Electromagnetic Theory
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Lecture - 60
Steady state sinusoidal response of T-line - II

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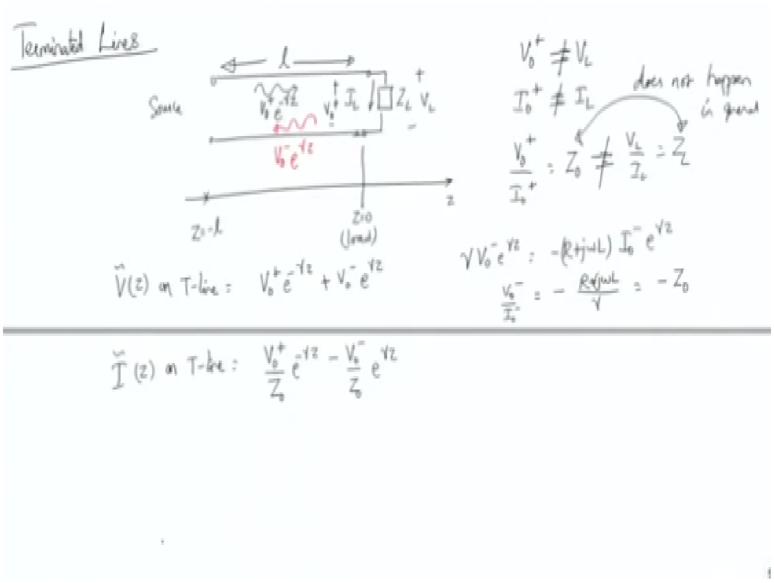

We now consider realistic case where the transmission line is actually terminated okay. So let us consider the case where the transmission line is terminated, which is what would happen in a realistic scenario right. So you have a transmission line, which would run for a certain distance and then it might terminate itself into a wave guide or it could terminate itself into a free space in rare cases or it could terminate into a different load or a second transmission line or an antinode.

Could terminate into anything on to this side, but then let us characterise whatever the termination by its terminal characteristic in the form of an impedance Z L. So through this load Z L, there will a current I L flowing and across the Z L will be the voltage V L okay. So here you will have the source side of the transmission line, we have a transmission line of length l okay.

This is the length of the transmission line and what you have to observe here is that on the transmission line okay. Initially when you are looking at there would be infinite wave right. So it would be v zero plus e power minus gamma z and you come all the way up to this point.

Now in dealing with these kind of situations, the convention is to actually take this as z equal to zero that is one assumes you are actually sitting on the load okay and takes that as the point of reference or plane of reference as z equal to zero.

This is a kind of older convention, but that is very popularly followed. So we are going to follow that one. So you locate your load at z equal to zero okay and then the input end of the transmission line will be located at z equal to minus l, this is how the z axis is going. Now initially, if you assume that there is only incident wave, incident voltage wave propagating, so this wave voltage would propagate propagate and would be the value at z equal to zero, it would just be equal to v zero plus.

But immediately, you are stuck by something, on the transmission line the voltage seems to v zero plus amplitude, but the just immediately next to that one there will be a voltage of v l right and because you have connected k v l dictates that v zero plus must be equal to v l clearly this should also indicate that I zero plus is equal to I L and if you take the ratio of V zero plus to I zero plus, which is basically giving you the characteristic impedance of the transmission line.

This would be z zero, this should be equal to V L by I L, which is equal to Z L. Now it is possible that one chooses Z L equal to Z zero okay indeed most of the RF designers spend their time in trying to make Z L equal to Z zero. But this is not always guaranteed right. So this is not always true that the load impedance will always be equal be equal to the characteristic impedance.

If that happens, then there is only incident wave and this incident wave is completely absorbed by the load. But this does not happen in general, does not happen or it is not true in general. So this V zero plus by I zero plus is equal z zero should not be equal to Z L, which further implies that I zero plus is not equal to I L and V zero plus is not equal to V L. So where have we gone wrong.

