if you look at, how would ‘E x’ be there at ‘x is equal to zero’ boundary? ‘E x’ is the one which is directed along ‘x’ direction. So for this boundary at x is equal to zero’, as well as for this boundary at x is equal to ‘A’, ‘E x’ component would be transverse.

Whereas ‘E x’ components, which are directed along ‘x’, but found near the boundary at ‘y equal to zero’ and ‘y equal to B’ would be the tangential electric fields. And we know that the tangential electric fields are continuous. So, we know that ‘E tangential’ is equal to zero, which means that the tangential continuity is guaranteed. And moreover because there are these electric fields, impinging on a conductor, the tangential electric fields must go to zero at conductor walls as well.

So, because this is a conductor and a dielectric interface, conductor will not allow you to have any ‘E x’, the tangential component of ‘E x’ there. It does allow you to have a normal component that would not help you in boundary condition. So tangential component of the electric field must go to zero and it must go to zero at two boundaries, ‘y’ is equal to zero as well as at ‘y’ is equal to ‘B’. What kind of a function between sin and cos will have a zero at the boundaries?

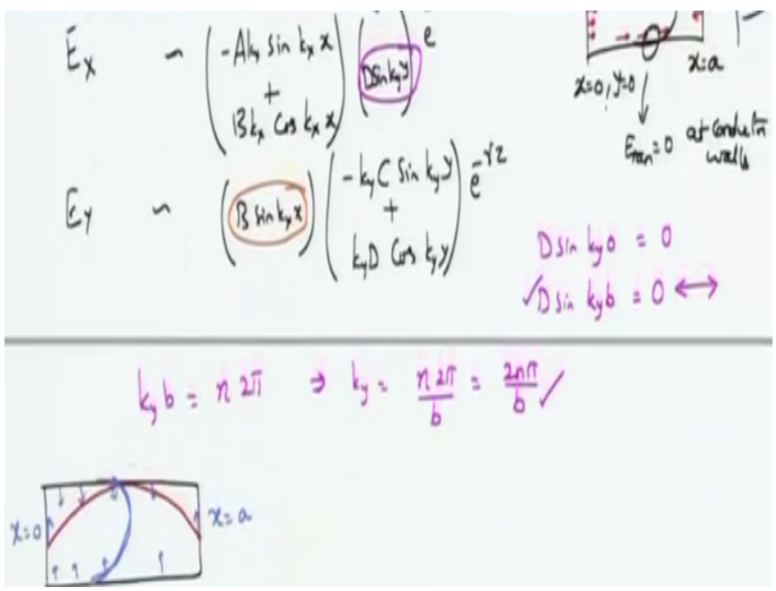
So if you have something at ‘x equal to zero’ that means to go to zero, sorry, ‘y equal to zero’ and ‘y equal to B’, if something has to go to zero, what kind of a function would go to zero? Why? The answer is a sinusoidal function. A function, which would be sin, will go to zero at ‘y equal to zero as well as at ‘y equal to ‘B’. If you do not want to follow this line of thought, all we have to do is to simply understand that ‘E x’ forms the tangential component at the boundary wall substitutes for ‘Ex’ at ‘y equal to zero’ and at ‘y equal to B’.

So if you substitute ‘y equal to zero’, since, this is not changed ‘y equal to zero’ will cause this term to be present C and then ‘y equal to zero’ will eliminate this term. All the other terms cannot go to zero. So which means that C is equal to zero. So ‘cos k y Y’ is gone from this solution. But from the boundary condition itself, you can clearly tell that the only solution that would remain here will be that of ‘D sin k y Y’, because only the sin function.

If you have to plot like this, only the sin function can go to zero at two boundary points, which we have taken as ‘y equal to zero’ and ‘y equal to B’. Can I do a similar thing for ‘E x’ at ‘x equal to zero’ and ‘x equal to A’? Unfortunately, I cannot really do that one. But I need to use a different boundary condition, in order to compensate for that one. There is another point that we need to mention here.

Although the form of a solution that eventually remained, was that of a ‘sin k y Y’ what this actually form allows us to find out what is ‘k y’.

