Electromagnetic Theory
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Lecture - 05
Vector Analysis II

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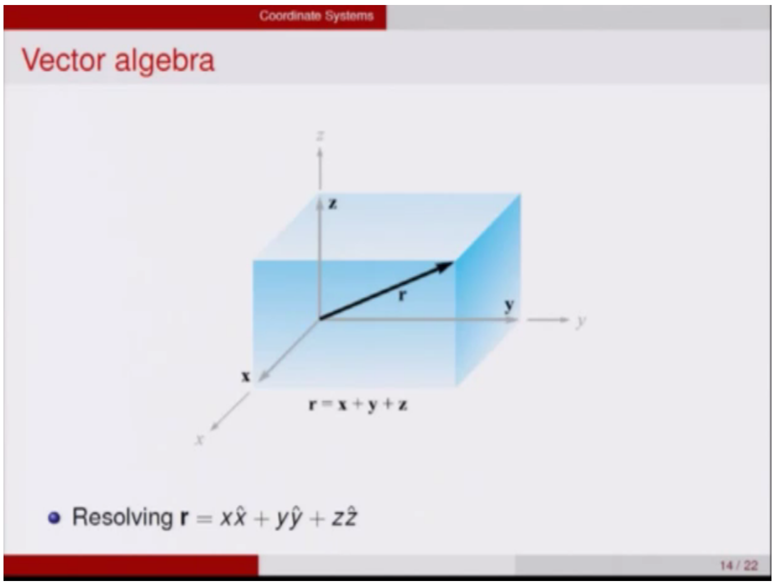

Now, that was coordinate system which allowed me to specify any point in space by with reference to a particular coordinate system by a set of three numbers, I was able to specify every point in space with respect to this coordinate system. Of course, I could have chosen any other coordinate system, but you know that if I choose the coordinate different system, the numbers associated with the points will change, but the points themselves will be physically same.

The points will not change, only the numbers that you associated will change if you change the coordinate system, okay. Consider associating vectors to this coordinate system. Now, that we have talked about how to associate point in a three dimensional rectangular coordinate system. I want to know associate vectors on this plane or I have to describe vectors on this space. So, how do I describe vectors?

Remember, all vectors are drawn with their origin sitting at the origin of the coordinate system such that I can now specify any vector, the end of point of any vector by 3 numbers. So in this particular case, I have a vector r which is called as a position vector and this position vector has its origin at o and it is terminating at some point, which is intersection of 3 planes. This plane that you are seeing in the front which is slightly dark is the x is equal to constant plane.

This plane that you are seeing here is the y is equal to constant plane and this plane that you are seeing here on the top is the z is equal to constant plane. So if you know bring all the 3 planes together, the intersection will be the point where the vectors r terminates. So now, I can draw a line from the origin until I reach the point here and that will define a vector r to me.

In other words, given any vector whose origin is at the origin itself and which is terminating at any other point in the space all I have to do is to set up the 3 planes, so that I can describe the end point of the vector, okay and this vector can now be specified by just 3 numbers or I can resolve this vector into 3 different vectors. These 3 different vectors, how do I resolve that? Remember, resolving a vector means you take a vector.

And then you look at the components of the vector along x direction and along y direction, this is what we did for the two dimensional case. Now, there is no reason why we cannot do this for the three dimensional case. So you take a vector, okay and look at its component along the x direction, look at its component along the y direction and look at its component along the z direction, so you can actually resolve any vector into x, y and z components.

So that is precisely what we are going to do and when you resolve a vector, you will actually be generating 3 mutually perpendicular vectors, just as in the two dimensional case, you generated 2 mutually perpendicular vectors. Here, you are going to generate 3 mutually perpendicular vectors. These vectors are x x hat plus y y hat plus z z hat, where x, y and z without the hats are the magnitudes or the points that are dependent on the particular vector r that you are considering and x hat, y hat and z hat are the unit vectors along x, y and z directions.