Is k v l, k c l wrong, well k v l and k c l are not wrong, what is wrong is that you cannot satisfy this boundary condition just with an incident voltage wave okay. Because of this scenario of termination with a different load than the characteristic impedance, there will be a reflected voltage wave with an amplitude of say v zero minus and propagating along minus z direction. So it would be v zero minus e to the power gamma z.

Therefore, the total voltage on the line, on the transmission line okay is the sum of forward going plus the backward travelling reflected voltage. This is something that we have seen right. Now for, when we assume that you had only v zero plus e power minus gamma z, we found out the relationship between d v by d z, or we found out the relationship between i of z and v of z and that ratio was actually z zero right.

Now, when you have v zero minus, if you assume for a minute that there is only backward propagating wave and then differentiate this one with respect to z what you see is that gamma v zero minus e power gamma z should be equal to minus R plus j omega L into the current, current will be I zero minus e power gamma z and then when you look at the ratio of v zero minus to I zero minus, you will see that this would be equal to minus R plus j omega L by gamma.

Again writing gamma has square root of R plus j omega L into G plus j omega C, you will see that this is nothing but minus z zero. This is very crucial okay. So the ratio of backward going or reflected voltage to the reflected current is actually equal to minus z zero. Therefore, If I know what is the voltage on the transmission line, I can write the current on the transmission line as I zero plus is nothing but v zero plus by z zero propagating in the same direction.

Therefore, I can write this as V zero plus by Z zero e power minus gamma z, but the ratio of v zero minus to I zero minus is actually minus z knot, therefore this is minus v zero minus by z knot e power gamma z okay. These are the voltage in currents on transmission line. Both voltage and currents actually consist of incident as well as reflected waves in order to satisfy the boundary condition. Have you actually satisfied the boundary condition, well let us look at what happens.

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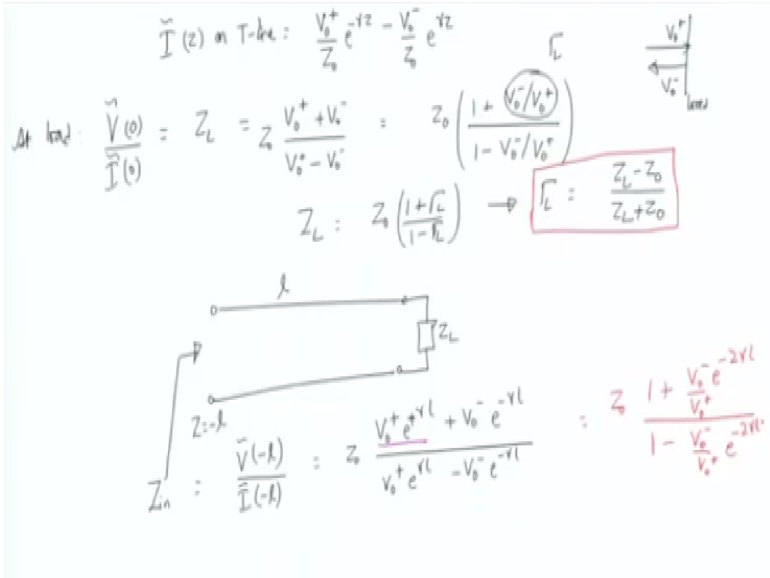

Let us assume that V at Z equal to zero, which is the load that you are considering right must be equal to Z L clearly right. So at the load side, the ratio of voltage to current must be equal to Z L and this should be equal to V zero plus plus V zero minus divided by V zero plus, this Z knot goes on to the numerator minus V zero minus okay. It is (()) (06:37) factor V zero plus out, so you get Z zero into one plus V zero minus by V zero plus divided by one minus V zero minus by V zero plus.

Now what is V zero minus, this is the incident V zero plus amplitude and this is the reflected amplitude at the load okay. So at the load your V zero minus by V zero plus is a ratio of reflected to incident voltage and this ratio is denoted by gamma L, this is the reflection coefficient as measured on the load L okay, on the load and the load is characterised by gamma L. So what you have is Z L given by Z zero into one plus gamma L by one by gamma L.

