Electromagnetic Theory

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Lecture - 56
Oblique incidence of waves (contd)

This is, what Snell's law is. But we still have an equation to solve.

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$$\frac{E_{r0}}{E_{i0}}, \frac{E_{t0}}{E_{i0}} \qquad ②: \; H_{i0} - H_{r0} = H_{t0}$$

$$\frac{E_{i0}}{\eta_1} - \frac{E_{r0}}{\eta_1} = \frac{E_{t0}}{\eta_2}$$

$$① \quad E_{i0}\cos\theta_i + E_{r0}\cos\theta_i = E_{t0}\cos\theta_t$$

We still have this equation 2 and equation 1 to solve because remember the goal for us is to actually find out the ratios of reflected amplitude of the electric field to the incident amplitude of the electric field and transmitted amplitude of the electric field to the incident amplitude of the electric field. So, we still need to find out these two ratios, which give you correspondingly reflection coefficient and transmission coefficient.

In order to get that one, we now employ this fact that the phase relations have to be same. Therefore, boundary condition 2, in terms of the magnetic H component becomes H i 0 - H r 0 equals H t 0. Now, the magnetic fields can be related in a very straight forward manner to the electric fields because H is equal to E / Eta. However, you have to be careful to use the appropriate Eta, Eta or the wave impedance.

Wave impedance for i and r will be Eta 1. And the reflected amplitude is E r 0, which is again Eta 1. This must be equal to the transmitted amplitude E t 0 divided by Eta 2 because transmitted is in the second medium. So, the boundary condition for magnetic field has reduced to a simpler equation and what will happen to the boundary condition for the electric field? You have E i 0 Cos Theta i + E r 0, Cos Theta r but Theta r is nothing but Theta i.

So, we already know that one. This must be equal to E t 0 Cos Theta t. So, now I have these two equations. I can kind of push this Cos Theta onto the other side. Because there is a simplification that I can actually make.

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$$\frac{E_{r0}}{E_{i0}}, \quad \frac{E_{t0}}{E_{i0}}$$

(2): $H_{i0} - H_{r0} = H_{t0}$

$$E_{i0} - E_{r0} = \frac{\eta_1}{\eta_2} E_{t0}$$

(1) $E_{i0} + E_{r0} = E_{t0} \dfrac{\cos\theta_t}{\cos\theta_i}$

$$\frac{E_{r0}}{E_{i0}} = \Gamma_{TM} \ (\Gamma_{\parallel}) = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \qquad \frac{E_{t0}}{E_{i0}} = T_{TM} \ (T_{\parallel}) = \frac{2\eta_2 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

So, you can push this Cos Theta onto this side and thereby reduce this equation to E i 0 + E r 0 equals E t 0 Cos Theta t / Cos Theta i. Now, you have two equations. So, here again you can push Eta 1 onto the other side. So, in this equation you can push Eta 1 onto the other side, so you get Eta 1 / Eta 2 E t 0. So, now I have two equations and therefore I can solve these two equations to obtain the ratio of E t 0 to E i 0 as well as the ratio of E r 0 to E i 0.

So, I can do that one. I am not going to do that calculation; I leave it as a small exercise for you in Algebra because it is not very difficult. You just have to add and subtract these two equations appropriately. When you do this small algebra what you will see is that the ratio of the reflected amplitude E r 0 to the incident electric field amplitude E i 0, which we will call as Gamma T M. Or equivalently Gamma parallel.

I will tell you in a moment what this Gamma T M and Gamma parallel mean. These are reflection coefficients and this is equal to Eta 2 Cos Theta t - Eta 1 Cos Theta i / Eta 2 Cos Theta t + Eta 1 Cos Theta i. Similarly, you have the transmitted field E t 0 to E i 0 ratio being written as some T T M. T standing for transmission coefficient or T parallel. I will tell you in a moment what this parallel and T M designations mean.

This is equal to 2 Eta 2 Cos Theta t / Eta 2 Cos Theta t + Eta 1 Cos Theta i. So, you can actually do this thing and you will be able to find out what these values are? So you have these two expressions.