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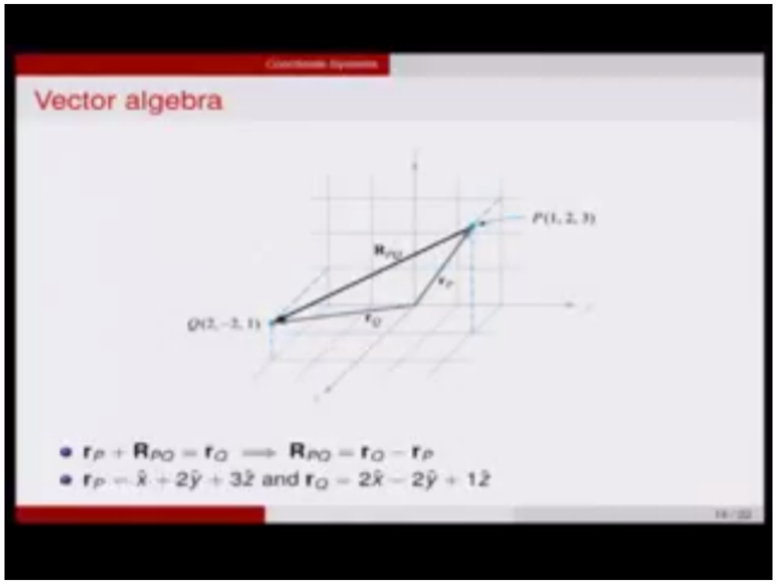

So this is resolution of vector. Now, we want to add 2 vectors or subtract 2 vectors. We have already seen how to do this graphically, but with the knowledge of coordinate systems, is it possible for me to do all this without drawing lines every time and without translating one vector onto the other vector, yes it is possible and to do that one, you have to, so this is an example in which we are going to do that one.

You have start imagining that every point or every point can be represented by a vector, okay, whose origin is at the origin and which terminates at the desired point. For example, I have 2 vectors here, r P and r Q, which are defining 2 positions, position P which is given by the coordinates 1, 2, and 3. How do I get to 1, 2 and 3? I move 1 unit along x, I move 2 units along y and then I move 3 units along z, right.

So I get to the point P which is 1 unit along x, 2 units along y and 3 units along z. So, this point is described equivalently by a vector r P. So, please note that this point in space which has its coordinate 1, 2, 3, can be equivalently defined by this vector r P, okay. Similarly, if I take another position Q which is at 2, minus 2 and 1, okay. I can define another vector r Q whose origin again is situated at the origin of the coordinate system and which is terminating or whose head is situated at 2, minus 2 and 1.

Again 2 units along x, minus 2 units along y, so you need to go to the left, so minus 2 units along y and 1 unit along z. Now, you have a vector R P Q, okay, which is a vector which is directed from point P to point Q. This vector is not defined in terms of the origin. How I am going to describe this vector R P Q. Here is where the vector addition will come to my help. I know that this vector r P plus this vector R P Q is equal to the vector r Q, okay.

This vector r P plus the vector R P Q is equal to the vector r Q. What we said about all vectors being at the origin means that every position is equivalently represented by a vector whose origin is at the origin of the coordinate system and which terminates at the particular point. However, the vector R P Q is a vector which is not defined from the origin 2. It is actually defined between the 2 points P and Q and you have to specify that those points for this vector.

However, you can specify the vector R P Q in terms of the position vectors r P and r Q. How do I do that one, you start you know from the vector addition that r P plus R P Q is equal to r Q that is r P plus R P Q is equal to r Q which implies that R P Q is equal to r Q minus r P, so it is a simple transferring r P to the right hand side, you get R P Q is equal to r Q minus r P and I already know how to represent the vector r P.

The vector r P can be represented, go back here, the vector r is represented by giving its 3 components x, y and z vectors which was x x hat, y y hat, and z z hat. Therefore, r P can be written as x hat, 1 into x hat is just x hat plus 2 into y hat plus 3 into z hat. Similarly, r Q vector will be 2 x hat minus 2 y hat plus 1 z hat. Now, if I subtract r Q and r P, I am going to get R P Q.