(Refer Slide Time: 17:06)



So what should be ‘k y’, because the substitute ‘D sin k y at y is equal to zero’ is equal to zero. That is alright. So, what this simply implies is that ‘k y’ cannot be found with this equation. But, there is a second boundary condition. The electric field quantity at ‘y equal to B’ must also go to zero. None of these will go to zero. So, the only way that you will have that second boundary condition satisfied is, ‘k y b’ is equal to zero.

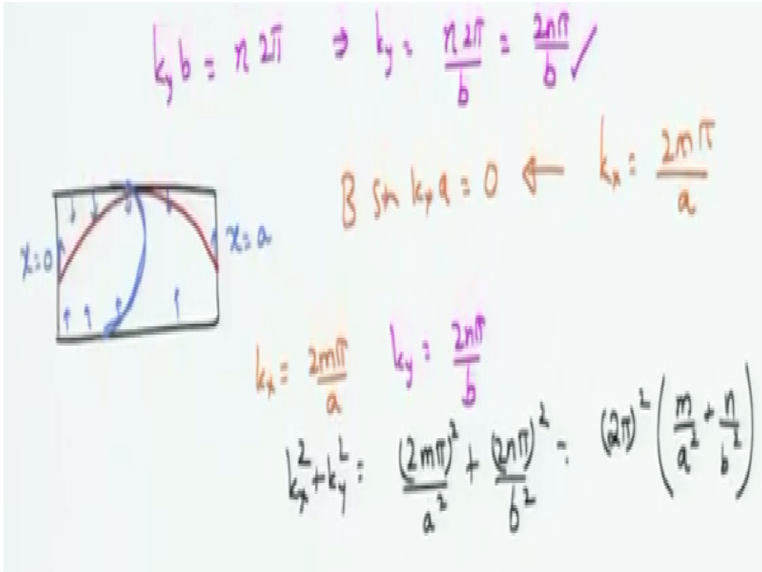
Now this equation has solutions, which can be used to obtain ‘k y b’, assuming ‘D’ is the constant that would not go to zero. And you do not want that one. If ‘D’ is also zero and ‘C’ is also zero, then the entire ‘E x’ is equal to zero. There would not be any electric field. So, the solution for this equation is that ‘k y into b’ must be some integral multiple of two pi. So, this gives you what ‘k y’ is. ‘k y’ must be equal to ‘n two pi by b’ or sometimes written as ‘2npi/b’. So this is ‘k y’. We have figured out what is ‘k y’ from the two boundary conditions.

Let us try to figure out if it is possible to find what is ‘k x’. For that, let us look at ‘E y’ expression. So ‘E y’ in the waveguide wall so this is my waveguide wall, ‘E y’ will go in this, something that is directed along ‘y’. So at ‘y equal to zero’ and at ‘y equal to B’ these components would be normal and therefore they would not be of any help to you in finding the corresponding boundary condition. So, they would not help you in that.

However, ‘E y’ is actually tangential to the two boundaries, at ‘x is equal to zero’ and at ‘x is equal to A’. So clearly one function, which would fit as a function of ‘x’ to zero values on the two sides will again be a sin boundary condition. So, you have a sin boundary condition here, you have a sin boundary condition here also. So these are the two sin boundary conditions for this particular waveguide mode that you will have.

Because this ‘E y’ goes to zero at the two boundaries and this is essentially the sin sort of solution that you are going to get, for ‘E y’ the only possible way that you will have that solution is when you have ‘B sin k x X’. Again this component ‘B sin k x X’ can be used to obtain what is ‘k x’.

