Electromagnetic Theory
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Lecture - 59
Steady State Sinusoidal Response of T-Line - I

So in this module, we will be discussing the study state sinusoidal response of transmission lines. We have derived the wave equations for transmission lines and actually saw that voltage and currents on the transmission line or more like waves, they actually propagate as waves okay. And we will consider sinusoidal response much as the same case as we considered the sinusoidal response for plane waves.

Because the transmission line equations are linear, if I know how the transmission line would respond for a time harmonic or a sinusoidal signal, then any other waveform can be expressed as Fourier series of consisting of this different sinusoidal signals and therefore I can easily obtain the response of for any other waveform okay.

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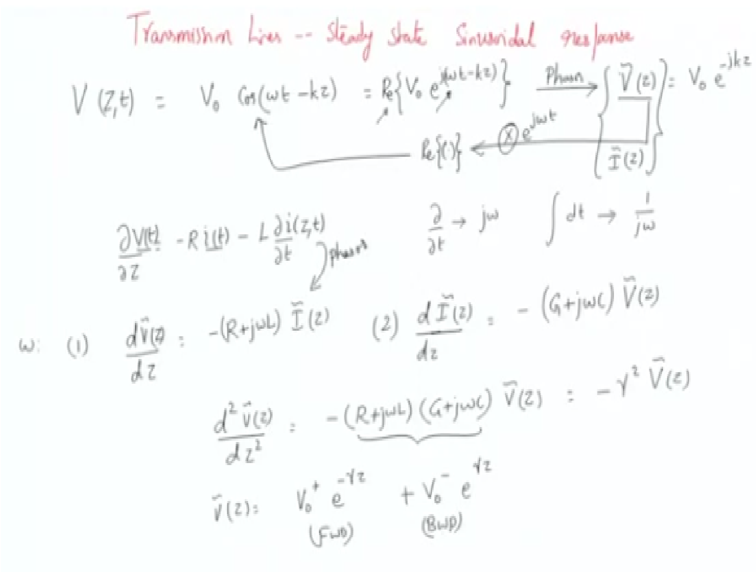

So this is in the frequency domain that we are going to consider, we will be talking about one sinusoidal excitation of the transmission line and then let us see how such time harmonic or sinusoidal voltages and currents propagate on a transmission line. Before we do that, we need to very very briefly review the concept of a phasor, because that is what we will be using in our transmission line equation.

Further, if you recall a phasor was basically a shorthand way of expressing voltage or a current without really specifying the frequency because we assume that a particular frequency has been chosen and for that frequency we do not want to always carry this cos omega t or omega t term. Therefore, this is a convenient way of representing those voltages and currents of the same frequency by dropping that omega t term like we will see how it is done.

So given a voltage on the transmission line, no at any point z on the transmission line and a function of time, it of a particular frequency as caused omega t minus k z, we can actually rewrite this one in terms of the complex notation as v zero, or rather real part of v zero e power j omega t minus k z right. V zero, we will assume to be real and then we have a shorthand notation, which we call as phasor and this phasor drops the term of e power j omega t as well as this real part okay.

So if the phasor for this one is basically the amplitude. In this case, the amplitude is actually changing with z that is along the transmission line and therefore your phasors are complex numbers, which are functions of z as well. Now how do you go from this phasor back to the original notation, well all you have to do is to multiply by e power j omega t and then take the real part of it right.

So when you multiple this one by e power j omega t, you will see that this would be v zero e power j omega t minus k z and then you take the real part of this to obtain the real time dependent voltage okay. So this is all about phasors and we will be talking about this voltage. Similarly, one can introduce the current phasor and all our further equations will be described in terms of this phasors okay.

So the first step in order to obtain the sinusoidal response of the transmission line would be to actually rephrase the equations in terms of these phasors and then solve those equations okay. So we will do that one, so in order to do that, let us actually start by recalling what the wave equation was. So del v by del z was equal to minus r i of t, so this is the case where you had both z and t and this is in the time domain waveform I am writing, minus L del i by del t.

