Electromagnetic Theory
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Lecture - 69
Waveguide Analysis

So in this module, we will begin analyzing waveguides that we discussed in the previous module. For the sake of simplicity, we will consider only rectangular waveguides, which means that they have this cross section which is rectangular.

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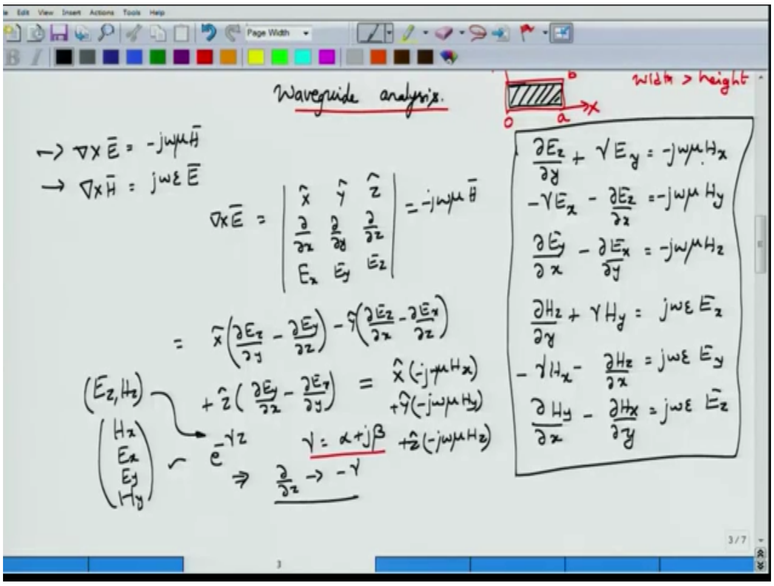

So, if you look at the cross section this is how they would look probably let us make it more rectangular rather than square. So, along x axis, along y axis, the width of the waveguide is a and the height of the waveguide is b, okay. In the traditional notation, the width is actually taken to be larger than the height, okay. And, we will see why that has to be done. Although in principle there is no difference between width and height.

I mean you can consider b as width and a as height and that does not really matter, okay. We will make one important assumption that is all the field quantities, electric fields that are there and magnetic fields, they would all have to propagate along z axis, okay. So, they would be propagating along z axis, but they could be varying with respect to x, y and in the other case they might also vary with respect to z.

For the rectangular waveguides though, we can assume that the electric fields would vary along, I mean vary with respect to x and y, but they would also propagate along z. Now, what sort of propagation should we assume about this one. Should we assume about this one sine propagation, sine propagation, a pulse propagation.

Well, what we will do is, we will assume that these waves are of a particular frequency, okay and they are propagating in the medium in which they can suffer both attenuation as well as just a propagation, so, there can suffer attenuation as well, okay. So, we capture that by writing down the propagation constant to be a complex number consisting of alpha and beta as real and imaginary parts.

As we have seen even for the transmission line case scenario. Alpha corresponds to attenuation, beta corresponds to propagation constant and the propagation of all these fields this should have been H y, okay, so I forgot H y. So, all these field components would be varying along e to par minus gamma z. And even the components E z and H z, they both would be varying as a function e to the par minus gamma z.

Now, the advantage for assuming this type of propagation is that I can replace the derivatives of any of these field quantities with respect to z by minus gamma times the field quantities. In other words, I can replace the differential operation by a simple multiplication operation with minus gamma, okay. The starting point for analysis would be that two Maxwell's equation, which relate the curl of electric field and curl of magnetic field to each other, okay.

In the second expression, in the curl for H, we have ignored whatever the conduction current is because we are going to assume that the region of interest for us to look at the waves is inside the waveguide. And inside waveguide, there is no conductor and there is no possibility of having a conducting current, okay. Well, if there is no current and how are these fields generated?

And the answer is that the currents are there, but they are not there inside the waveguide, but they are there on the waveguide walls. We will also of course assume that the material that fills this waveguide is all completely perfect in the sense that there are no leakage currents. Therefore, we can assume safely that there is no currents inside the waveguide and anyway these are you know hallow cavities.

