Electromagnetic Theory

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Lecture - 55
Oblique Incidence of Waves

In this module, we will discuss oblique incidence of waves. We have already discussed normal incidence of waves and we found that when a wave is normally incident from one dielectric to another dielectric medium, some part of the wave gets reflected, some part of the wave gets transmitted. When you have a wave incident on a conductor, of course the entire wave gets reflected.

We will be looking at oblique incidence of waves on a conductor some time later, when we discuss wave guides. For now, we will assume that we have two dielectrics that are separated. These two are perfect dielectrics and they are described by their equivalent wave impedances and wave is going from one dielectric to another dielectric. And then we will see, what could be the reflection coefficient and transmission coefficient, which will tell us how the, what is the amount of wave is reflected.

And what is the amount of wave that is transmitted. Now, compared to normal incidence, the formulas that we will develop will be slightly complicated because we now have to consider the angle at which the wave is coming to the second medium. So, when the wave is incident normally, there was no problem, the formulas were quite simple.

However, when you have wave's incident at an angle, then one has to consider explicitly that angle of incidence of the wave, in order to calculate the reflection coefficient. So in that way, this is slightly more tricky. But along way, we will also discover two laws, which is something that you might have studied in your tenth or eleventh standard. These laws are called as Snell's laws. There are two laws, one law is the law of reflection and the other one is law of refraction.

Law of reflection from Snell's law states that the angle of incidence must be equal to angle of reflection, and angle of refraction tells you that the angle of refraction is different. But then the sign of the angle of incidence to the sign of the angle of refraction are related to the refractive indexes of the other, of the two mediums, which are participating in this wave deflection phenomenon.

So something like n 1 sin theta 1 is equal to n 2 sin theta 2 is what probably you might have seen, where 'n' 1 and 'n' 2 are refractive indexes, not the wave impedances. So, we will see that these two laws are not really laws per se, but they are simply consequences of boundary conditions. So, when you apply an appropriate boundary condition, then that particular law will automatically come out.

So, we begin with oblique incidence, we will consider, we will see that there are two cases to consider, however we will consider only one case in our study in this module. The other case, I will leave it as an exercise to you. The analysis is quite similar for the other case also. So you might actually take it as an exercise and then complete all the steps that are there to arrive at the reflection coefficient.

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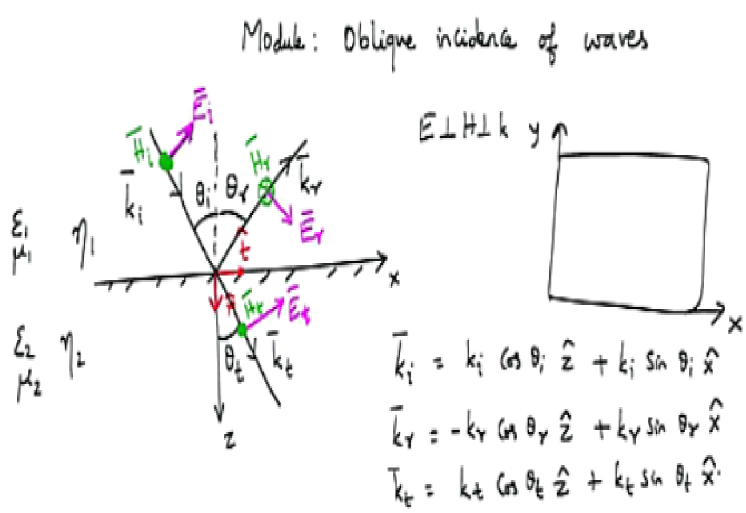

So, we have a surface. This surface is characterized by 'Z' equal to zero. That is this particular surface is at z is equal to zero. And then the wave is propagating along 'Z' direction. So, this is my surface, this is my X axis, the other axis for this one is the Y axis and this surface separates two dielectric medium. So, this is the direction along wave is propagating ‘Z’ and let us call this as ‘X’.