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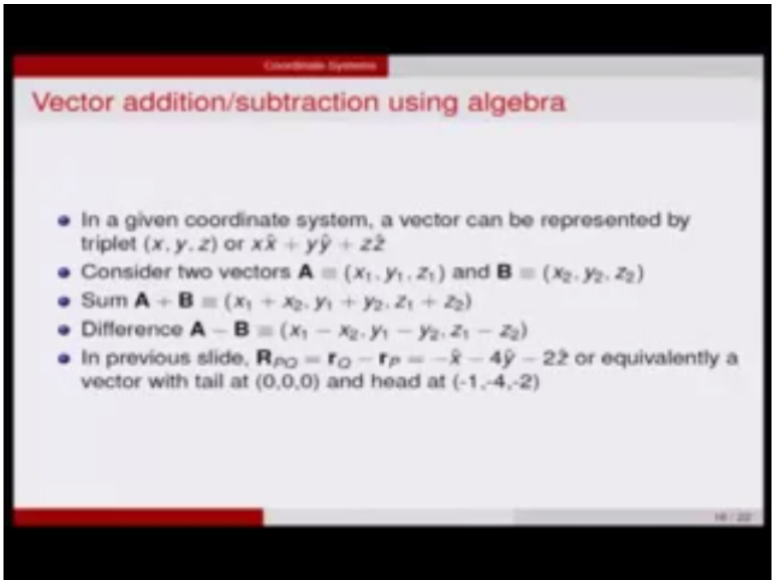

How do I subtract that? In a given coordinate system, a vector is represented by 3 points x 1, y 1 and z 1 or equivalently x 1 x hat plus y 1 y hat plus z 1 z hat and similarly vector B, the sum of the 2 vectors is given by the vector whose components are simply added like this, so you have the x component added to get x 1 plus x 2, y component is y 1 plus y 2, and z component is z 1 plus z 2.

Similarly, to get the difference between A and B vector, you have to subtract x 2 from x 1, y 2 from y 1 and z 2 from z 1. These 3 points define the end point of the difference vector A minus B. So if you go back to the previous slide, R P Q was a vector directed from P to Q and that was given by r Q minus r P and you can subtract the individual components, you are going to get a vector which is minus x hat minus 4 y hat minus 2 z hat equivalently it can be represented as a vector with tail at 0, 0, 0 and head at minus 1, minus 4, and minus 2, okay.

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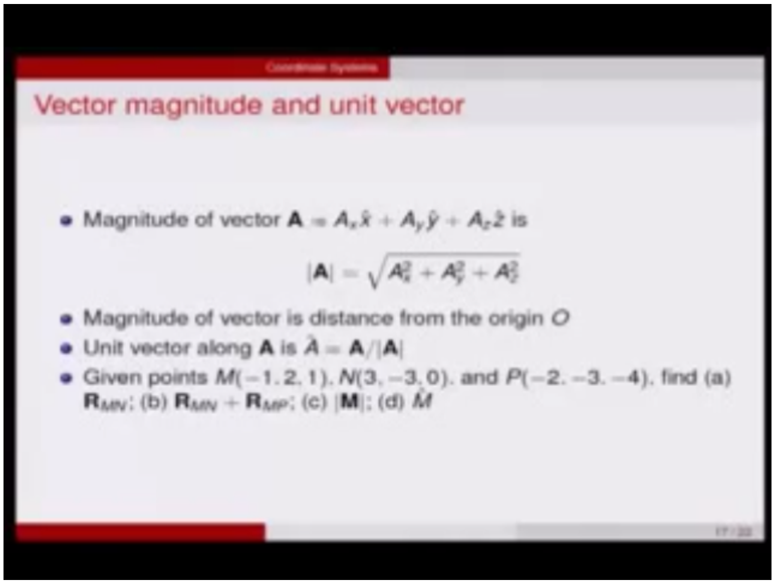

This is how you are going to subtract the 2 vectors. So, we have seen how to define a vector in 3 dimensional Cartesian coordinate system. So given a vector A we can resolve this vector in to 3 vectors which are mutually perpendicular, so when we say mutually perpendicular, we mean by if you take pairs of vectors, these pairs of vector will be perpendicular with each other. So this vector A can be resolved into 3 vectors, A x x hat plus A y y hat plus A z z hat.

And we call this A x, A y and A z as the x component of the vector A, y component of the vector A, and z component of the vector A and the vector that we formed along the X direction will be the component times the vector. Now one of the things that we have seen in Coulomb's law is that we not only require the vector, but we also require its magnitude because the magnitude gives you the separation between the 2 charges.