And for the voltage range that one chooses there would essentially be no leakage inside the air dielectric and hence there will not be any current inside, the conduction current will not be there. So, whatever the current that would be there that would be displacement current and I have also replaced or I am actually using without telling you that these are phasor quantities, right.

So, I am not explicitly indicating the time dependence, we are only interested in the spatial dependence because time dependence is taken to be the phasor form. So, del by del t become j omega, integral becomes 1 by j omega and because of that reason we have changed, we have written this as j omega. We will also assume that mu and epsilon are constants, okay, unless we specify them to be non constant, okay.

We also know how to obtain curl of electric field. You know this is simple determinant of this 3 by 3 metrics and fortunately this works only for, I mean unfortunately this works only for the rectangular waveguides and that is probably the reason why we are choosing to work in the rectangular field, waveguide. Because the formulas are little tractable compared to the rectangular waveguide.

Although there is nothing specific, there is nothing special about a rectangular cross section. In fact, circular cross sections are much more widely used than rectangular cross section in several applications, okay. So, going back to the rectangular cross section, we have this curl of electric field consisting of three different components x component, y component and z component and this should be equated to the right hand side.

The right hand side also contains three components H x, H y and H z. All these components getting multiplied by minus j omega mu, right. So, this curl of e must actually be equal to minus j omega mu H, so which means that it should be x hat minus j omega mu H x plus y hat minus j omega mu H y plus z hat minus j omega mu H z, okay. Unlike the plane wave situation, none of these H x, H y are going to be - I mean none of these H z will not be zero.

E z will not be zero in general and moreover H x and H y will not be independent of x and y, so you cannot just remove them out like that. They do depend on x and y, okay because there are bounding boxes, if you think of this along x direction as well as along y direction. There are conductors, if you go along x, there are two conductors, if you go along y, there are two conductors, of course they are not really two conductor.

But there is a single conductor, but there is a sort of you know imagination that there are two conductors out there, okay, you can think of that way in that. Okay, now here are a set of equations, which I have written, okay. This set of equations are obtained by equating the appropriate or writing down the curl expression for electric field as well as curl for the magnetic field.

And then equating them correctly, okay, so, equating them to the corresponding components. So you can see that the first three set of equations are simply, this equation, curl of E is equal to minus j omega mu, because minus j omega mu H x must be equal to del E z by del y, you can see here, minus del E y by del z. But, I already know what is del by del z. Del by del z is minus gamma and there is a minus sign already present here, so that becomes plus gamma E y.

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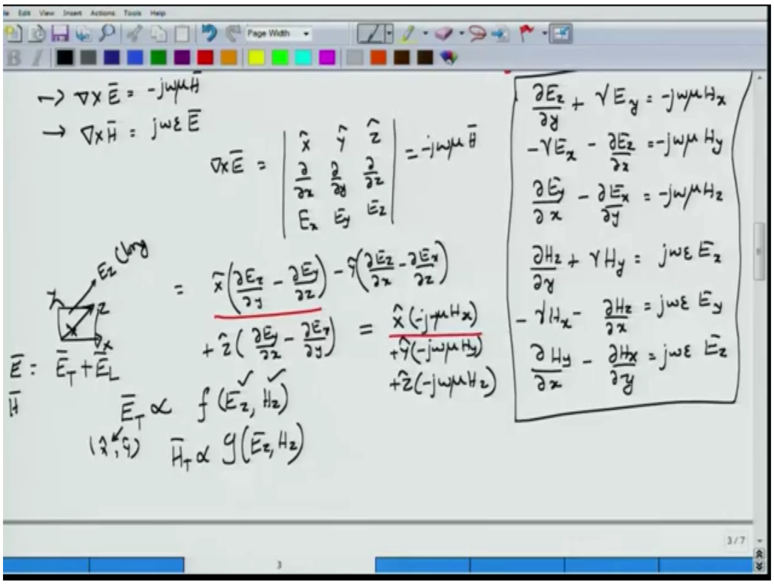

I hope you understand this first equation, carefully look at this two so, maybe I can help you in that one by erasing off this part, hope you can rewrite that one, minus j omega H z. Now consider each component, okay, so, consider for example this component and then equate that to this one, okay and then replace del by del z by minus gamma and you will actually get the first three equations.