So clearly ‘X’ cross ‘Y’ should be along ‘Z’. So the appropriate direction for ‘Y’, you have to assume. The ‘X’ and ‘Y’ planes, if you look from the top view, this would what you would find. So this would be say ‘X’ and this would be ‘Y’ and then the ‘Z’ axis will actually be going into the page. So this is what the surface would look. So above the surface is dielectric medium having wave incidence eta 1 or equivalently epsilon 1 mu 1.

And below the surface is eta 2 having epsilon 2 and mu 2 as their respective consequent parameters. In this module, we will assume that eta 1 and eta 2 are real. But there is no real requirement of assuming that one. The formulas will not depend on whether there is some amount of sigma 1 here and some amount of sigma 2. When sigma 1 and sigma 2 are present, then, eta 1 and eta 2 become complex.

The equations are all valid. But unless it will slightly complicate our situation and therefore we are not going to consider that sigma one1 and sigma 2. If you, if the problem explicitly ask for it, we can actually take into account. So this is all the geometry that we have described. One can also describe a normal, which is directed from one surface to the other surface.

So let us say that if the normal is directed from so along this red arrow, if the normal is directed from first surface to the other surface, it could be pointing to the ‘Z’ axis. This is the surface normal. And the directions ‘X’ and ‘Y’ will be tangential. So if this is the normal direction and then the vectors along ‘X’ and ‘Y’ can be called as tangential vectors. So any vector on the ‘X’ ‘Y’ plane can be tangential vector.

Now, we have a wave, which is incident. We are assuming that this particular surface ‘Z’ is equal to zero and this is the origin that we are considering. And the wave gets incident at a certain angle. Now how do we measure this angle? There are two ways of measuring the angle. One with respect to the tangential that is with respect to the surface or the other one would be with respect to the normal to the surface.

So, this normal to the surface measurement is quite common, especially in optics. So, we will follow that notation and call this angle measured with respect to the surface normal as the angles. So, this ‘theta i’ indicates that this is angle of incidence. So which means that this line which I have drawn with an arrow is actually the incident wave vector, which is ‘k i’. And as the wave hits this medium.

There will be a reflected wave that is generated, which will be at an angle of ‘theta r’. And there will be a reflected wave ‘k r’. So this ‘k i’ and ‘k r’ are the incidental reflected waves. There will also be some amount of waves, which is transmitted into the second medium, which is described by the transmitted wave vector ‘k t’, making an angle of ‘theta t’. The subscript ‘i’ stands for incident, ‘r’ stands for reflected, ‘t’ stands for transmitted. We are going to assume plain waves.

So I know that ‘E’ must be perpendicular to ‘H’, which must be perpendicular to ‘K’ for a plain wave. And this must be there for all the three waves, for incident, reflected and transmitted waves. So, what direction should I give for ‘E’? There are two directions that I can give. One case is when the electric field is in the same plane. So, let me use probably a different line here. So, this would be the direction of the incident electric field.

Associated with this must be the direction for the magnetic field. So you have ‘E’ cross ‘H’ must be in the direction of ‘K’. Since ‘E’ is directed along this way, my ‘E’ cross ‘H’ means that ‘H’ must be coming out of the page. So ‘H’ is coming out of the page and that ‘H’ vector is ‘H i’. Then, I also have a reflected electric field. For the reflected electric field, I will assume that it is directed along this way.

Again you do not have to really worry about these directions. If the directions are different, the equations will already tell us. This is like you know the notation for KCL and KVL. You can assume certain direction for the current. And if the equation tells you that the assumed direction is correct or not. If the assumed direction is current, then the current will come out to be positive, if not, the current will come out to be negative.

So at which case, you can simply change the direction of the current, so very similar thing is happening. Do not particularly worry about how to draw this ‘E i’ or how to draw this ‘E r’. The equations themselves will tell you whether this ‘E r’ should be reversed in this way or ‘E r’ should be kept in the same way. This is just convention. You don’t have to really worry about this. However, having made an assumption the ‘E r’ is in this direction, B consistent.