So when you have 2 charges separated by a distance r, we define a unit vector along the line that joins the 2 charges. But at the same time, I also want to know what is the magnitude of the separation? To get the magnitude of the separation, we need to define the magnitude of vector, we normally denote the magnitude of vector by putting it in between 2 vertical lines, so this denotes the magnitude of vector.

Sometimes, we simply write down the number without vector notation that would also denote the magnitude of vector and this magnitude of vector is given by square root of A x square plus A y square plus A z square which you can think of as a three dimensional generalization of Pythagoras rule. So magnitude of the vector actually tells you the distance of the vector from the origin that means at what point the head of the vector is located with respect to the origin.

So, the distance from the origin of the vector to the tail point or the end point of the vector is the magnitude of the vector. What is the magnitude of the unit vector? 1, so do not forget that. How do I define a unit vector along a particular direction? So, I have a vector A which is starting with the origin O and terminating at some point in the three dimensional space. How do I define a unit vector along that? What is a unit vector?

It is a vector which is directed at a particular direction, any direction if the vector is directed along x direction then it will be a unit vector x hat, if it is directed along y, it would be a unit vector along y hat. Now if I have a unit vector directed in general along A, I can denote this as some A hat and what is that, it is a vector, so I need to give the vector A itself, but since the magnitude of the vector has to be unit.

I will have to divide this A by its magnitude, it some sort of normalization, okay. So if I have 10 x hat as the vector along x direction and if I divide the 10 x hat by 10, which is the magnitude along x, I will get x hat, which is the unit vector along x. So the same rule if I want to specify unit vector A, I specify this by writing A and putting a hat on top of it, I can sometimes use bold letter A with a subscript A or I can use u A, all this different notation you will be seeing in literature.

So, unit vector along a particular direction is given by the vector divided by its magnitude. This is the unit vector along a vector A in its particular chosen coordinate system, okay. Here are some exercises for you, I have given you three points. So, one point is M at minus 1, 2, and 1 and the point N at 3, minus 3 and 0 and another point P which is minus 2, minus 3 and minus 4. What I would like you to do is to first find a piece of paper and draw the 3 axes and try to represent this points M, N, and P on that piece of paper, okay.

So I wanted to get some familiarity with representing the point, so take this as an exercise and locate the points M, N and P. Now, remember to every point M, N or P, I can actually define a vector whose origin is at the origin and with whose head is at M, N or P. I want you to find out this vector R M N. What is this vector R M N, it is a vector which is directed from M to N, okay and what is that vector given us R N minus R M.

This is in general. If I have a vector R I F, okay, it would be a vector which is directed from the initial point to the final point F and then that vector will be given by the position vector R F minus position vector R I. So R M N, I want you to find out. Similarly, I want you to find out R M P and then add to this R M P to R M N, need to do these both graphically as well as using the coordinate system values, okay.

Finally, find the magnitude of the vector M, also find the unit vector along the direction M, okay. How do I find the unit vector along the direction M, you know what is the vector M and divide by its magnitude and what is the magnitude? You already found that out in the previous exercise, okay. So you do all this exercise to gain some familiarity with using the vectors, okay. So, we have taken a large (()) (14:12) from Coulomb's law.

We were discussing Coulomb's law and then we started discussing vector analysis, went to coordinate system, all this was required because electromagnetic is essentially a three dimensional subject, okay. You need to know vector analysis, you need to know how to setup coordinate systems if you want to have good success in electromagnetics courses, okay.