This is what we had, but with phasors what happens is that this operation of del by del t can be replaced by multiplication by j omega and the operation with respect to integration can be replaced by one by j omega okay. So with this substitutions in mind and then replacing this v of t by the corresponding phasors, what we get is an equation, which states d v of z by d z is equal to minus r plus j omega L into I of z.

Please note that this v of z is the phasor. Of course in this case, it is not i, it is just i of z t similarly, it is i of z t here and v or z t. These are the real time expressions for voltage and currents okay. But when we go the phasor notation right, when we employ the phasor notation all the time dependant terms drop out because we are assuming that all the voltage and currents on the transmission line are varying with a particular frequency omega and then we use this phasor form to drop this omega t dependents okay.

And del by del t will be replaced by j omega and this is the first order partial differential or rather in this case full differential equation that you are going to get, which describes how the phasor voltage changes along the transmission line. A similar equation can be written for the current phasor, which is d I of z by d z given by minus g plus j omega c okay times v of z.

Hopefully, this makes sense because these conductance times voltage must give you the current and then j omega c into v is nothing, but c into del v by del t, which would essentially be the current through the transmission line. So these are the two coupled first order differential equations. In order to solve them analytically what we do is we differentiate this equation once more with respect to z and then employ the second equation.

So we differentiate equation one, use equation two to obtain a second order differential equation for the phasor voltage and that is given by minus r plus j omega L G plus j omega c into c v of z. There is a convenient shorthand notation for this one okay. We will call this, this is the most symbol that we use, this shorthand notation says that this is some complex gamma and this gamma square is equal to R plus j omega L into G plus j omega c time v of z.

Clearly, this second order differential equation has a solution, which will be consisting of both forward going wave as well as backward going wave right. So for the forward going wave, you have v zero plus e power minus gamma z and for the backward propagating wave, you have v zero minus e power plus gamma z. So this is my forward wave and this is my backward travelling wave.

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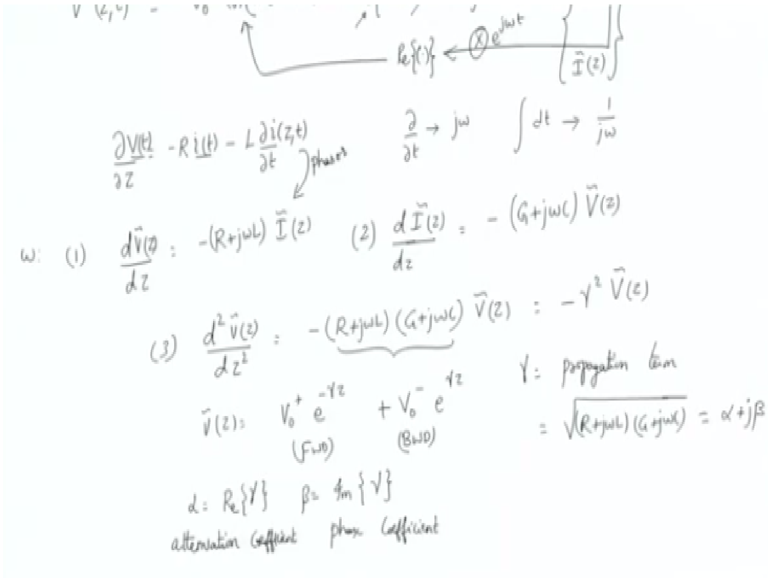

So what is gamma, gamma is the complex propagation constant, although this is not really a constant, so may be a better term would have been to use propagation term and this gamma is given by square root of R plus j omega L into G plus j omega c clearly because gamma square is this quantity R plus j omega L into G plus j omega c. So taking the root will give you gamma right.

And this inside will be a complex number and square root of that complex number will also be a complex number therefore we can write gamma in general as two terms, real part as well as imaginary part. So alpha being the real part of gamma and beta being the imaginary part of gamma. These are called as attenuation coefficients or attenuation constant sometimes, although, again this constant is not really constant.