You can actually use this equation, turn it around to find out what gamma L is, gamma L is given by Z L minus Z knot by Z L plus Z knot. This equation is very important. So let me box this equation out okay. This equation actually is giving you the reflection coefficient on the load in terms of the load Z L and the characteristic impedance Z zero.

Clearly as you change Z L or Z zero the value of gamma L will also change alright. Now we might also be interested in something that happens over here okay. So let us say this is the load that we have connected and this is the transmission line of length L that I have, so z equal to minus L corresponds to the input terminals of the transmission line right.

So at the input terminals of the transmission line what is the impedance Z in, this impedance Z in seen looking into that input terminals of the transmission line should clearly be the voltage at the source side or at the input terminals to the ratio of voltage to current right, so current at the input terminals.

Now using the expressions for voltage on the transmission line and current on the transmission line, substitute z equal to minus l, what you get is v zero plus e power minus gamma l plus v zero minus sorry this becomes plus gamma l because you are substituting z equal to minus l and then you have v zero minus e power minus gamma l divided by v zero plus, there is a z zero here, e power gamma l minus v zero minus e power minus gamma l okay.

Factor this entire v zero plus e power plus gamma l out, so what you get here is z zero into one plus v zero minus e power minus two gamma l divided by v zero plus divided by one minus v zero minus by v zero plus e power minus to gamma l. But I already know what is v zero minus by v zero plus, that is precisely equal to gamma l.

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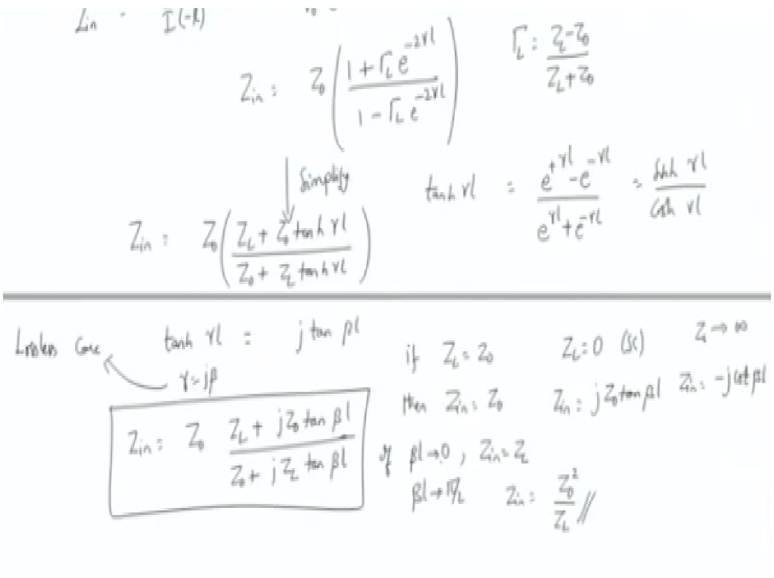

Therefore, the input impedance Z in, is equal to Z zero into one plus gamma L e power minus two gamma l divided by one minus gamma L e power minus two gamma L okay. Now you re-multiple this one by e power gamma L on both sides okay and substitute for gamma L as Z L minus Z zero plus Z L plus Z zero and simplify the equation. This is not very difficult, this is just little tedious but you can do it in couple of lines.

So when you simplify this, what you get is the input impedance to be equal to Z zero into Z L plus Z zero Tan hyperbolic gamma L divided by Z zero plus Z L and hyperbolic gamma L, where Tan hyperbolic gamma L is actually given by e power plus gamma L minus e power minus gamma L divided by e power plus gamma L plus e power minus gamma L. This is actually the ratio of hyperbolic sign gamma L to hyperbolic cos gamma L okay.