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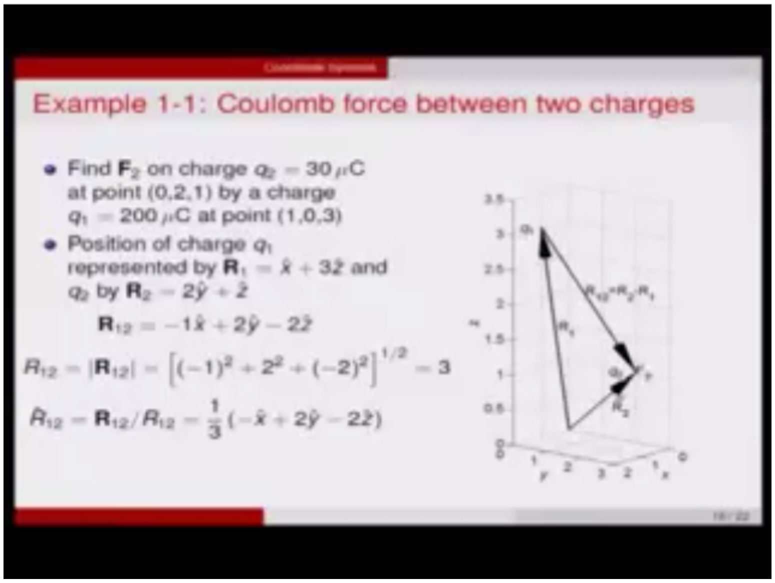

So, we will now come back to Coulomb's law and try to work out a problem. Hopefully, whatever that we have learnt in the vector analysis and coordinate systems will come to our help and help us find the solution to this particular problem. What is the problem, it is essentially a simple problem, we have an origin, okay, and we have 2 charges, one charge at q 1, the other charge is at q 2.

The charge q 1 is 200 micro Coulomb's, micro is a prefix which means that 10 to the power minus 6. So, this is 200 into 10 to the power minus 6 Coulomb's or 200 micro Coulomb's and this q 1 is located at point 1, 0, 3, see you can go 1 unit along x. So here in this particular figure, I have rotated the axis to show you everything hopefully in a clarified manner. This is the x axis, this is the y axis, and this height is the z axis.

So to locate point 1, 0, 3, I have to move 1 unit along x, there is no movement along y, so y is equal to 0. Now, I have to move along z to z is equal to 3 to reach point q 1 or charge q 1 whose value is 200 micro Coulomb. Please note that this is the positively charged particle which I have put here, okay. Similarly, we have a charge q 2 with a value of 30 micro Coulomb's, okay. It is also positively charged and it is located at 0, 2, 1.

So, 0 means x is equal to 0, so you are now looking only at the intersection of y and z planes that is the point 0, 2, 1 is in the y, z plane, okay. So you can see here that I have to move 2 units to y and 1 unit to z in order to reach point q 2. Now, the question is q 1 is exerting some force on q 2, find the force at q 2. So, Coulomb's law will help us find this, but Coulomb's law requires us to give both magnitude as well as direction.

Magnitude is the easier part, okay. Direction is something that we want to understand. If I denote the point, where the charge q 1 is placed by a vector R 1 and if I denote the point where the charge q 2 is placed by a vector R 2, then I know that because Coulomb's law also tells me that the force will act along the line that joins the 2 charges q 1 and q 2, right. It acts along the line that joins the charges q 1 and q 2.

I define R 1 2 as a vector which is starting at q 1 and ending at q 2, okay. In this direction is the force F 2, okay. So, I have R 1 position vector for charge q 1, R 2 as the position vector for charge q 2 and R 1 2 is the vector starting from q 1 to q 2 and I know clearly that R 1 2 is given by R 2 minus R 1 where R 2 is the position vector of charge q 2, R 1 is the position vector of charge q 1. Why because remember R I F, I is the initial point, F is the final point.

So the vector that would be described by R I F is actually given by the position vector of F minus position vector of I, so R F minus R I, okay. So, what is R 1? R 1 is the point which corresponds to charge q 1 which is 1, 0, 3, this is in the x, z plane, so you can see that this is how x will increase, this is how z will increase, so this plane is the x z plane and in this x z plane, R 1 is given by x hat plus 3 z hat. Similarly, R 2 is in the y and z plane, okay.

So, R 2 is given by 2 y hat plus z hat and R 1 2 is given by R 2 minus R 1, remember how to do R 2 minus R 1, I have to look out the components, the y component of R 2 subtract the y component of R 1 from the y component of R 2, subtract the z component of R 1 with z component of R 2, subtract the x component of R 1 with x component of R 2, okay. You can do this subtraction.

And you will see that R 1 2 is a vector that is in the three dimensional, it has all the 3 nonzero components. It is given by minus 1 x hat plus 2 y hat minus 2 z hat. This minus 1 is just for convenience I have written, this is minus x hat plus 2 y hat minus 2 z hat and that is this vector R

1 2. Now, force acts along the line is fine, but what we want is not the force acting along that one.