Because it depends on the particular frequency that you are looking at and similarly you have a phase coefficient okay. I choose to call them as coefficients because these are just coefficients or phase terms would probably have been better fit over here. The idea is that these are functions of frequency and we need to express that particular thing okay. There are some cases where it will not be a function of frequency, then we will be talking about those things as special cases okay.

The bottom line here is that we were able to obtain a second order differential equation, which is equation three over here for the propagation of the voltage phasor. We solved the equation and saw that there would be forward as well as backward waves. Of course, we have not mentioned how the backward wave could be generated, that is something that we are going to do very soon okay.

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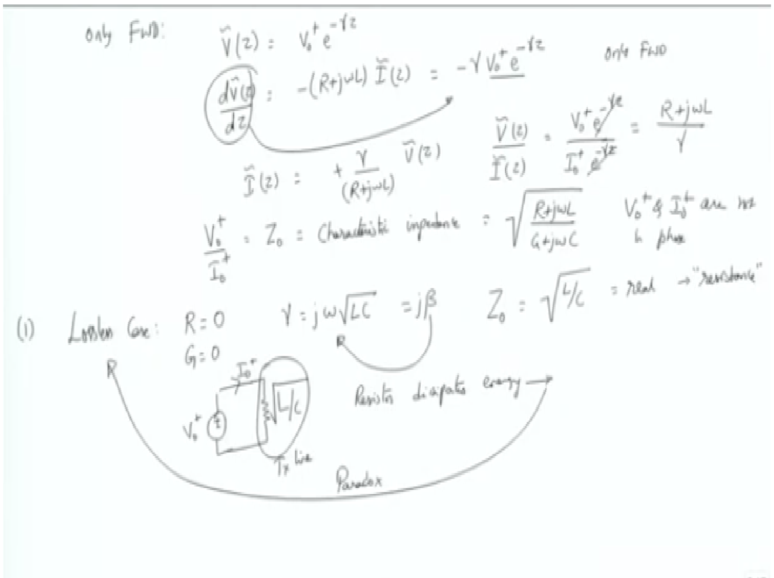

So let us for now assume that there is only forward going wave okay. So let us assume that only forward going wave is there which means that the voltage phasor is v zero plus e power minus gamma z and then differentiating this voltage phasor and equating it to the current term, what you get is minus R plus j omega L into I of z. So this equation, when you differentiate this v of z with respect to d z, will give you minus gamma v zero plus e power minus gamma z.

Clearly, what we have is I of z being given by minus gamma or rather, because on minus on both sides will cancel, you have gamma by R plus j omega L into the phasor v of z, because this is nothing, but voltage phasor. Now if you take the ratio of voltage phasor to current phasor and recognise that the voltage phasor has an amplitude v zero plus and the current phasor could similarly have an amplitude of I zero plus okay.

And the same e power minus gamma z kind of propagation, what you are really looking at is the amplitude ratios of forward going wave okay. This is very crucial, amplitude ratios of only forward going wave and that ratio can be seem to be R plus j omega L by gamma. But I already know what is gamma, gamma is nothing, but R plus j omega L into G plus j omega C.

Therefore, the ratio of v zero plus to I zero plus, which is the ratio of forward going voltage amplitude to the forward going current amplitude is given by this has to have units of impedance, because there is a voltage by current and this impedance is a characteristic of the line, because it depends on the parameters R, L, j and C, which further depend on the geometry of the transmission line right.

Because l for a coaxial cable is different and l for a two wire line is different. C for a coaxial cable is different, C for a two wire line is different, which is again different from a microstrip line okay. So this ratio is called as characteristic impedance okay. Although in this case this is impedance, for lossless line this becomes characteristic resistance okay. We will see that one very shortly.

And what is the z zero, this is R plus j omega L by gamma right. But substituting for gamma as square root of R plus j omega L into G plus j omega c will give you this expression okay. This expression is very important and in general it describes the fact that v zero plus and I zero plus are not in phase, which means that not the entire available voltage and current is contributing to the power, some power is actually getting lost.