So this is the input impedance and you can actually, you will be little surprised to see that input impedance depends on two things right. It of course depends on Z not if you leave that out. It also depends on the load impedance Z L. So if I connect different loads, then that load would look very different at the input terminals of the transmission line okay and it would also depend on what is the length of the transmission line that connects the load to the input.

In fact, this observation that the length of the transmission line and Z L can influence what the input impedance of the transmission line, it is extensively used in analysis for RF and microwave circuits okay. So it is in fact used to transform the impedance Z L by choosing an appropriate value of L and appropriate value of Z zero to whatever value of Z in that you want, so as to maximise the power transfer okay.

We have a simplified expression for Z in for the lossless case. For the lossless case what happens is that, tan hyperbolic gamma L actually becomes j tan beta l why because gamma is equal to j beta for a lossless case, there is no attenuation. So substituting this into the expression for input impedance will give you Z in Z zero and Z L plus j Z knot tan beta L divided by Z knot plus j Z L tan beta L okay.

So this is the expression for input impedance. Now let us look at some of the cases. First of all, if Z L is equal to Z knot, what happens to input impedance, you substitute Z L is equal to Z knot here. So it becomes Z zero plus j Z zero Tan beta l, Z zero plus j Z zero Tan beta l. So clearly Z in will be equal to Z zero right. If Z L is equal to Z zero, then input impedance will exactly be equal to the characteristic impedance.

It is completely independent of load impedance and it is completely independent of the length of the transmission line okay. Suppose Z L is equal to Zero that is you short circuit the load side. With Z L equal to zero, the input impedance becomes j z zero tan beta L right. So this is the input impedance. When Z L is open circuited, then Z in will be equal to, so with Z L being open circuited this term becomes larger.

This term becomes larger, Z L cancels, what you have is minus j cot beta l okay. Further, when beta l goes to zero right that is when the length of a transmission line is made very very small then this tan of beta l goes to almost zero; therefore, this will go away and what you have is Z in equal to Z l that is probably not much of a surprise in this expression okay. But what might be surprising to you is when you take beta l to be pi by two okay.

So when beta l becomes pi by two tan of pi by two becomes very large. So in this case, this will survive, that j Z zero tan beta L in the numerator and in the denominator j Z L tan beta L will survive and input impedance becomes j tan beta l will all go away, that become Z zero square by Z l okay. So this is what would happen.

On the other hand, when Tan beta l goes to zero that is when beta l goes to pi, tan beta l goes to zero, then z in again goes to z L. Oh yeah, we have already considered that particular case when beta l goes to zero Tan beta l goes to zero or when beta l goes to pi again it would essentially be the same situation okay. So these are some of the cases.

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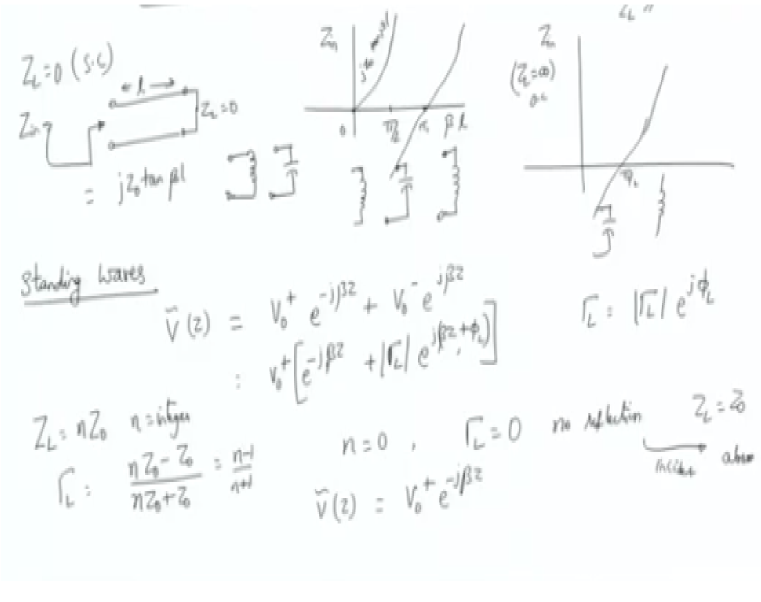

Let us actually look at these two cases, which seem to be very interesting okay. First consider Z L equal to zero, which is placing a short circuit on the load and input impedance for a L length transmission line, so this is the situation that we are considering okay. This Z L equal to zero and this is a piece of length L of the transmission line and this is your input impedance seen looking into this okay.