Similarly, writing down curl of H, you will get the next three equations. I really urge you to write down these equations you know without looking at the answers first, and then write down them and convince yourself that this is alright. Now, you may ask, why am I doing this. I mean what is the need for doing this? The terms of that, I can actually solve for the electric and magnetic field by expressing all of them, you know the transverse components.

What are the transverse components? Transverse components are x and y components for the electric field in terms of E z and H z. This E z and H z are called as the longitudinal components, okay. So, you have three axis, right? So, this is the z axis, this the x and y. Do not worry if my notations are not really consistent. What I want to show you is that there are three axis out there.

And the electric field components in the x and y axis together are called as the transverse components and the component of the electric field along this one is called as the longitudinal component. So, in the plane, okay, so the electric field that you will have will be called as the transverse component and therefore the total electric field can actually be split into transverse and longitudinal components, okay. Longitudinal only E z is there.

Similarly, you can split H as well as into its transverse and longitudinal components, okay. So, I want to express E x and E y and in terms of z and H z. I also want to express H T transverse components as some function of E z and H z, okay. Some function let us say this is g, function g, okay. So, if I express them then if I know how E z and H z are to solved then once I have that I can find out the expression for E x, E y, H x and H y from these two solutions, right.

So, only two variables, E z and H z need to be known in order to completely specify E x, E y, H x and H y. Again, there is no specific reason, why one has to choose the horizontal components, expect that for the more nomenclature, it becomes easier for us to talk of TE or TM modes, if I know E z and H z, right. So, for a TE mode, there will not be any component of the electric field along the longitudinal.

So E z will be equal to zero, then you are left to solve only for H z and use that obtain the transverse electric and magnetic field components, okay.

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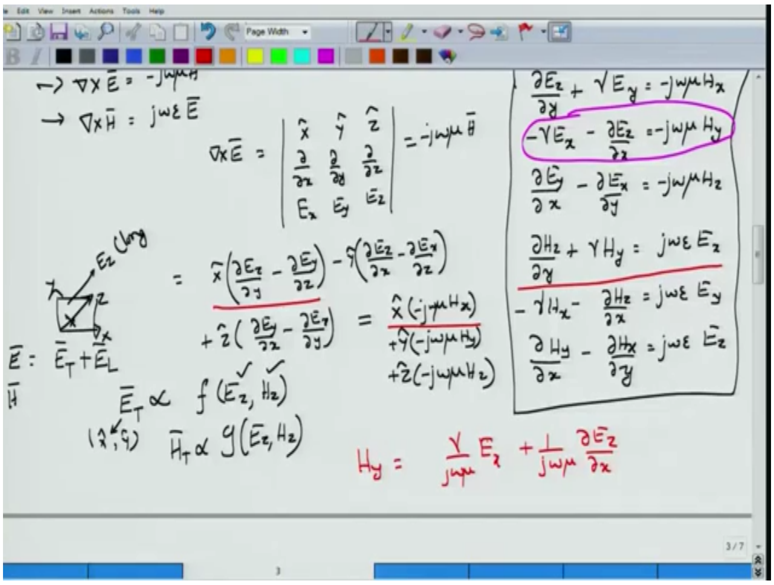

Now, how do we do that? Consider for example, the second equation, okay. So, let me highlight that equation, okay. I use this equation to solve for H y. What do I mean by that? I simply take this minus j omega mu on to the left hand side and then write down what is H y, okay. So, you can see H y is actually equal to gamma by j omega mu E x, okay, by taking minus one by j omega mu down and you have also plus one by j omega mu del E z by del x.