If you make an assumption that ‘E i’ is in this way, then you have to have ‘H i’ coming out of the page. Similarly, if I assume that ‘E r’ is in this way, then ‘H’ must be going into the page. ‘H’ must be going into the page and that can be represented by putting a cross here. And this would be your ‘H r’. So that ‘E’ cross ‘H’ will be pointing in the direction of the reflected wave propagation.

Finally, for the transmitted case, we can assume them to be the same directions as the incidence wave because that is kind of very natural to assume. Both of them are propagating in the same direction. Therefore, the ‘H’ field for the transmitted wave would also be coming out of the page. So ‘E’ cross ‘H’ will be pointing in the ‘K’ direction, ‘K’ transmitted direction. Before we proceed further, let us write down what are these ‘K i’, ‘K t’ and ‘K r’ are.

So what is ‘K i’? ‘K i’ will have two components. One component will be along ‘Z’ direction and other component will be along ‘plus X’ direction. What are those components? ‘K i’ is equal to, magnitude of ‘K i’, which I am denoting simply by ‘K i’, cos theta i for the ‘Z’ direction plus ‘K i’ sin theta i for the ‘X’ direction. Similarly, for ‘K r’ you will have magnitude of ‘K r’ and then cos theta r, but this is actually along minus ‘Z’ direction.

Because this ‘K’ is propagating along minus ‘Z’ axis. So it is obvious because incidence must be propagating along plus ‘Z’ axis, but as reflected move must be propagating along minus ‘Z’ axis. However, that ‘X’ component of ‘K r’ will not change the direction. So this would be plus K r sin theta r along ‘X’ ‘K t’ is very simple. It follows exactly the ‘K i’ kind of signs. So, this is given by K t cos theta t ‘Z’ direction, plus K t sin theta t X hat.

So these are the different ‘K’ vectors. Now I also need to write down what is the electric field vector. So electric field will also have two components. It will have ‘E x’ component as well as ‘E z’ component. So it will have ‘E x’ component and then it will have ‘E z’ component. Similarly, the transmitted field will also have ‘E z’ and ‘E t’. The reflected field will also have ‘E z’ and ‘E x’. So what are these components?

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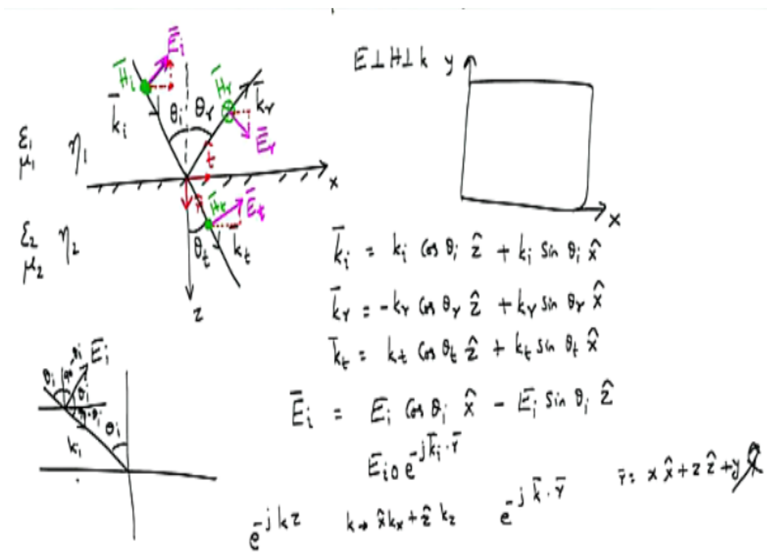

Well, to find out these components, let us slightly expand our view here. So this angle is ‘theta i’. I know that this must be ninety degrees. So this is my ‘E i’. If I now draw a horizontal line like this and a vertical line to simulate the origin that I am considering. This line will make an angle of ‘theta i’. So this angle is ‘theta i’ because this is the angle made by the line ‘K i’ with respect to the normal, which is along ‘Z’ axis.