And this input impedance seems to be purely real. This is as good as or as distinct as having an inductor or a capacitor as an equivalent transformed load right. If you actually graph how Z L changes as a function of L or as a function of beta L. So that you can see that at beta L is equal to zero, this Z in will be equal to zero and then at beta L is equal to pi by two between the zero and pi by two the value of Z in keeps on increasing and it looks something like this.

So this fellow j Z zero tan beta l and the equivalent impedance or equivalent reactants in this case would be that of inductor. So the input impedance would look more like an inductor okay, it would look exactly like an inductor for this particular case. At pi by two; however, beyond this up to pi right, you have this kind of a behaviour. So in this region your input impedance looks like a capacitor.

So the load looks like a capacitor. Here again the load looks like an inductor okay. Similar thing would happen when you consider Z in for Z L equal to infinity that is for open circuit. In this case, they would actually change slightly okay; it would start from capacitor, go through and then become an inductor okay. So here it starts of being a capacitor and then becomes an inductor at say pi by two okay.

So this is for the case of open circuited input impedance. In fact, by choosing whatever the value of L, you can choose whatever the required inductance or capacitance. In fact, you can generate a required amount inductance and capacitance by choosing this L appropriately okay for a short circuited or open circuited load alright. We now come to very interesting scenario okay of standing waves okay.

What we mean by this standing waves. Let us first look at what is the voltage phasor on the transmission line. Voltage phasor on the transmission line is for a lossless line, we have V zero plus e power minus j beta Z plus v zero minus e power j beta Z, but clearly this V zero minus is related to V zero plus, you can actually take out this factor V zero plus e power minus j beta Z if you want.

What you get is e power minus j beta Z plus gamma L, because this V zero minus is gamma L and gamma L itself could be complex because gamma L is Z L minus Z zero by Z L plus Z zero, it could be complex, so it can be written in terms of its magnitude and a certain angle right. So you have e power j beta Z plus phi L, phi L, where phi L is the phase angle. So what we have written is reflection coefficient gamma L as magnitude of gamma L times e power j phi L.

So this is what you would actually see okay on the transmission line. This is the voltage phasor okay. Let us consider some cases. When Z L is equal to n Z knot where n is a integer okay. So in this case what would happen to gamma L, gamma L will be n Z knot minus Z knot by n Z knot plus Z knot, this is nothing but n minus one by n plus one. This would actually be a completely real quantity for n being an integer.

Suppose n is equal to zero, in this case gamma L is actually equal to zero and then the voltage phasor V of Z will simply be equal to V zero plus e power minus j beta Z right. This implication that gamma L equal to zero or no reflection is simply telling you that Z L equal to Z zero and the entire incident voltage is observed by load okay entire thing is observed by load okay.