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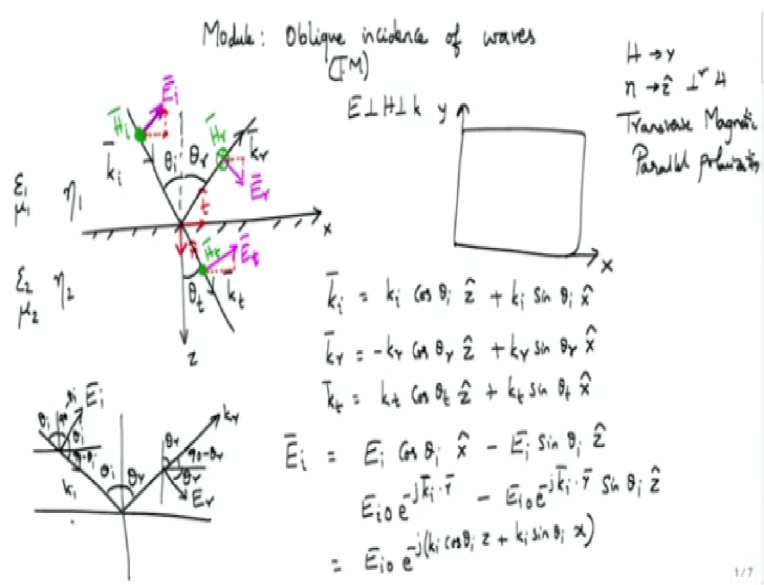

Now, why have I call this as T M and parallel. Let's go back to what the original form of the electric fields were? So, in the oblique incidence that we consider it was true that electric field E was perpendicular to H. And H was perpendicular to k. But a different thing has also happened. H is in the direction of Y. It could be in the direction of + Y or -Y. But H is along Y. And the interface is along Z direction.

And what we have just seen is that, I mean if you look at this diagram very closely. H is along Y and the interface n which is along Z, is perpendicular to H. Does there is no component of the H field in the X and Z planes? In the X and Z plane there is no component of H field and therefore this is the case where we have or this is the case of what we call as Transverse magnetic. In contrast, there is an electric field component in the XZ plane.

So, electric field component although it is perpendicular to the k vector, it is not perpendicular to Z. There is a component of E along Z direction and Z direction is the normal to the interface. And therefore, we say that the electric field is in the plane of incidence. The plane of incidence being formed by the k vector and the normal to the interface. The k vector and normal to the interface in that plane, you actually have an electric field sitting there.

Therefore, this is called as parallel polarization. Wherein electric field is parallel to the plane of incidence. Electric field is parallel to the plane of incidence and magnetic field is completely perpendicular to the plane of incidence. Therefore, this is called as T M case. So far what we have been discussing is the transverse magnetic case T M. And because of that reason we have this designation of reflection coefficient being Gamma T M or transmission Coefficient being T T M. So please keep these expressions in mind.

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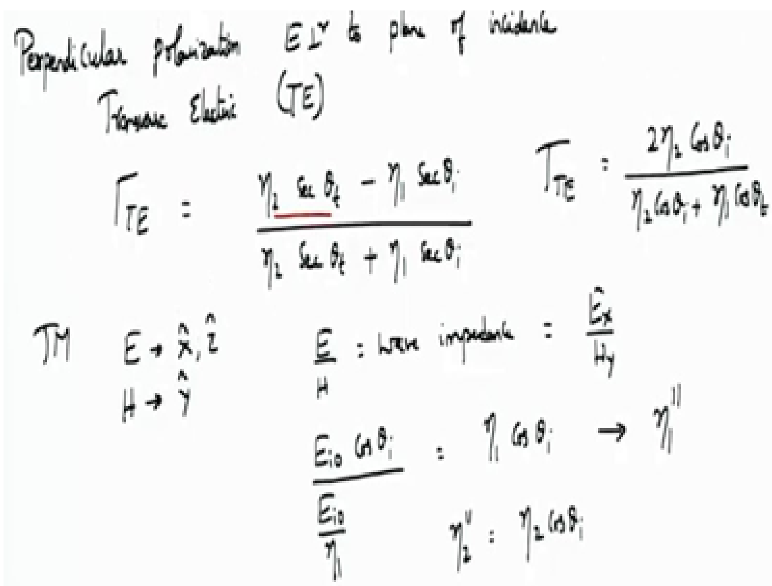

Similarly, there is another situation which is called as perpendicular polarization. As you can clearly imagine perpendicular polarization would be the case, where electric field will be perpendicular to the plane of incidence. So, electric field is perpendicular to the plane of incidence and therefore this is called as perpendicular polarization and because electric field is perpendicular to the plane of incidence that will be transverse to the plane of incidence.