So if this is ‘theta i’, I already know that this ‘K’ and ‘E’ must be ninety degrees. Therefore, this must be ninety minus ‘theta i’. I also know that this two are ninety degrees. So therefore this must be ‘theta i’ and this must be ninety minus ‘theta i’. So, these are the different angles that I have obtained for ‘E i’. And I can use these angles to actually write down the expressions for ‘E i’. So let me write down what is the incident electric field ‘E i’.

Please remember I am writing everything in the facer notation. All time dependences are assumed to be e to the power J omega t. It is an interesting fact that when you incident light, the frequency does not change. What really changes is the wavelength. The frequency of the incident, reflected and transmitted waves will remain same. So you have 'E i', which is at an angle of 'theta i' with respect to the horizontal that is along 'X'.

So, you can now write down what are these components. So 'E i' is given by magnitude of 'E i' cos theta i along 'X' axis and along 'Z' axis you have minus 'E i' sin theta i, because this is 'E i', sin theta i along 'Z' axis. But, this 'E i' itself will varying as some 'E i zero', 'E' to the power minus jK i dot r. Now, what is this K i dot r here? Well, I know that if a wave this propagating along 'Z' direction and has wave component along the 'Z' direction.

And wave vector having only the 'Z' component, then the phase factor can be written as E power minus jKz. However, since 'K' actually has two components, it has 'X' component as well as the 'Z' component and we are describing the wave propagation in region, where 'X' and 'Y' both are changing, the corresponding phase factor should be e power minus jK dot r, where 'r' is the position vector, xX hat plus zZ hat.

There is no 'Y' component and therefore I am not put that one down here. But in theory, this actually has yY hat. But because 'K' is described only having 'x' and 'z' component, that component along 'y' does not really matter. So with this, the component along 'x' will be E i zero e power minus jK dot r.

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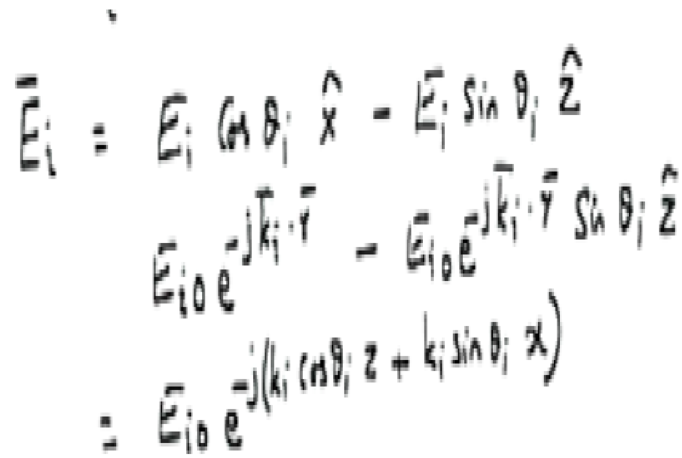

So, let us rewrite what is that ‘K i’ dot r correctly. So I get E i zero e power minus j, now ‘K i’ I already know, which is given by (K i cos theta i z plus K i sin theta i x). So, this is the component that we were looking for, for the ‘x’ component. Similarly, there will be a ‘z’ component, which is E i zero e power minus j K i dot r and then sin theta i Z hat. So this is the incident electric field.

What about the reflected electric field? What about the transmitted electric field?

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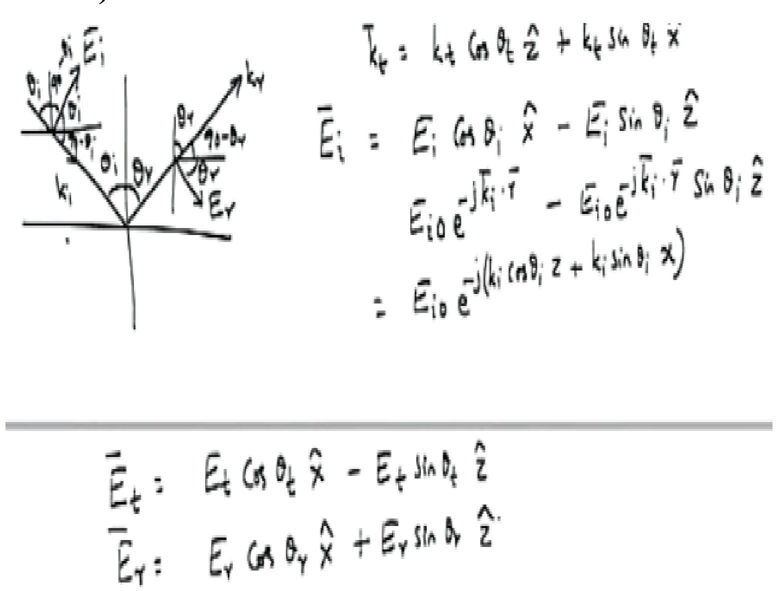