I do not how E z varies with x. I only know how E z varies with z. I also know how E z varies with the other, I mean I only know all the other components varying only with z. I do not know how they vary with x or y, okay, but it does not matter. Now, I know what is H y. Is there H y in the other group of equations, right? So, there is H y in this equation wherein you have del H z by del y plus gamma H y is equal to j omega epsilon E x, okay.

And, if you look at H y, H y is expressed in terms of E x and H, E x and E z. So, if I can write down this, you know substitute for H y into this expression and rearrange it and pull this gamma by j omega E x to the right hand side then I will have E x solely in terms of E z and H z, okay. So, I can do that, I mean let me try doing that one. I cannot use this second equation because I do not know how H y would vary with x.

If I try doing that one by taking the derivative with respect x and I will also have to write down del E x by del x and this becomes the second order equation del square E z by del x square, so, I do not know how to do that. Therefore, I do not use this equation and there is no H y in the second equation and there is only H y in this equation, which I can use, okay.

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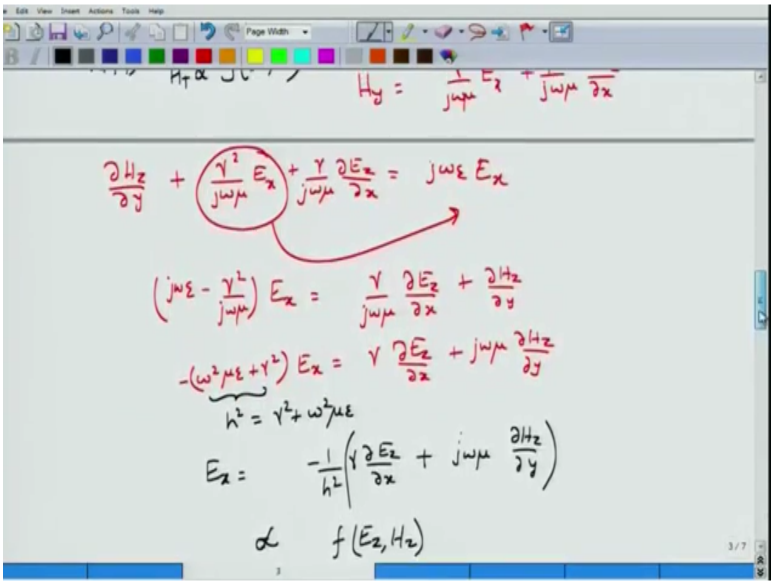

So, let me use that equation and write down substitute for H y, so what do I get. I have del H z by del y that hasn't changed, right, plus gamma square by j omega mu E x plus gamma by j omega mu del E z by del x must be equal to j omega epsilon E y, right. So, with this I can actually pull this gamma square by j omega mu epsilon to the, sorry, this is actually E x, right. So, this is E x not E y, okay.

So, I can pull this on to the right hand side and then pull everything on to the left hand side, rearrange the two equations. What I get is j omega epsilon minus gamma square by j omega mu into E x must be equal to gamma by j omega mu del E z by del x plus del H z by del y. I can rearrange this equation in the brackets, I mean expression in this brackets.

So, I get minus omega square mu epsilon plus gamma square, okay times E x must be equal to j omega gets mu gets multiplied, so I have gamma del E z by del x plus j omega mu del H z by del y, okay. Call this omega square mu epsilon plus gamma square as h square, okay. So, h square is equal to gamma square plus omega square mu epsilon, okay. So, if I call this as h square.

Then E x will become minus gamma by h square del E z by del x minus j omega mu by h square, this is h square times del H z by del y, okay. I can of course pull this minus sign outside and then call this h square as a common factor. So, what I get is minus one by h square gamma del E z plus j omega mu, right. So, this is what I get for E x. So, it is very clear that E x has been expressed in terms of, you know proportional to some functions of E z and this is H z, right.