And therefore this is also called as transverse electric polarization or simply T E polarization. So you have T M polarization and you have T E polarization. You can actually do the same kind of expressions. You can derive the expressions; you can apply boundary conditions, write down the expressions, apply boundary conditions and then do some simple algebra to obtain the reflection coefficients for the T E case as well.

And you can see that the reflection coefficient as you can derive it will be equal to Eta 2 Sec Theta t - Eta 1 Sec Theta i / Eta 2 Sec Theta t + Eta 1 Sec Theta i. The transmission Coefficient for the T E case is also given by 2 Eta 2 Cos Theta i / Eta 2 Cos Theta i + Eta 1 Cos Theta t. So, you have these expressions for reflection coefficient of T E case. So, you can observe that for T M case, it is Eta 2 Cos Theta t, whereas, for T E case it is Eta 2 Sec Theta t.

The reason for these two is that for the case of T M the electric field components have both X as well as Z component, whereas the magnetic field has only the Y component. And the ratio of E to H must be the wave impedance, this must be the wave impedance. But wave impedance is actually defined only for the transverse components, because the direction of propagation we have taken to be Z axis.

The component of E along Z needs to be not considered and you need to define the wave impedance in terms of its transverse components, which is E X to H Y. And if you do this one for the incidence waves what you will see is that E X component if you go back. So if you go back to the E X component you can see that this would be E I 0 Cos Theta i. And then the H component will simply be E i 0 / Eta 1.

So for the T M case, what you have is, E i 0 Cos Theta i being the actual component of the electric field along the X direction divided by the magnetic field strength E i 0 / Eta 1. Therefore, for incident wave impedance or wave impedance for the incident as well as for the reflected waves is given by Eta 1 Cos Theta i. Let us call this as Eta 1 parallel. That double line parallel simply indicates that we are considering parallel polarization or equivalently transverse magnetic polarization.

What would be Eta 2 parallel? That is the wave impedance for second medium that would be again the ratio of E X to H Y. And this will be given by Eta 2 Cos Theta i.

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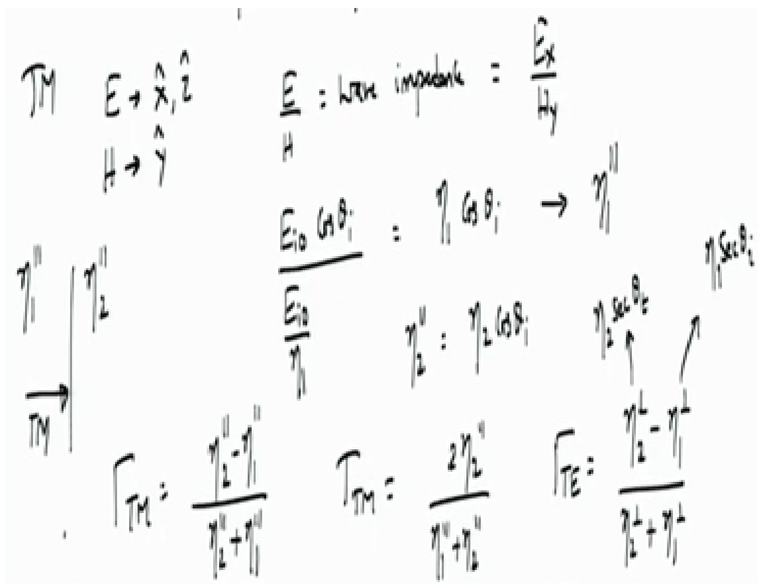

Now in terms of this Eta 1 parallel and Eta 2 parallel you can actually write down this expression for the reflection coefficient in a much more simpler way Gamma T M will be equal to Eta 2 parallel - Eta 1 parallel / Eta 2 parallel + Eta 1 parallel. So, this is equivalent of having an effective wave impedance of Eta 1 parallel, effective wave impedance of Eta 2 parallel and having your wave incident normally.