Transmitted electric field can be written down quite simply because this is of the same directions. So ‘E t’ will be equal to, ‘E t’ the magnitude of this fellow, times cos theta t X hat minus E t sin theta t Z hat. Please remember that in this ‘E t’, you have to replace this ‘K i’ by ‘K t’. What about ‘E r’? Well, to get to ‘E r’, let us go back to this diagram and take a look at ‘E r’ in slightly more detailed fashion. So, this is ‘E r’.

Again I am going to write down these two lines here. I know that this angle is actually ‘theta r’ because this angle is ‘theta r’ made by ‘K r’, with respect to the normal. So this is ‘theta r’. And therefore this is ninety minus theta r. I know that ‘K r’ and ‘E r’ are ninety degrees and therefore if this is ninety minus theta r, this should certainly be equal to ‘theta r’. So what are the components for ‘E r’ I have? ‘E r’ will be E r cos theta r along X, E r sin theta r.

So you will actually have E r cos theta r along X, but for ‘Z’ it could be positive, it could be plus E r sin theta r Z. Alright. So we have three components here, ‘E i’, ‘E t’ and ‘E r’.

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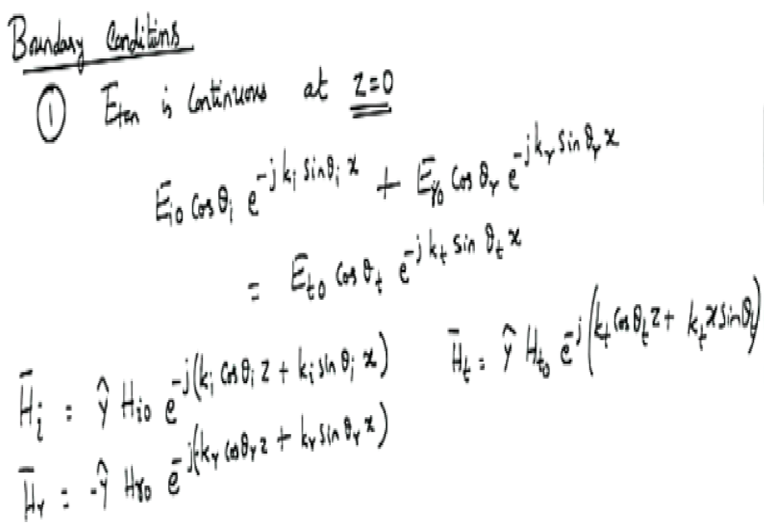

$$E_{i0}\cos\theta_i\, e^{-jk_i\sin\theta_i x} + E_{r0}\cos\theta_r\, e^{-jk_r\sin\theta_r x} = E_{t0}\cos\theta_t\, e^{-jk_t\sin\theta_t x}$$

$$\bar{H}_i = \hat{y}\, H_{i0}\, e^{-j(k_i\cos\theta_i z + k_i\sin\theta_i x)} \qquad \bar{H}_t = \hat{y}\, H_{t0}\, e^{-j(k_t\cos\theta_t z + k_t x\sin\theta_t)}$$

$$\bar{H}_r = -\hat{y}\, H_{r0}\, e^{-j(k_r\cos\theta_r z + k_r\sin\theta_r x)}$$

Next what we need to do is to apply boundary conditions. So, when we apply boundary conditions, the first boundary condition that I am going to apply is that the tangential electric field is continuous across the boundary. Why do I say that the tangential electric field is continuous across the boundary? Because for this boundary case that you are considering, there are no free charges or no currents.