(Refer Slide Time: 19:20)

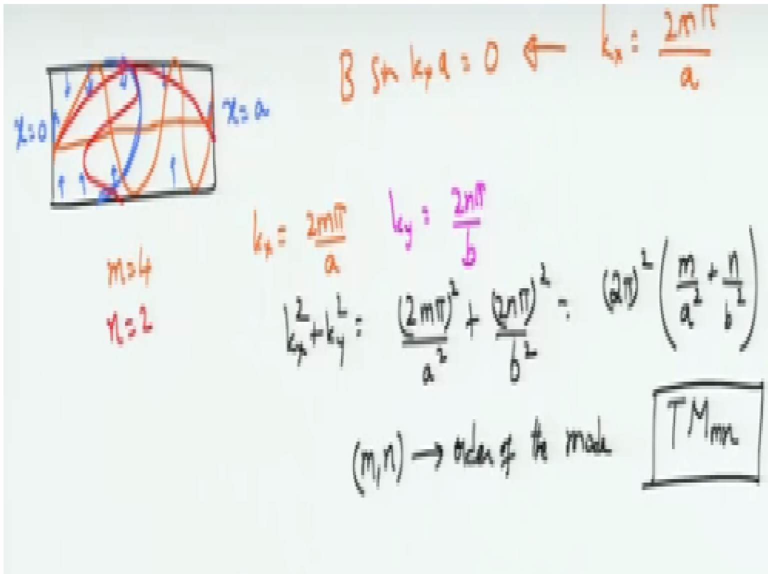


When you put ‘x’ equal to zero, nothing, you do not get anything because ‘B sin k x into zero’ is equal to zero. It would not give you any help. However, when being ‘sin k x A’ is equal to zero, this equation have a solution because ‘k x’ will now be equal to ‘2mpi / a’. So, we know what is ‘k x’. ‘k x’ is ‘2mpi / a’. You know what ‘k y’ is. ‘k y’ is equal to ‘2npi / b’. Therefore, I know what is ‘k x square + k y square’, now?

This is simply, ‘2mpi whole square/ a square + 2npi whole square/ b square. I can rewrite this one by taking out ‘2pi whole square’ as constant and then leaving behind ‘m/a square + n/b square’. Now you might ask what are these ‘m’ and ‘n’ are. Well. These are the orders of solutions. For example, with ‘m’ is equal to one, it means that there is one half cycle, ‘n’ is equal to one, there is one half cycle.

Because the same boundary condition can be obtained in a slightly different manner as well.

(Refer Slide Time: 20:42)

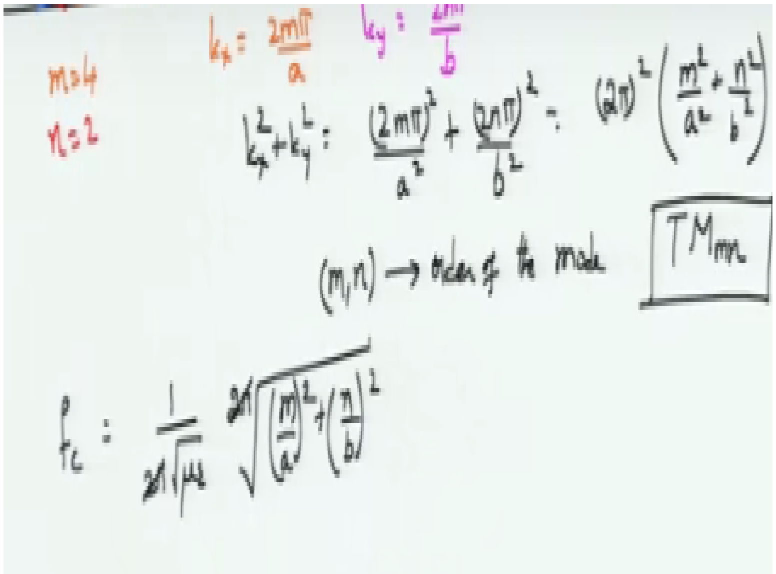


So, I can actually obtain a zero here, have a maximum reach in between, brings to minimum reach a maximum and then actually have one more zero in this way. So how many half cycles are there? There is one half cycle, two half cycle, three half cycle, four half cycle. So along this direction, which is ‘m’, ‘m’ will be equal to four. Similar things can be done for ‘x’ direction also. So, you can actually have two half cycles, so ‘n’ will be equal to two.

So, I will actually have one half cycle, another half cycle, so I will have two half cycles. So this combination of ‘m’ and ‘n’ corresponds to the order of the mode. And we call, so this is the order of the mode and since this corresponds to the TM mode, we denote this mode or we denote this possible solution as TM mn. So, this is the waveguide mode designation. We of course need to find out whether ‘m’ is equal to zero, ‘n’ is equal to zero solutions are possible or whether they would have to be ruled out.