Remember that the wave that is actually transverse magnetic polarized. Its magnetic field is perpendicular to the plane of incidence. But for this effective impedances that you are considering, you can think of a normal incidence of the transverse magnetic field with the wave impedances of Eta 1 parallel and Eta 2 parallel. Similarly, you can write down for T T M as 2 Eta 2 parallel / Eta 1 parallel + Eta 2 parallel.

This expression again looks very similar to the normal incidence. You can also write down the expression for Gamma T E, which happens to be Eta 2 perpendicular - Eta 1 perpendicular divided by Eta 2 perpendicular + Eta 1 perpendicular. Where Eta 1 perpendicular is actually Eta 1 Sec Theta i and Eta 2 perpendicular is Eta 2 Sec Theta t. These wave impedances are actually functions of the angle of incidence and angle of reflection.

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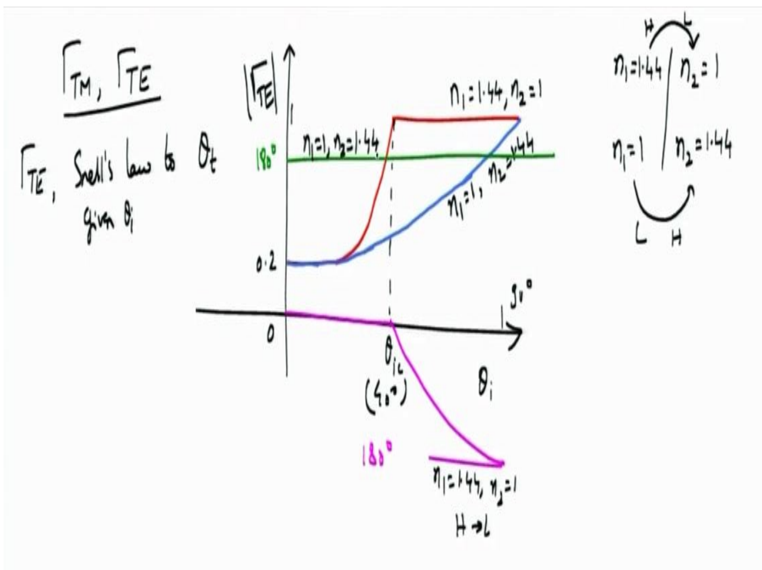

Let's try to sketch this Gamma T M and Gamma T E, to see whether we can actually discover something interesting. So let's try to sketch this Gamma T E first. For two different cases, I am going to sketch this one. You can actually write a short MATLAB script to do this thing. So you already know what is Gamma T E and then you can use Snell's law to calculate Theta t for given values of Theta i.

So for example if you can consider the X axis to be Theta i, angle of incidence. So in the incidence angle and then vary it over say from 0 to 90 degrees and on the Y-axis you plot the magnitude of Gamma T E. For let's consider the case, where the first medium has a refractive index n 1 of 1.44. And the second medium has a refractive index of 1, which is air. And the other case where n 1 is air and n 2 is 1.44.

This 1.44 is typical refractive index of a silica glass and then in 1 case you are going from a higher refractive index to a lower refractive index. And in this case, you are going from a lower refractive index to higher refractive index. So, if you plot them you will see a result that would look like this. So, initially for few angles of incidence you know the value of Gamma T E magnitude for both cases would remain the same.

However, after a certain angle of incidence, there is, you know, the magnitude of reflection Coefficient simply goes off to 1. So here let's say it starts at around 0.2. You can actually substitute this expression for Gamma T E and verify that these angles are correct? Or you can plug in to a small MATLAB script in order to do this thing. But, at a certain critical angle something has happened, the reflection coefficient magnitude has gone up to unity.

Now, if reflection coefficient magnitude is unity. What does it mean? Will there be any reflected power? Well, the entire power is reflected, nothing is actually getting transmitted. So, if the reflection coefficient hits one, then nothing is getting transmitted. And everything is getting reflected. Now, what was this case for whether it was the case for 1.44 and 1? You are right. This is where when n 1 is equal to 1.44 and n 2 is equal to 1.