And therefore on the surface, there is nothing happening. So, tangential electric field is continuous. So this continuity of tangential electric field means that you need to first identify what are the tangential fields. For the incident wave, the tangential field is ‘x’, for the reflected wave the tangential field is ‘x’ and for the transmitted wave also the tangential field is ‘x’. So, it is the ‘x’ components, which are actually tangential.

The ‘z’ components are normal; therefore, they do not have be written down that. Now, where are these tangential electric fields continuous? They are continuous at the entire boundary z equal to zero. So in these expressions, you have to go back and write them down to be zero. So, let us apply this boundary condition. For the incident wave, the total electric field in the region one is incident plus reflected.

So, the incident wave itself is given by E i zero cos theta i and e power minus j K i sin theta i x. ‘z’ is zero. And the corresponding component here is E i zero cos theta r and e to the power minus j K r sin theta r x. This is the electric field in, sorry this is not E i zero, this is E r zero and this is cos theta r, which is correct, this is K r sin theta r into x. So, what about the fields in the region two that is in the second medium? It is only the transmitted electric field.

So, for the transmitted electric field, this is E t zero cos theta t e to the power minus j K t sin theta t along X. So, this is the boundary condition that we have. This boundary condition is slightly different or radically different in the way if you think about it, from the case of a normal incidence. In the normal incidence, there was no ‘x’ component. So on the boundary, when you have a normally incident wave, if you satisfy wave boundary conditions at one point, you would have satisfied boundary conditions at all other points.

So that is very important. So you satisfy boundary conditions at one point, you would satisfy boundary conditions at all other points. However, in this case, we have a situation, where there are these phase factors. So, these phase factors show that they will be different at different values of ‘x’. So satisfying boundary conditions at one point is not enough. You have to satisfy boundary conditions at all points.

Now, this boundary condition does not really seem to be helping us much. It will actually help us. We will come back to that one later. So before we go to the next boundary condition, let us actually write down the corresponding expression for the magnetic field. So, what would be the incident magnetic field? The incident magnetic field will be sum H i zero, so it could be sum H i zero and this would be the magnetic field 'H' along 'y' direction.

The reflected field has 'H' along minus 'y' direction and the transmitted field has 'H' along plus 'y' direction. So, I can write down this 'H' as H i is equal to H i zero, which is the amplitude of the incident magnetic field, and then e to the power minus j. What about the 'K' vector? It is K i cos theta i z plus K i sin theta i x. So, it could be e power minus j K i cos theta i z plus K i sin theta i x. So, this would be for the incident field.

What about the reflected magnetic field? Reflected magnetic field is along 'minus y' direction. So, let me write down the 'y' direction for this one. So, this is minus y direction. So, it could be H r zero e power minus j K r cos theta r z and this is the minus sign here, plus K r sin theta r x. And finally for the transmitted field, which would be along 'plus y' direction, it would be H t zero e power minus j K t cos theta t z plus K t.

So, the transmitted H field into the second medium is given by its polarize along 'y' direction. H t zero into the power minus j and the K vector is essentially the same as, I mean the same direction as the incident K vector. So this is K t cos theta t into z plus, you have K t x sin theta t. So, we now have one equation in which we have used the tangential electric field continuity at the interface z equal to zero. And we have obtained one equation.