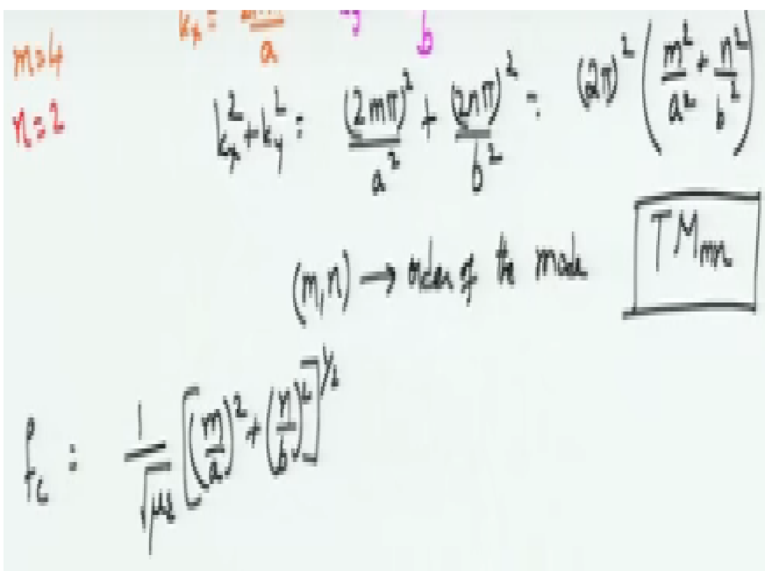
So, what possible values are ‘m’ and ‘n’ are there, will come from looking at all the components of electric fields. So, once we have obtained this, we can actually simplify those equations. Remember we had written something about the cut off frequency. So, even without doing all these calculations, we had written something about the cut off frequency. And cut off frequencies ‘F c’ equals ‘1/2pi square root mu epsilon k x square plus k y square’.

(Refer Slide Time: 22:12)



So, go back to that expression and write this ‘F c’ equals to ‘1/2pi square root mu epsilon, now I know what is k x square plus k y square’, that is ‘2pi whole square m/a square + n/ b square’. So taking square root of this, ‘2pi’ will be factored out. And inside I have ‘m / a square’, sorry this is actually ‘m square / a square’ because ‘m square by a square’. So you actually have ‘m square / a square’, ‘n square / b square’, which can be rewritten as, ‘m / a whole square plus n / b whole square’. This ‘2pi’ actually gets cancelled. So ‘F c’ and there is a square root up here.

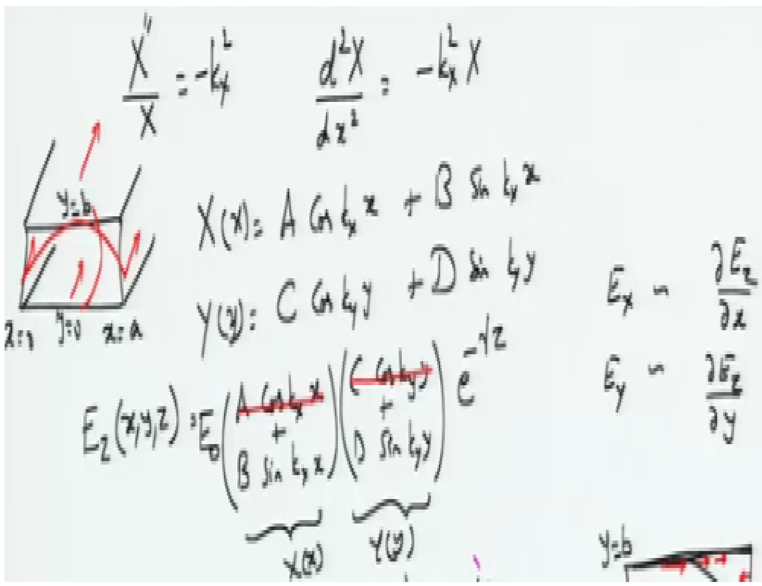
(Refer Slide Time: 22:53)



So ‘F c’ is this fellow ‘m / a whole square + n by b whole square’ under root. So this is the expression for cut off frequency. So we still have to figure out what is ‘E z’ completely. We have figured out what is ‘E x’ the ‘y’ dependence, ‘E y’ the ‘x’ dependence, the ‘z’ dependence is already known. For ‘E z’ what could be the corresponding boundary condition? Because I need to still decide whether A, B, C, D are all there or some of them are zero.