Obviously, that power must be getting lost in R and G terms because R represents the imperfect conductors, which make up the transmission line conductors and G represents the imperfect dielectric that feels the conducting region okay. So this is your characteristic impedance. Immediately, one can identify different cases okay.

So let us consider the case, which we call as lossless case okay. In the lossless case, we assume that the conductors are perfect, which means that there is no drop or resistance in the conducting wires and R is equal to zero. Similarly, we will assume that the dielectric that feels the conducting surfaces is also completely ideal, which means that there is no leakage or conduction through that.

So G is also equal to zero. Substituting this, what you get for gamma. Gamma will be j omega square root L C means that there is only phase term with beta being omega into square root L C okay. What happens to z zero, the characteristic impedance, well characteristic impedance becomes square root of L by C and this is completely real, which means that this is exactly like a resistance okay.

So if this is resistance then there will be power dissipation and that actually brings us to a nice paradox right. We said that this a lossless case, but then we saw that the characteristic impedance turns out to be like a resistor right. So if for example, this is your v zero plus and there is a current I zero plus, the equivalent transmission line thing would actually look as a resistor square root L by C, but we know that a resistor dissipates is a passive device, which basically dissipates energy correct energy or power.

It dissipates energy, but if a line is dissipating energy then why do we call it as a lossless case. Is there a paradox somewhere over here, how can a lossless line be equivalent to a, this is the equivalent of a transmission line right. How can a lossless line be equivalent to a resistor. This paradox if you think about it carefully actually can be resolved in many ways, but two ways one has to understand.

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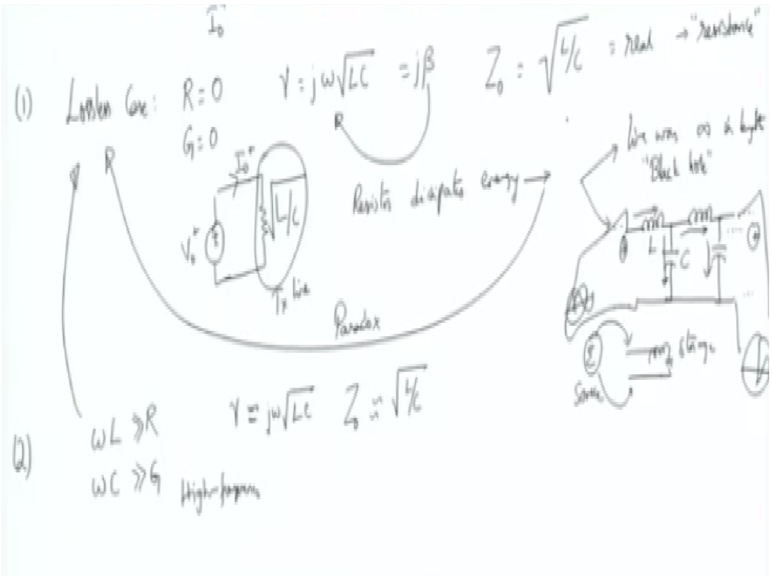

For now, we are considering only forward going wave, which means that there is no end to the line. The line was actually infinite right. So line was actually infinite in length or it was matched with z zero will come to matching thing later. For now, we will assume that this line is infinite in length and for an infinite length, if the voltage source connected at the input terminals of a transmission line generates a certain current.

And therefore puts out a certain amount of energy into the line, that energy keeps propagating propagating propagating. It would never come back to the source right. So it is as though the transmission line acts like a black hole right, sucking up all the energy and returning nothing. So, in terms of the source, it is clearly a lossy transmission line, because source has put out energy, but that energy is not coming back to us.

Well, technically the energy would be coming back to us because you know it goes all the way to infinity, only thing you have to wait for an infinite time, which is practically saying that, that amount of energy is simply lost on the transmission line okay and that is what is captured by having a real resistance as an equivalent resistance for the transmission line. Now if you are not convinced with this solution, try this.