Now, what would be the graph for Gamma T E for the other case that is when you go from lower to higher refractive index? Again at 0, they would start to be the same and they would actually continue to be same until a few degrees of angle of incidence. And there after there is no change, the magnitude of reflection coefficient gradually goes and becomes equal to 1. This kind of behavior is expected. This is what you would expect as angle of incidence is slowly increased.

You can go back to this expression for Gamma T M and you keep changing, Theta i. Calculate the new value of Theta t and therefore, obtain what is Gamma T M, find the magnitude and start plotting that one. So, when you do that thing you will see that gradually as you go the magnitude of reflection coefficient will hit 1 at 90 degrees. So, this particular thing, where this kind of a critical thing is happening this is around 40 degrees.

For this particular set of numbers, this is around 40 Degrees. Do not go by the very accurate number, these are approximate numbers are enough for us. But the characteristic thing that you have to notice is that, just after the critical angle, magnitude of Gamma T E has hit 1 and continues to be 1. Whereas for the other case, when you go from n 1 equals 1 and n 2 equals 1.44, this change over to magnitude of reflection coefficient, changeover towards 1 happens very gradually.

So, if you plot the phase of this, you would see that for 1 case the phase remains constant at 180 degrees. So, if you call this as 180 degrees and for the other case, you will see that the phase actually changes over. So it initially is almost 0 and once you hit the critical angle, the phase gradually goes towards 180 degrees. So, what is this case for? This is the case, where n 1 is equal to 1.44, n 2 is equal to 1. You are going from high to low refractive index.

And this 180 degrees whether it is + 180 degrees or - 180 degrees, it does not really matter. But that is remaining constant. This is the case where n 1 is equal to 1 and n 2 is equal to 1.44.

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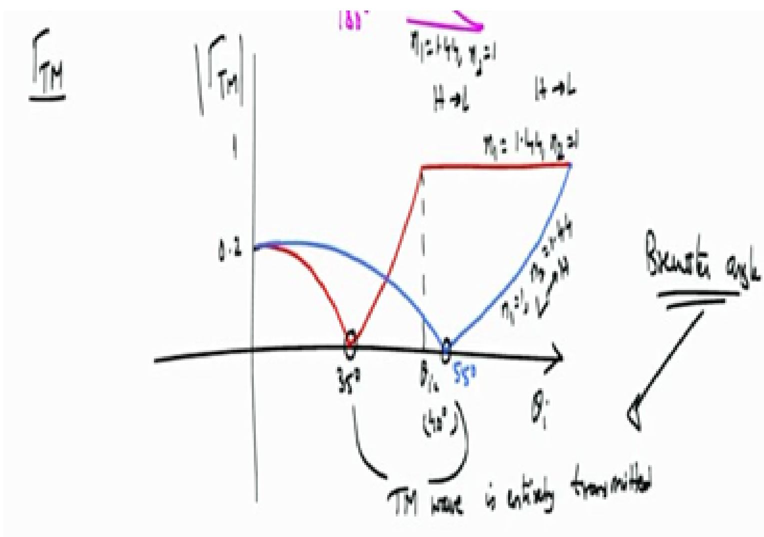

We will come back to what this Critical Angle is, in a few moments. But before that let us try to plot Gamma T M. For the same set of values if you are to plot Gamma T M, you will see something very interesting happening. So, I am plotting Gamma T M magnitude. And initially the magnitude of the reflection coefficient would start of somewhere. This would be 0.2 because it has to be the same for both.

And for some reason, I mean for this one the magnitude of reflection coefficient keeps dropping, drops and then rises and then becomes constant. It reaches the magnitude of unity at a certain critical angle. This critical angle is again around 40 degrees. Whereas, there is a case where the magnitude of reflection coefficient is going to 0. This is around 35 degrees and this is the case, where you had n 1 is equal to 1.44 and n 2 is equal to 1.

This was the case, where you went from high to low refractive index medium. What would happen in the other case? Well in the other case, again you start with 0.2 and thereafter, it actually goes down. And nothing whatsoever happens and slowly asymptotically you will reach the magnitude of reflection to be equal to 1. And the angle at which this would go to 0 is around 55 degrees and this is the case, where you are going from low to high.