So in this case, it might seem that ‘C’ is zero already and ‘A’ is zero already, but that cannot be true because there is ‘C’ here also. So we need to just figure out, which one is zero and which one is not zero. To do that one, you said boundary conditions should help us. So, with ‘E z’ boundary condition, what would, how would that look like.

(Refer Slide Time: 23:46)

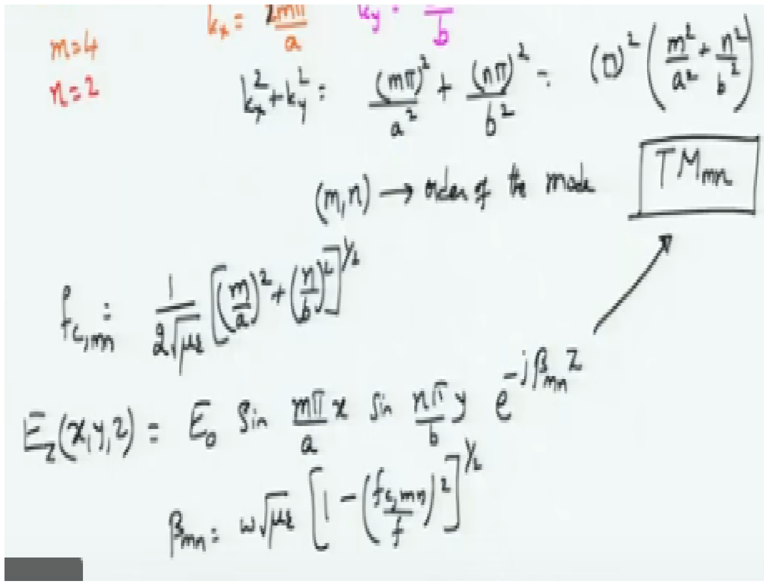


Go back to the picture of the cross section. In this case, may be the cross section does not really help. So you need to write down the wave itself. You will have this scenario. So, this is your wave equation. So, you actually have two conductors here or it seems that you have two conductors, one is at ‘x equal to zero’ and one at ‘x equal to A’. And you have two bottom conductors, one at ‘y equal to zero’, one at ‘y equal to B’.

What could be the direction of the electric field ‘E z’, at these walls? Well, ‘E z’ is directed in this way, so on the wall, it would be sideways directed. I mean, it could be in the sideways, directed along ‘z’ axis. So this would form as tangential component there. It would also form a tangential component to the left. It would also form a tangential component on the top and tangential on the bottom.

So, which means that there would be a sin type of solution here and another sin type of solution in these two directions. So, clearly this could rule out ‘A Cos k x X’ and ‘C cos k y Y’ and the solutions that you are going to get will be consisting of ‘sin k x and sin k y’.

(Refer Slide Time: 24:52)



So your electric field E z, because of the two boundary conditions, will look like some constant ‘E zero’, which we had written sin, ‘k x’ we have already figured out, that is nothing but ‘mpi / a x’. You know what is ‘k x’. Actually for this case, we do not have to write ‘k y’ as ‘2npi/b’, even an ‘npi’ would be alright. So let me cross out these two. I did not realize this one, because sin of something equal to zero will be valid for every pi, not every 2pi.

So I can have to scratch out all these two here. So it is quite unfortunate that I have to do that one over here. So we will actually have a two sitting down here because ‘k x square + k y square’ will be equal to ‘mpi square / a square + npi square / b square’. So, I am going to remove this 2 here and simply write down this as ‘mpi square and npi whole square’ like this. Again this 2 will have to be removed. ‘pi square’ would remain.