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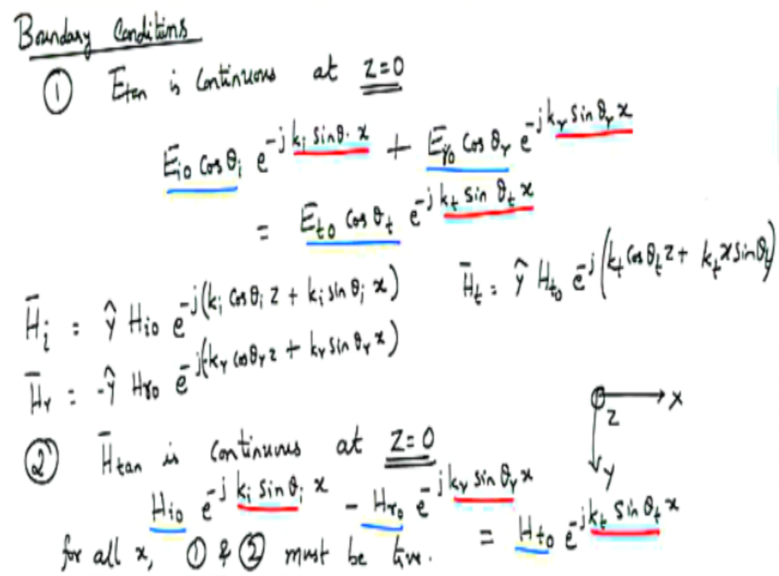

We can similarly use the second boundary condition, which states that the tangential electric field component must also be continuous. Why should the tangential electric field component be continuous? Because this is the case where we are considering the two boundaries to be perfect dielectrics. So because of this perfect dielectric, there is no chance of having a current sheet between the boundary on the interface, there are no chances of having a current sheet.

And therefore tangential ‘H’ field is also continuous. And in, if you look at the expressions for incident, reflected and transmitted H fields, we will see that all these fields are tangential to the interface. Because the interface is actually characterized by the vectors ‘X’ and ‘Y’, whereas ‘Z’ is the direction along which the wave is propagating and Z equal to zero is the boundary which actually separates the two medium with incidences eta 1 and eta 2.

Therefore ‘Z’ is the normal to the interface as you can see over here, whereas ‘X’ and ‘Y’ define the interface plane itself. So, for that H y being on the tangential component on the interface that will also be continuous, so this again continuity condition or the boundary condition must be imposed at ‘Z’ equal to zero on the interface. So when you impose that one, what you see here is that H i o at Z will be equal to zero.

Therefore, you just are left with the phase term, which is e power minus j K i sin theta i x minus Y direction. Therefore, this would be minus H r zero e to the power minus j K r sin theta r x. This must be equal to H t zero e to the power minus j K t sin theta t into x. So, now we have two equations here, one equation for the electric field tangential condition, one for the magnetic field condition.

And what you have to see is that these two conditions must be true for all values of ‘x’. In the case of a normal incidence, it was possible for us to equate these two continuity conditions or the boundary conditions at a single point on the interface because the phase factors were completely independent of the interface point. But in this case, you have the phase points ‘x’ coming into picture and this equation must be satisfied for all values of ‘x’.

So for all ‘x’, 1 as well as 2 must be true. Now you can convince yourself that the only way in this statement can be true that is for all interface points, the boundary conditions have to be true. It can only happen when the corresponding phase terms they are all equal. So, if K i sin theta i is equal to K r sin theta r, which is equal to K t sin theta t, that is the first condition that needs to be satisfied.

And the second condition is that these sums H i o minus H r o must be equal to H t o. Similarly, for the case of this electric field condition, the phase terms, so the phase terms must all be equal and the corresponding amplitude terms must also be equal. So, E i o cos theta i plus E r zero cos theta r must be equal to E t zero cos theta t. So you have these two conditions and these two corresponding amplitude conditions.

So, you can substitute them and what you get is very interesting.