That is n 1 is equal to 1 and n 2 is equal to 1.44. You are going from low to high value of refractive index. So you can see that there are two things that are happening for T M in addition to what was happening for the magnitude of reflection coefficient hitting one beyond a certain Critical Angle. The other peculiarity is that the magnitude of reflection coefficient is actually going to 0.

Which means that at these two angles the wave is entirely transmitted. That is T M wave is entirely transmitted nothing is reflected. This angle at which this happens is called as Brewster angle. So, Brewster angle is the angle at which the entire T M wave is transmitted and nothing of T M polarized wave actually gets reflected. It is a simple matter to actually derive the expression for Brewster angle.

Let me go through the Derivation for you. If I leave out some steps, I am pretty sure you can fill it up. So, this Brewster angle is normally defined only for T M polarization because there is a pathological case, where T E polarization also exhibits a similar behavior. However, for that to happen the magnitude medium must be different. The values of Mu must be different. And there is something that people do not normally use.

Because these are used at very very high frequencies, light frequencies, whereas at those frequencies magnet materials are extremely, extremely lousy. So you do not normally encounter a Brewster angle for T E polarization. You encounter a Brewster angle for T M polarization. Now, when will the magnitude of reflection coefficient go to 0, when this expression for Gamma T M goes to 0, which means that Eta 2 Cos Theta t must be equal to Eta 1 Cos Theta i.

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$$\eta_1 \cos\theta_i = \eta_2 \cos\theta_t$$

$$\cos\theta_i = \frac{\eta_2}{\eta_1}\cos\theta_t = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}\cos\theta_t = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\cos\theta_t \rightarrow \text{Square } \cos\theta_t$$

$$\sqrt{\varepsilon_1}\sin\theta_i = \sqrt{\varepsilon_2}\sin\theta_t \Rightarrow \sin\theta_t = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i$$

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i}$$

$$\cos\theta_i = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sqrt{1-\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i}$$

$$\cos^2\theta_i = \frac{\varepsilon_1}{\varepsilon_2}\left(1-\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i\right)$$

So you have Eta 1 Cos Theta i equals Eta 2 Cos Theta t. But, I already know what is the relationship between Cos Theta i and Cos Theta t. Because Cos Theta i is Eta 2 / Eta 1 Cos Theta t, which can also be written as square root of Mu 1 Epsilon 1 / Mu 2 Epsilon 2 Cos Theta t. But, we have already said that Mu 1 is equal to Mu 2 typically. Therefore, this is nothing but square root of Epsilon 1 / Epsilon 2 Cos Theta t.

But I also know from Snell's law that Square root of Epsilon 1 Sin Theta i must be equal to square root of Epsilon 2 Sin Theta t. This implies that Sin of transmitted angle Theta t is equal to square root of Epsilon 1 / Epsilon 2 Sin Theta i. But what you want is a Cos Theta t. But I already know the relationship between Cos and Sin. So, Cos Theta t is equal to 1 - Sin Square Theta t under root.

So, this is Cos Theta t. So, you can write down this one, this is square root of 1 -, now Sin Theta t is this fellow, so you can write this as Epsilon 1 / Epsilon 2 Sin Theta i. And in this equation, you can actually Square Cos Theta t, so when you square that one what will you get? You will get Cos Theta, so let me write down Cos Theta i is equal to square root of Epsilon 1 / Epsilon 2. So, this is Cos Theta i.

So, Cos Theta i is Square root of Epsilon 1 / Epsilon 2 times Cos Theta t. Cos Theta t is Square root of 1 - Epsilon 1 / Epsilon 2 Sin Theta i. So, this implies that Cos Square Theta i must be equal to Epsilon 1 / Epsilon 2 into 1 - Epsilon 1/ Epsilon 2 Sin Theta i. Is it Sin Theta i or Sin Square Theta i? This is Sin Square Theta I, we forgot a Square up here. So this is the expression for Cos Square Theta i.

Now you can write down Cos Square Theta i as Sin Square Theta i bring that terms together and then you can simplify the expression and solve for Theta i.