So, our objective has been achieved for E x. Now, you have to do this procedure for E y, H x, H y. Rather than doing all of that, I will simply give you the values, you can actually check for yourself you know, all you need to do is to find a corresponding component and then find another equation, substitute, rearrange all that equations and you will be able to obtain what is E y, okay.

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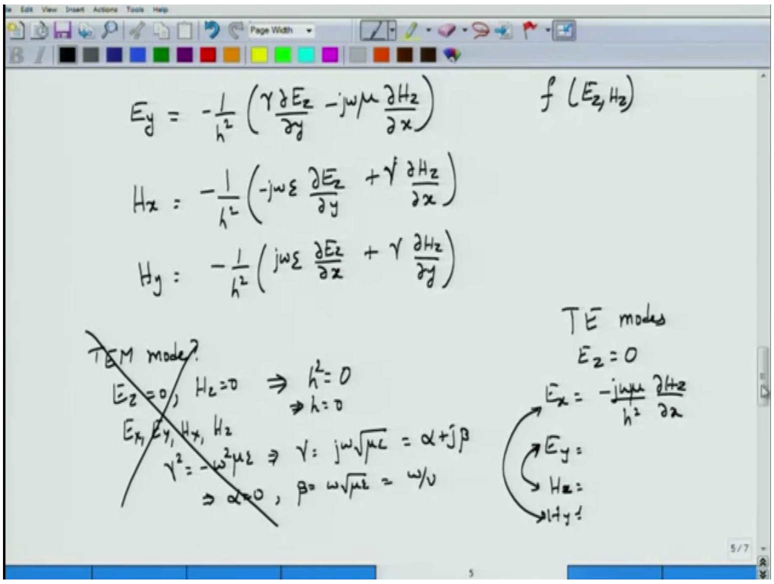

You will be able to obtain what are, all the different terms, but I will just write down for E y, okay that would be gamma del E z by del y minus j omega mu del H z by del x, okay. So, this is one expression for E y, similarly I have expression for H x as well as H z. These are given in the text book, so you can refer to that minus one by h square minus j omega epsilon del E z by del y, okay plus gamma del H z by del x.

Finally, you have H y right. H y is minus one by h square j omega epsilon del E z by del x plus gamma del H z by del y, okay. So, with all these things known to us, right, so with all these fellows expressed some function of E z and H z. Let us actually write down the wave equation of E z, write down the wave equation for H z, solve them and then use those expressions in this expression for E x, E y, H x and H y, okay.

To obtain, right, so once you have done that one, all you have to is to substitute for E z and H z after calculating them and substitute and obtain the value for E x, E y, H x and H y. So, the objective of getting the transverse components in terms of longitudinal components and you know the relation between the two has been achieved, okay. What do we do from here? Where do we go from here?

First of all, let us see what possible modes that we can obtain, okay. So, let us first try if any wave we can obtain TEM mode. Well, what do we need to obtain a TEM mode? In TEM mode, you have E z equal to zero, H z is also equal to zero, right, so this is the TEM mode, right. For this mode in which both E z and H z is equal to zero, what will happen to del E z by del x or E z by del y? What will happen to these components? They would all go to zero.

So, all the components that are mentioned in this brackets will have to be equal to zero and if you want a non-zero value of E x, E y, H x and H y, the only way you can obtain that is to make h square is also equal to zero, okay. Because you now have zero by zero and that zero by zero will make you know a small quantity by small quantity. We will keep everything to be finite.

So, if you want E x, E y, H x and H z and all of them to be independent of x and y and propagating along z which forms the TEM mode then the condition is that h square should be equal to zero or equivalently h is equal to zero. Now, h equal to zero means, gamma square must be equal to minus omega square mu epsilon, which also means that gamma must be equal to j omega square root mu epsilon.

We know that this is alpha plus j beta. This simply implies that alpha is equal to zero, beta is equal to omega square root mu epsilon or omega by v, okay. So, we have seen this condition, you know for a TEM mode, this is what you need to have a lossless TEM mode condition is required for the waves to be existing. Of course this does not really happen for a waveguide, so, you can conveniently and safely scratch this off from your list of allowed modes.