What I want is not the statement that force is acting along the line R 1 2, what I want is a unit vector along the line R 1 2. What is the unit vector along the line R 1 2? That will be the vector R 1 2 divided by its magnitude. I have already found out what is R 1 2, the magnitude of the vector R 1 2 is its x component square plus y component square plus z component square, all under root, okay.

And if you do that one you will get minus 1 square plus 2 square plus minus 2 square, which will be 3. So, the unit vector along R 1 2 that is along this vector R 1 2 is 1 by 3 minus x hat plus 2 y hat minus 2 z hat.

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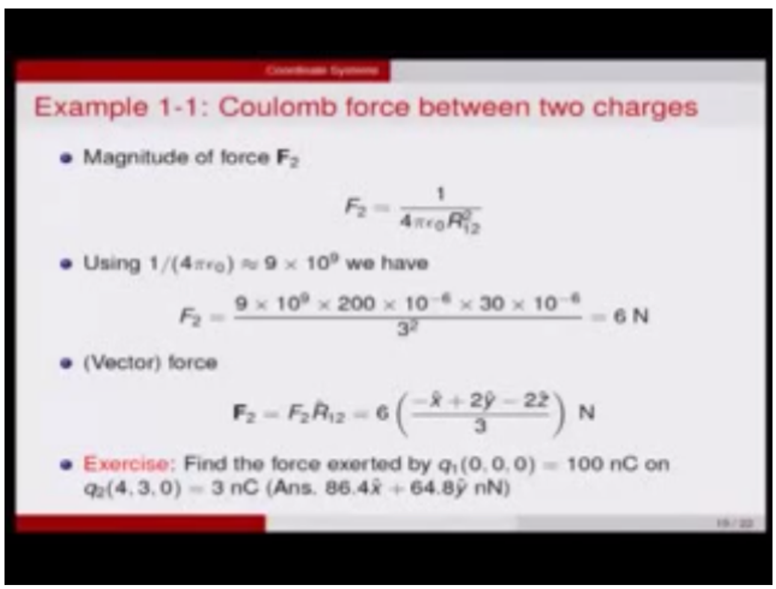

I hope that you derive this one for yourself and convince yourself that this is correct, okay. So coming back to the force part, I have already found out the unit vector. So now, I know that if I take the magnitude of the force and then multiply it by the unit vector along the line R 1 2, I will get the full vector force F 2, right, a vector force F 2 I will get. What is the magnitude of the vector force F 2?

The magnitude of the vector is 1 by 4 pi epsilon 0 R 1 2 square, why epsilon 0 because there was no specification on the medium, so you just take the medium to be air or free space and we know that for air or free space, epsilon R is equal to 1, so I have 1 by 4 pi epsilon 0 R 1 2 square. Now, here is an approximation that you can use frequently 1 by 4 pi epsilon not can be approximated as 9 into 10 to the power 9 because epsilon not is in the order of 10 to the power minus 12.

And when you multiply 4 pi and the corresponding value for epsilon not, this is an approximate value for 1 by 4 pi epsilon 0, we can put that approximation. If you have a good calculator, you do not have to do this approximation, but if you decide to do this approximation, you can just write it over here, multiply the magnitude of the charges, remember the force on a charge q 2 is proportional to the product of the charges, charge magnitude q 1 and q 2 and inversionally proportional to the square of the distance between the 2.

The distance between the 2 is 3 square that is R 1 2 square, so you are going to get 6 Newton's, okay. This is the magnitude of the force. However, in terms of the vector, I need to multiply the magnitude by the unit vector, so this is the unit vector in brackets and I am multiplying that by the magnitude. So this is the force F 2 that is acting on charge q 2 because of charge q 1. Now that you have seen this example, re-look at the example once more, understand all the steps that are there in this example.

And then try your hand at finding the force exerted by a charge q 1 which is placed at the origin of three dimensional coordinate system 0, 0, 0 and has a value of 100 nano Coulomb's on charge q 2 which is placed at 4, 3, and 0, and with the value of 3 nano Coulomb's and you can verify that the answer is approximately 86.4 x hat plus 64.8 y hat nano Newton, it is a very small value because the charge magnitudes are also quite small and the distance between the 2 charges are quite large, okay.