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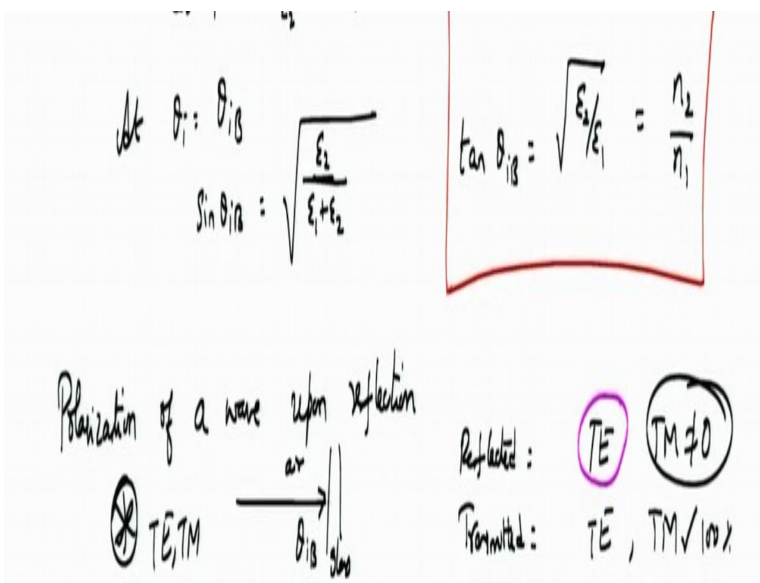

So, at Theta i is equal to Theta i B, which is the Brewster angle, you get Sin Theta i B is equal to square root of Epsilon 2 / Epsilon 1 + Epsilon 2. So if this is Sin Theta which is opposite side by hypotenuse for a given angle Theta i B, you can also write down this expression in terms of Tan

Theta and Tan Theta i B will be equal to Square root of Epsilon 2 / Epsilon 1 or equivalently this is n 2 / n 1. This expression is very simple. You can used to calculate what the Brewster angle is?

And Brewster angle actually has lot of applications. Two applications I am going to discuss. The first application is polarization of a wave or light upon reflection. Suppose you have a wave which is unpolarized that is it has both T E and T M components. Now, if your incident this one at an angle of Theta i B, what would happen to the reflected and the transmitted waves? For the reflected waves, there will be T E waves but there won't be any T M waves.

There is no T M wave because for the case of incidence angle with Brewster angle the entire T M wave gets transmitted. Therefore, if you look at what is transmitted wave. Transmitted wave will be some portion T E as well as complete portion of T M. So, 100 percent T M is transmitted, whereas now if you look at reflected wave, reflected wave is only transverse electric. Therefore, what we have actually achieved is that you took an unpolarized light beam.

And then made it fall at, on to a particular, for example if this is air then this is a glass window. This window is typically called as a Brewster window. So on to the Brewster window if you drop an unpolarized light, the reflected light will be entirely T E polarized, single polarization whereas transmitted fields could be both T E and T M polarization.

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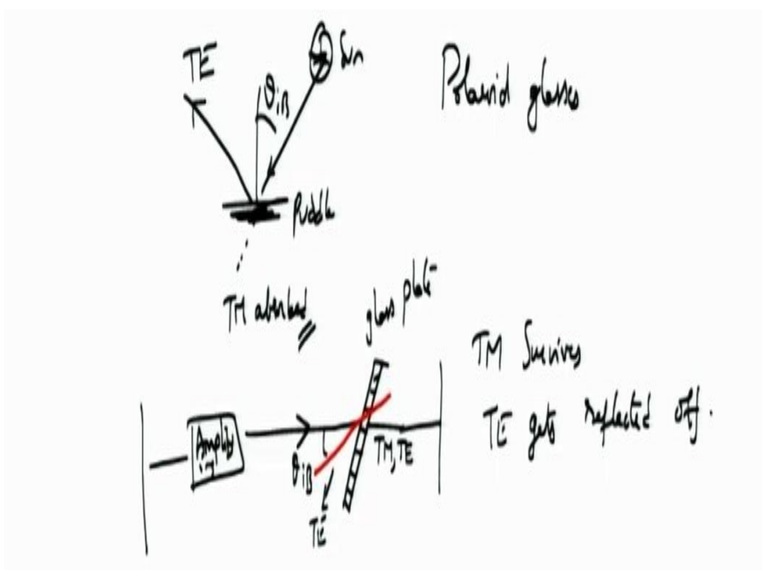

This is the idea which people use when you actually have to make a Polaroid glass. So you have a pedal or something. And then you have light from Sun which is falling. If this falls at an angle of Theta i B or close to angle of Theta i B. The reflected light will be completely T E. Because T M gets absorbed by this pedal. So this is my sun, this is pedal. So, in this pedal T M wave is completely absorbed and the nothing of that T M is actually reflected.