So, then what about other modes? Can I have TE modes? It is possible. For TE modes, E z must be equal to zero, and what will happen to E x in this case? E x will be minus one by h square, right. E z component is zero, so j omega mu into del H z by del y, so you get minus j omega mu by h square del H z by del x. Similarly, you can write down what is E y, okay and write down what is H x and H y, okay.

You will actually see that E x and H y pair can be found, okay. E y and H x pair can be found just as you would form in a plane wave scenario, okay. Can I have TM modes? Certainly possible. So, for TM modes, all that we need to ensure is H z is equal to zero and then E x, E y, H z, Ex, E y, H x and H y will simply be functions of E z itself, okay. So, this is also possible. So, since we are not writing down the expressions let me write down this over here.

TM modes implies that H z is equal to zero and these are also valid mode patterns for us, okay. Now, we need to solve the equations, okay because all we have done so far is to establish that we can write down E x, E y, H x and H y in terms of E z and H z and then we can characterize the modes that is possible set of solutions of this Maxwell's equation in this waveguide problem to have two times of modes, TE modes and TM modes, okay.

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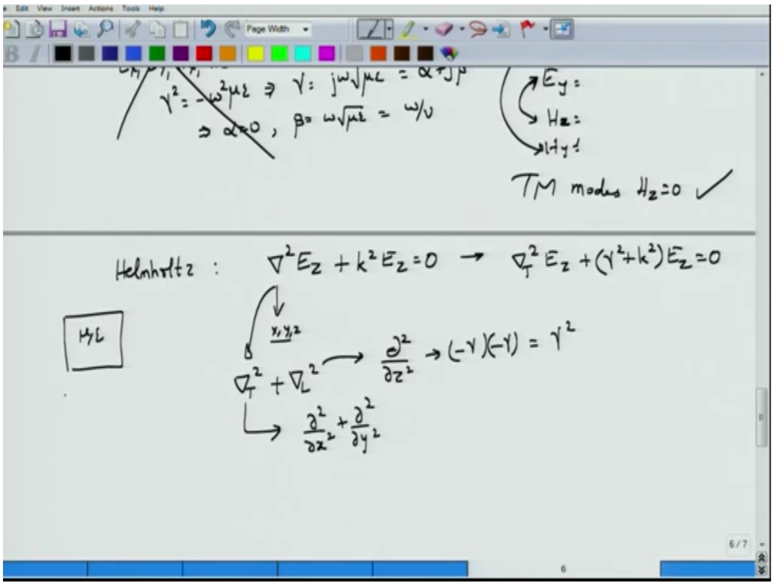

So, for these modes we need to actually obtain what is E z and H z, okay, but before we do that one, we need to first of all obtain an expression for E z itself, okay. How do I do that? Well, that is actually quite simple, right. I know from Helmholtz equation, right, I know from Helmholtz equation that del square E z for any component of the electric field or magnetic field I can write this Helmholtz equation.

So, del square E z plus k square E z must be equal to zero. What is del square here? Del square is nothing but the Laplacian operator operating on the component E z. What does a Laplacian operator consist of del square by del x square del square by del y square and del square by del z square. So, this particular operator has components for x y and z or rather has differential operators with x y and z, okay. x and y are transverse; z is the longitudinal one.

In fact, I can split this del square itself into del transverse and del longitudinal, okay. There is no inherent advantage of doing this, but this is something that you would find in my grove literature quite often. So, transverse component of the Laplacian is given by del square by del x square plus del square by del y square. The longitudinal component is actually del square by del z square which is nothing.

But minus gamma multiplied by itself because remember each del by del z constitutes a minus gamma, so multiplying this one what you get is gamma square, okay. So, this equation can actually be rewritten in terms of transverse components, transverse Laplacian del transverse square E z plus gamma square plus k square, E z is equal to zero, okay. So, this is what you wanted to write down.

But, I already know what is k square, right because I have this waveguide, right, which is filled with mu and epsilon material.