Actually, a lossless line would, you know, if you go back to the sectioning of a transmission line that we did, would consist entirely of L and C right. So this is how an equivalent transmission line circuit would look for the case of a lossless transmission line and what you see here, you will see some inductance per meters, capacitance per meters, but this inductances and capacitances are precisely energy storing elements right.

So which means that whatever the energy that is given to the transmission lines okay, by the source okay, so if you connect a source and put out some energy, that energy actually goes in the form of storage okay. So the lossless line actually simply starts storing the energy rather than dissipating that energy. So again from the source point of view, this kind of energy is lost on the transmission line, but energy is not really lost in the sense.

Because energy is getting stored in the transmission line okay, now this equivalent circuit also must tell you why there should be delay between voltage at this point and voltage at this point okay. Why should there be a delay. If you hook up an oscilloscope over here okay and then see what you actually get. So for example, this is the voltage waveform and then you hook up an oscilloscope here, you would see that the voltage waveform would actually be delayed right.

This delay is now or this delay can be explained qualitatively because the equivalent circuit of a transmission line consists of L and C right, which means that there has to be some current for charging the inductor and some voltage across the capacitor for charging the capacitor and this current has to flow, charge the inductor, charge the capacitor. Then this has to flow charge the inductor, charge the capacitor and so on.

Because of this charging process which takes time, there will be some amount of delay in the voltage at two different points on the transmission line okay. Now that we have seen this lossless case, let us look for slightly more realistic case okay. The realistic case what we have is omega, the operating frequency is so large that the term omega L is much larger than R and similarly the term omega C is much larger than G.

In this case what happens, gamma will be approximately j omega square root L C and z zero will be approximately square root of L by C okay. So this case which we can call it as high frequency case okay is almost similar to that of a lossless case. In the lossless case, there was only propagation, no attenuation. The characteristic impedance was real and the same thing continues for high frequency line also.

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In practice, you do not really get such nice high frequency lines or you know, although you do get high frequency line so cannot complete ignore R and G, but there is another kind of condition, which is called as Heaviside's condition and this case is slightly interesting because in this case you actually adjust the transmission line parameters in such a way that L by R will be equal to C by G.

Do you recall, what L by R is, L by R is the time constant for an R L circuit right. So for an R L circuit the charging time is characterised by the time constant L by R and similarly for a capacitive circuit, the charging time is characterised by C by G and one by G is kind of R therefore this is R C time constant. So what we are seeing is, series time constant okay must be equal to parallel time constant okay.

Where this parallel in simply indicates that this is capacitor and a conductor okay. If these two time constants are actually equal, then this case is called as distortionless case and the line is called as distortionless line. What do you mean by this is that if you launch a particular way form on the transmission line okay at say z equal to zero, which corresponds to your source at z equal to l, which corresponds to your load okay.

The waveform would be delayed of course because there is a transmission line, so it's get delayed and then its amplitude might also d k okay indicating that the attenuation is not zero, but the waveform actually would be the same. So you can see some delay, you can see some amplitude laws, but you will see that the waveform actually has not changed its shape. So whatever the waveform that you have put out at the load the same waveform shape would be available at the load as well okay.

For this particular case, I will leave it as an exercise, you can show that gamma will consist of both alpha as well as beta and we are interested in what is alpha. So gamma is alpha plus j beta where alpha can be shown to be equal to R square root of C by L or G square root of L by C. Similarly, you can show that z zero for this case is approximately square root of L by C and this is again real okay.