So, only T E wave gets reflected. This is the idea behind Polaroid glasses. So, light coming from Sun, it is made to fall at a Brewster angle, the reflected light will be T E polarized and the transmitted field will be T M polarized light. T M + some amount of T E polarization. So, these are used to anti-glare cases. A different application of Brewster angle, actually based on the same idea is that in many laser systems, you actually have to select only a particular polarization.

So, in order to select a particular polarization what we do is, we put a glass Plate up here. So, this glass plate is position such a way that light that is incident onto the glass plate and that is normal to the glass plate will be at an angle of Theta i B, the Brewster angle. So, when this is at a Brewster angle then what happens? The Light that is getting reflected will be entirely T E. The light that is passing through will be T M + some amount of T E.

So, there is some reflection and some amount of T E that is passing through and over the round trip propagation many many times, when it comes back and forth, back and forth it's only T M, which survives and T E wave gets reflected off, thereby giving you a single polarized or a single polarization of the laser light. These are the two applications that you would normally come across.

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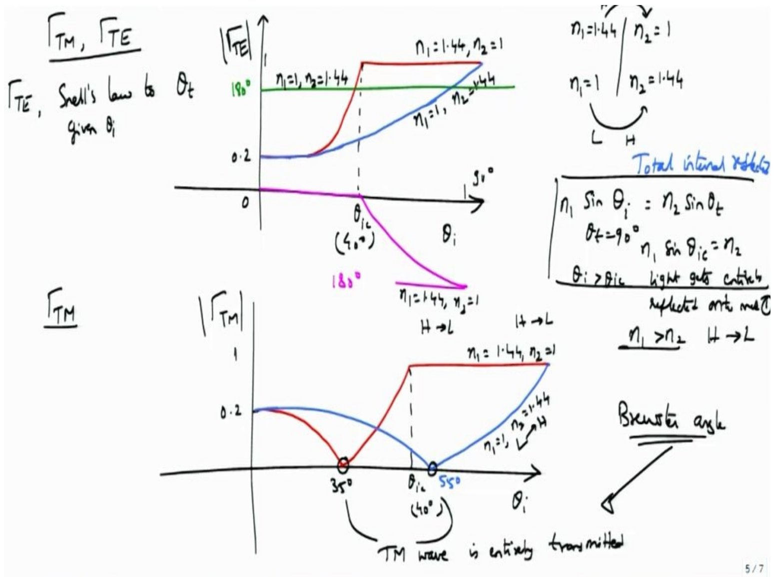

Now to briefly mention what is really happening with this Critical Angle? What actually is happening is critical angle is this, we have the expression for refractive index and the angle of incidence and reflection. So, you observe here that when you have an angle Theta I, such a way that the reflected wave will have an angle Theta t is equal to 90 degrees. Then what would happen, so this at a critical angle let's say, what would happen is that the angle of transmission has become 90 degrees.

And if you move angle of incidence greater than this angle of critical angle then what happens is that light or wave gets entirely reflected back into the first medium. For this to happen it is necessary that n 1 be greater than n 2. That is you need to be moving from high to low refractive index. And only when you move from high to low refractive index, you will see that as the angle gets greater than critical Angle, light gets entirely reflected on to medium 1.

And this phenomenon of light getting reflected into medium one is called as total internal reflection. Now, total internal reflection does not always mean that entire light gets reflected that is actually some amount of light into the other region as well. But that light does not actually carry any power because the fields there will be 90 degrees out of phase with respect to each other.

Because of that reason there won't be any power carried by them. Those waves are known as evanescent waves. And these evanescent waves are important, when you have to couple light from one waveguide to another waveguide in an optical communication system. So, we will be talking about this total internal reflection when we talk about optical waveguide so at this point, let us actually not talk about that one.