And when you take out this ‘under square root’, ‘pi’ would come out and ‘pi’ would cancel there. But here you have ‘F c’ as ‘1 / 2 square root mu epsilon, m / a whole square + n / b whole square’. So similarly, I have in this ‘z’ condition, I have ‘sin k x X’ and ‘sin k y Y’. There are again two constants, B and D. But I do not have to worry about those constants. I can push all those constants into ‘E zero’ itself.

So I can push those constants into ‘E zero’ itself and then just write the functional dependence. So ‘sin of (mpi / a) x, sin (npi / b) y, e power minus j beta mn into z’. Again I am denoting even the propagation constant by two subscripts ‘m’ and ‘n’ to denote the corresponding order of the mode. But what is ‘beta mn’? ‘Beta mn’ is nothing by ‘omega square root mu epsilon into one minus F c mn / F whole square.

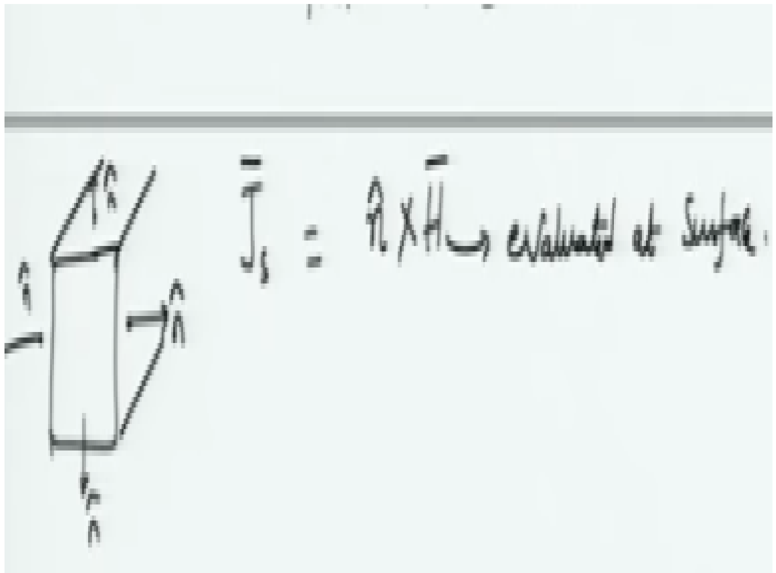
You might ask why did I write down ‘F c mn’? Well because ‘F c’ itself is dependent on ‘m’ and ‘n’. Therefore, it seems correct to include, even for the cut off frequency, which depends on the order of the mode, the corresponding subscripts. So, it depends on the corresponding subscripts, so again different values of ‘m’ and ‘n’, the cut off frequencies are different for the different orders, which will also make it different for the propagation constant beta.

I can leave this part here and leave finding out ‘E x’ and ‘E y’ as an exercise for you. We have already done most of the work. You just have to find out amongst these two, which one you believe because ‘A cos k x X’ has gone. So in this case ‘A sin’ will go away. You have ‘b k x cos k x’ and ‘sin k y Y’. So for ‘E x’ this will go away and for this one, this would also go away. So this is what you would essentially be left with, ‘sin along x’ and ‘cos along y’.

So, I will not write down this, you guys can write down. There is also some proportionality constants that you need to fill in. You can look at the text book for the formulas. There is not much of this one to gain with, then inside to gain with. Whatever we need to understand the modes, we have already done so. One final point, electric fields are tangential on the conducting walls of the waveguide.

So, if electric fields are impinging on the waveguides in the form of a wave, they have to introduce or they have to induce a surface current or surface current density.

(Refer Slide Time: 28:32)



What is the relationship between them? You remember the surface, in the skin depth analysis we have written down this surface current density or the surface current density ‘J s’, we wrote down in terms of that transverse component of magnetic field. We showed that ‘J s’ was actually equal to transverse component of ‘H’ and if you define the normal component of ‘n’ pointing from each walls, for example, this is my waveguide wall.

If I point ‘m hat’ along outside of the wall, at this point also outside of the wall, here also outside of the wall and here also outside of the wall, then I can write down the surface current density ‘J s’ as ‘n cross H’. n cross whatever the H that we had applied and of course this ‘H’ must be evaluated at the surface.