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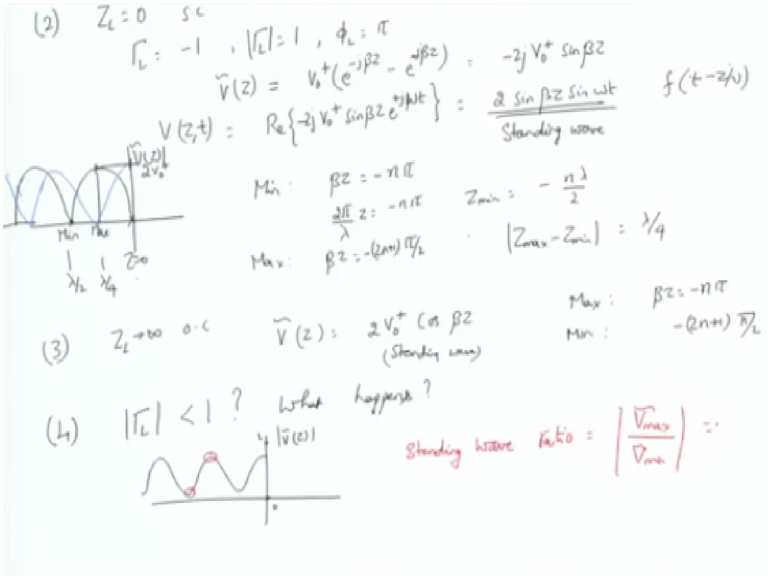

Consider a second case which is very interesting. You short circuit the load side okay, so if you short circuit the load side, your gamma L will be equal to minus one because Z L is equal to zero, so you get gamma L of minus one you can, if you want find out what is the magnitude of gamma L, which will be equal to one and the phase angle phi L will be equal to pi.

Now substitute these expressions into the voltage phasor expression on the transmission line. So what you see is, V of Z is equal to V zero plus e power minus j beta Z minus e power plus j beta Z. But this is nothing but minus two j V zero plus sin beta Z. So the phasor that you see is actually minus two j V zero sin beta Z and if you now try to find out what is actually V of Z and time, that is in terms of both space and time.

You see that this is real part of minus two j V zero plus sin beta Z e power j omega t right. The real part of this, assuming that V zero plus is real is nothing but you split this e power j omega t into cos omega t plus j sin omega t, but cos will not give you the real part that j sin omega t j and minus j will cancel with each other. What you get is two sin beta Z sin omega t. This is certainly not in the form of some function t minus Z by V.

So this is not in the form of a propagating or a travelling wave. This is an example of a standing wave okay. Because you go back to this one in the case of n equal to zero and gamma L equal to zero situation, your V of Z t will be equal to some V zero plus cos omega t minus beta Z, this is an example of a travelling wave whereas the case where you have just seen this Z L equal to zero short circuit is an example of a standing wave okay.

If you plot the voltage phasor, magnitude of the voltage phasor for this particular case at Z equal to zero is where the load is and at that load point sin beta Z will be equal to zero and thereafter it will go as magnitude of sin beta Z right with a maximum amplitude of two V zero plus okay as you can see from this expression. You have to observe that there is a minimum at the load, then there is one more minimum at as different location on the transmission line right. This is also a minimum.

There is actually a maximum in between two minimum that is obvious right. So if you go from minima to minima there has to be one maximum, but where are these minima located well, the first minima is located at Z equal to zero and subsequent minima must be located whenever the sin beta Z goes to zero. When does sin beta Z go to zero, beta Z goes to zero when the sin beta Z goes to zero when beta Z is actually n pi.

Because at n equal to zero that is at load Z equal to zero, you have a minima at pi right. When n is equal to one again you have a minima. The minus signs simply indicated that the load is counted at Z equal to zero and all the further things are counted backwards. You know the transmission line extends from Z equal to zero to Z equal to minus L. You can find out this relation. You can better put this relationship in a slightly different way.

This beta can be written in terms of wave length as pi by lambda okay. Times Z will be equal to minus n pi okay and you can actually see where the minima are located, minima are located at minus n lambda by two. So n equal to zero being the fundamental position where the minima is, thereafter at every lambda by two there is a minima. Where will be the maxima, maxima for this one would occur whenever beta Z equal to pi by two right.

It cannot be n pi by two, it has to be two n plus one pi by two and this has to be a minus and these are the locations for maxima. In fact, the distance between maxima to minima, magnitude must be equal to lambda by four. So if this is at Z equal to zero or in terms of lambda this is zero and then next minima occurs at a lambda by lambda by two. The maximum must occur at lambda by four.