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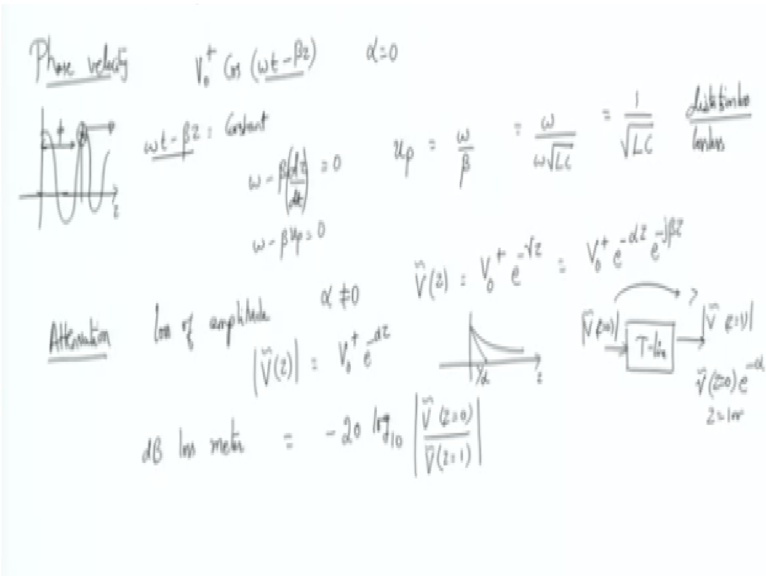

There is one last thing, which I want to discuss before we go to the next topic in at hand and this is called phase velocity. Remember our waves are going as V zero plus cos omega t minus beta z, assuming that this is on a lossless line alpha is equal to zero. So this is how the voltage actually propagates on the transmission line.

Now if you pick a particular point, which is characterised by having a constant argument that is omega t minus beta z is equal to some constant right. So you have your waves, which are going as a cos omega t right. So you have this wave and if you pick a particular point okay, which is characterised by, so this has to be both z and t, I am just showing you as a function of z.

So if you pick a particular point at which this argument omega t minus beta z is constant and then look at how this particular constant point actually moves in time then the velocity with which it moves will give you the phase velocity, because this would be the phase with respect to some reference phase and what you are looking at is how this phase itself is changing okay as omega as t and z increases.

To obtain the expression for phase velocity, you can simply differentiate this one with respect to time and you get omega minus beta d z by d t, beta is assumed to be independent of time. So this would be equal to zero and this d z by d t is the velocity of this constant point. So this velocity is denoted by u p and u p is given by omega by beta because this is nothing, but omega minus beta u p equal to zero.

So you can push this on the right hand side and then interchange the left and right hand side to see that u p is equal to omega by beta. For a lossless case, we have already seen that beta is nothing but omega square root L C, so which means that u p, the phase velocity is given by one by square root L C okay and this is again the characteristic of lossless line, low less line or the high frequency line as well as for the distortionless line okay.

Why for the distortionless line, because in the distortionless line what you have is beta being some constant omega times something okay. So it would actually be again real and this is the reason why you actually have no distortions. If you phase velocity would start to depend on omega, then there would be distortion something that we will come back to this scenario later okay, when we discuss group velocity okay.

We talked about attenuation right, so we said that attenuation is nothing but loss of amplitude okay, loss of voltage amplitude or current amplitude and then if attenuation which is defined by alpha or which is denoted by alpha is non zero then the voltage phasor v or z goes as assuming again only forward going wave, you have v zero plus e power minus gamma z is a general voltage phasor on the transmission line. This becomes v zero plus e power minus alpha z e power minus j beta z.

Suppose you are interested in finding out the magnitude of v of z, this magnitude will be equal to v zero plus v to the power minus alpha z and you can see that this magnitude actually decays with a decay constant of alpha right or one by alpha. The slope is one by alpha and as the wave propagates along the transmission line its amplitude starts to decay okay. We sometimes are interested in finding out what is the loss per unit length.

Per unit length would be denoted by per meter and the loss is typically talked about in terms of dB or units of decibel okay. So what is this decibel loss per meter, decibel loss per meter is simply minus 20 log to the base 10, what is the input voltage that we gave, v of z equal to zero, divided by v of z equal to one meter okay. So this is what we mean when we say that this is dB loss per meter.

Clearly, if the transmission line is lossy, the voltage that we give at v at z equal to zero will be greater than the voltage at z equal to one. Because this numerator is greater than denominator, this would represent the loss. Why should it be greater because v of z equal to one magnitude will actually be equal to v of z equal to zero times e power minus alpha because you are propagated z equal to one meter okay.

So therefore you have the voltage at a later point in the transmission line to be less than what is the input voltage and that ratio is characterised by dB loss per meter okay.