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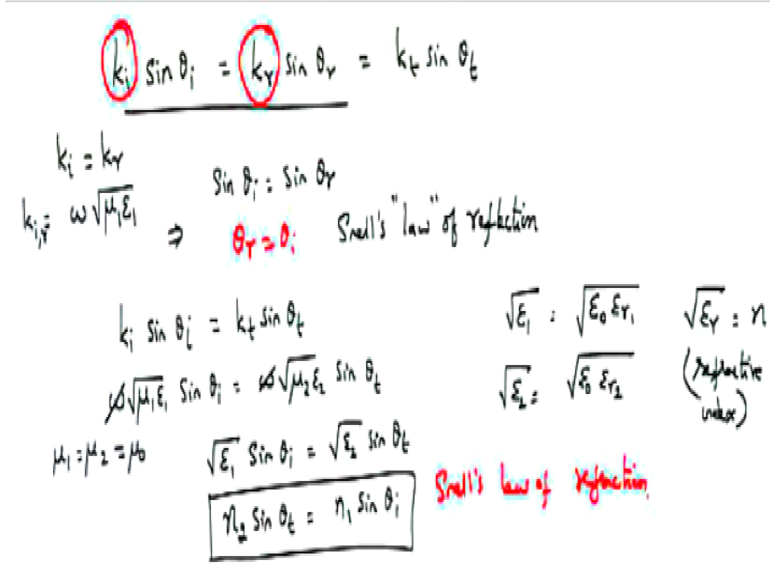

Let us first write down the fact that the phase terms must be equal to each other. So you have K i sin theta i is equal to K r sin theta r, which must be equal to K t sin theta t. Consider first this equation, where K i sin theta i is equal to K r sin theta r. So in this equation, if you observe carefully, you will see that the magnitude of K i must be equal to magnitude of K r. Why is that so?

Because the K vector is given by omega into square root of Mu epsilon, corresponding to whether the medium is incident medium or the second medium. So, K i and K r, both will be characterized by the same medium constants mu 1 and epsilon 1 and they have the same frequency. Therefore, the magnitudes of K i and K r must be equal. They are actually corresponding to the same medium.

And because they are on the same medium, their corresponding magnitudes must be same. So because their magnitudes are same, so K i and K r can be removed from this equation and what you get is a very interesting equation, which states that sin theta i must be equal to sin theta r and since theta i can be only between zero to ninety degrees, so it could either go from normal incidence to parallel or almost gracing incidence.

So, because of that reason sin theta i is equal to sin theta r, also implies that theta r must be equal to theta i. And this equation, you would recognize as Snell's first law or Snell's law of reflection.

What it says is that the angle of reflection, as measured from the normal to the interface is equal to angle of incidence. So, this is the first Snell's law or Snell's law of reflection. So, now you see that the Snell's law is not really a law in itself.

But this is simply the fact that there is a boundary condition to be satisfied and for the boundary condition to be satisfied at all points on the interface, the corresponding phase terms must be equal to zero, which automatically gives you Snell's law. So, let me put this one in quote to indicate that this is not really a law like a Newton's law of motion. There is a second thing that we need to consider. We still have K i sin theta i is equal to K t sin theta t.

Again write down what is the corresponding expression for K i and K t, in terms of the material Constance. So you will be able to write this as omega square root mu 1 epsilon 1 sin theta i is equal to omega square root mu 2 epsilon 2 sin theta t. Omega is the same on both sides, they will go away. Typically, what happens is that mu 1 is equal to mu 2 and both are non-magnetic, which means that they are actually equal to mu 0 and they can be removed out of this expression.

So, what you now are left with epsilon 1 sin theta i is equal to square root epsilon 2 sin theta t. Now if you recall that square root of epsilon 1 can be written as epsilon 0 epsilon r 1 under root and square root of epsilon 2 can be written as square root of epsilon 0 epsilon r 2, assuming that the medium are dielectric with no losses. You can see that square root of epsilon 0 cancels on both sides and remembering also that square root of epsilon r stands for refractive index of the medium.

So the relative permittivity is actually refractive index of the medium. What you get is the second form of Snell's law or Snell's law for refraction. So you have n 2 sin theta t is equal to n 1 sin theta i. So, this expression is called Snell's law for refraction. So this is your Snell's law of refraction. So, you have two Snell's law. One Snell's law stating that angle of reflection measured with respect to normal is equal to angle of incidence measured with respective normal.

The second law is the law of refraction, which states that n 2 sin theta 2 must be equal to n 1 sin theta i, where n 2 and n 1 are respectively the refractive indexes of the second and the first medium.