A very similar relationship you can find for Z L tending to infinity that is for the open circuit here though the voltage phasor becomes two V zero plus cos beta Z. This is still a standing wave okay, they still not away which is propagating, but if you were to plot this V of Z on the as a function of Z, you would actually see that initially there is a maxima then there is a minima, then there is a maxima and a minima and it goes like this okay.

Then it goes like this. For this particular case when there is an open circuit. So you can see that there is initially a maxima where the minima for this one is located and minima is located for the open circuit, where the maxima for the short circuit is located. Again you can find out when this would be minima and maxima. Location of minima will be when beta Z is equal to n pi or rather minus n pi.

Because at n equal to zero cos of zero is maximum at n equal to one and at pi cos of pi is minus one, but the magnitude is one; therefore, this is actually maxima right because these are maxima. So maxima are located at beta Z equal to minus n pi minima, on the other hand are located whenever this cos beta Z goes to zero, which means there has to be pi by two, three pi by two and so on. So this would be minus two n plus one by pi two.

Again the difference between or the distance between maxima and minima will be equal to lambda by four okay, so these are the standing waves that you would see on a transmission line, now let me actually just give you something to think about okay. What would happen when gamma L magnitude is less than one okay, so just think about this, the answer is simple but I do hope that you first think about this very clearly.

And then convince yourself that in the case of gamma L being less than one, this is how the voltage phasor magnitude looks like. It will not reach the maximum of two, it will not reach the minimum of zero, but it would be located somewhere in between okay and this is what would happen for gamma L less than one.

And sometimes where actually interested in finding out what is the ratio of this maximum magnitude to minimum magnitude okay. This ratio is called as standing wave ratio okay. Standing wave ratio gives you the ratio of maximum voltage phasor to minimum voltage phasor okay. When will the voltage phasor be maximum, whenever if you go back to the expression for the line okay.

So you can see that this is the expression, in this case maximum will occur whenever you have this beta Z plus phi L being to zero okay. So you can take this out so then this would be e power j two beta Z phi L. So when you factor this out what you get is one plus mod gamma L into something. When that phase factor is equal to one you get a maximum value okay and that maximum value is given by one plus mod gamma L.

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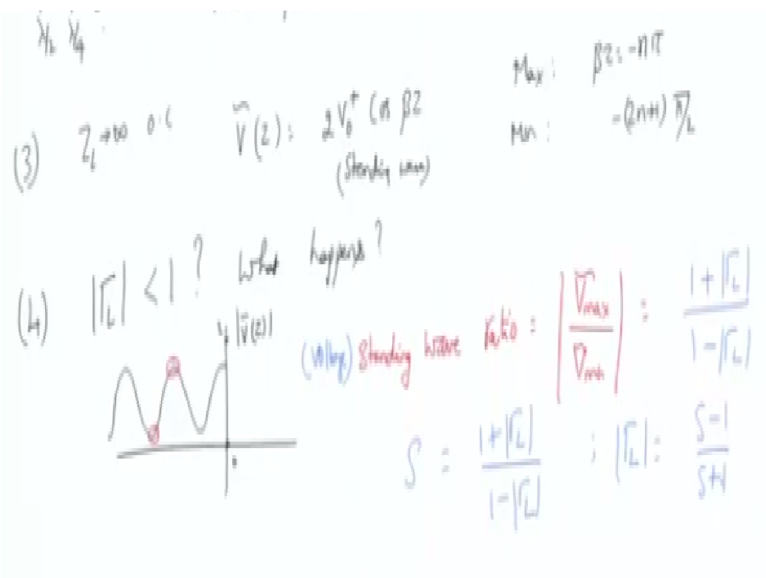

Similarly, the minimum value is given by one minus mod gamma L okay and therefore the ratio of these two, which is the voltage standing wave ratio is given by one plus mod gamma L by one minus mod gamma L. This is denoted by the symbol S and is given by one plus mod gamma L by one minus mod gamma L. In terms of S one can invert this relationship and say gamma L is equal to S minus one by S plus one.