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$$\alpha_{dB/m} = -20 \log_{10} e^{-\alpha} = \alpha\, 20 \underbrace{\log_{10} e}_{0.434} = 8.686\, \alpha_{Np/m}$$

$$\boxed{\alpha_{dB/m} = 8.686\, \alpha_{Np/m}} \qquad \boxed{\alpha_{Np/m} = 0.115\, \alpha_{dB/m}}$$

Now you substitute for v of z equal to one and evaluate this expression what you get is minus 20 log of 10 v of z equal to zero is simply v of z equal to zero okay it could be v zero plus and then what you get is e to the power minus alpha. Now you can rearrange this one or you can you know take the log to this thing, what you get is alpha into 20 log to the base 10 times e, but this log 10 to base e is nothing but point 434, you can check the log tables.

And what you see is that alpha, the dB loss in meters is actually given by 8 point 686 times alpha okay. If I call this one as alpha dB per meter okay and then this alpha which is coming from the actual expression of e power minus alpha will be denoted by neper per meter and this is the relationship between dB per meter, loss of the signals in dB per meter to loss of signals in the natural units neper per meter okay.

So it is worthwhile to represent these equations, so I mean highlight the equation, so I am going to write this again, alpha in dB per meter is 8 point 686 alpha in neper per meter and you can actually interchange this equation also and if you know what is alpha in neper per meter, you can write down what is this, this would be around point 11 okay times because it would be 1 by 8 point 686, so it is approximately point 115 something times alpha in decibel per meter okay. So given one alpha you can convert that into the second alpha.

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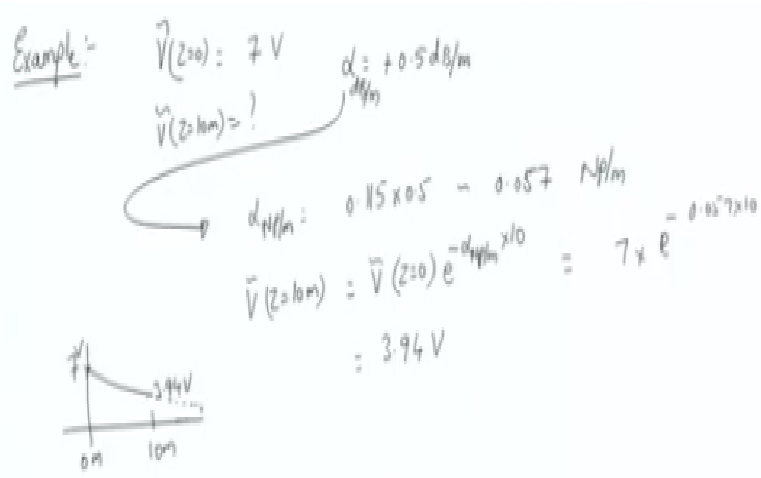

Let us try out an example okay to just show you what attenuation things can do, let us try out an example let us assume that voltage at z equal to zero at the input terminals is 7 volts, what we want is voltage at z equal to 10 meter okay. Given that alpha is point 5 dB per meter okay. So the loss is actually point 5 dB per meter, you can substitute for this point 5 dB per meter loss and then you can solve it in two ways, one you can solve it alpha in dB per meter itself but in this case, it probably is easier to first convert this into neper per meter okay.

So you can consider this dB per meter, convert this into neper per meter, when you convert that, that would be point 115 into point 5, which is approximately point 057 okay neper per meter and then voltage at z equal to 10 meters will be the voltage at the source side times e power minus alpha neper per meter into 10 because z is equal to 10 right. So you substitute this, this is 7 into e to the power minus point 057 into 10 okay.

And if you use your calculator, you will see that this is nothing but 3 point 94 volts okay. So you can see that the voltage at z equal to 10 meter okay as actually is dropped from a value of 7 to 3 point 94 volts okay. So from 7 to 3 point 94 volt it would continue to drop if the transmission line is actually extended beyond 10 meters.