Now before leaving this exercise you ask yourself one question, does it make sense? Does this answer make sense? Well, the answer makes sense because look at where the charge q 1 is placed, it is at origin 0, 0, 0, okay. Now, charge q 2 is placed in x y plane only. It is placed at 4 and 3, its z component is 0. So the force that must be acting on q 2 must originate from that origin because the q 1 charges of the origin and will be directed to q 2 and this force will only be in the x y plane.

There is no chance for the force to be in the z plane for this problem, okay. So in this problem, you expect only the x and y components for the solution and that make sense. You have to always do a sanity check on whatever the answers that you find, so that you are seeing that everything what you have done is correct, so this sanity check if you keep doing it, you will not make mistakes while deriving the answer, okay.

I want to now introduce Coulomb's law and change the notation that we use slightly, so previously we had use A R 1 2 or A R as a unit vector, but when we go to different charge distributions, it becomes easier to use a different notation, there is a reason why we go to other different notation, I will tell you when we go to that notation later. We keep some source charges, okay. These charges I am still labeling them as q 1, but I am also specifying the position of the source charge.

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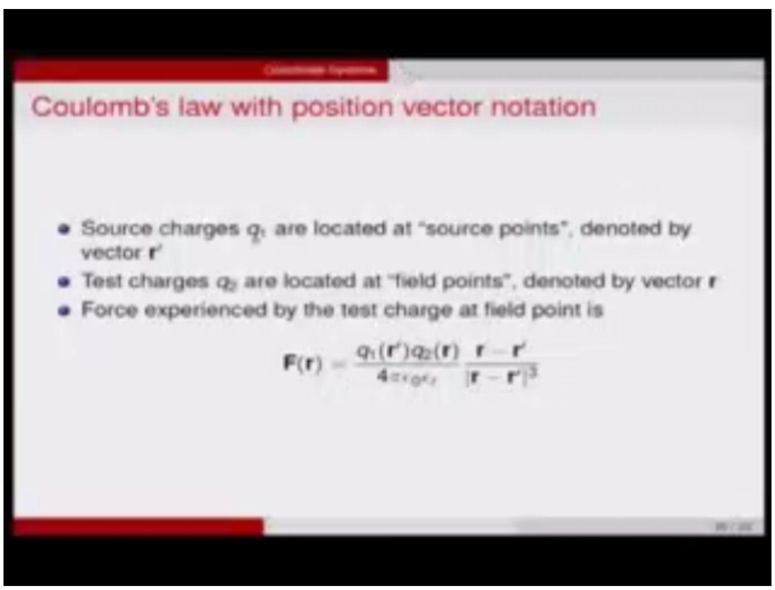

The source charge q 1 is at position R prime. R prime is a position vector from the origin to the source point and the test point, okay. Previously what we considered as charge q 2 which is defined by the position vector R, is what I am calling as a test charge, okay. So this is known as the source point, this R without any prime is known as the field point. This is the point where I am interested in finding the force, okay.

You will soon see why this is called field points, but for now just take the fact that the position vectors without a prime are called as field points, position vector with prime are called source points. So this part has not change, all that we have done is to identify the position of the charges with q 1 is at R prime, q 2 is at R. What is the unit vector? Unit vector is the one that joins R prime and R.

So it is a vector that is directed from R prime that is from the source point to the field point and we know that vector is given by R minus R prime divided by its magnitude, okay. The magnitude of the vector is R minus R prime magnitude. The separation between the 2 vectors is magnitude square. Magnitude square part will make R minus R prime square, but gets multiplied by R minus R prime, so as to give you R minus R prime q, okay. So keep this in mind.

Now you will frequently find as we go to the next lecture, you will frequently find that you are dealing with only one-point charge or it dealing with one charge distribution located at a point in space. You will be dealing with many charges or a continuous distribution of the charge, so for that case, we have know how to obtain the force when you have more than one charge. So let us I have 10 different charges.

How do I obtain the force? The answer comes in the form of superposition of forces.
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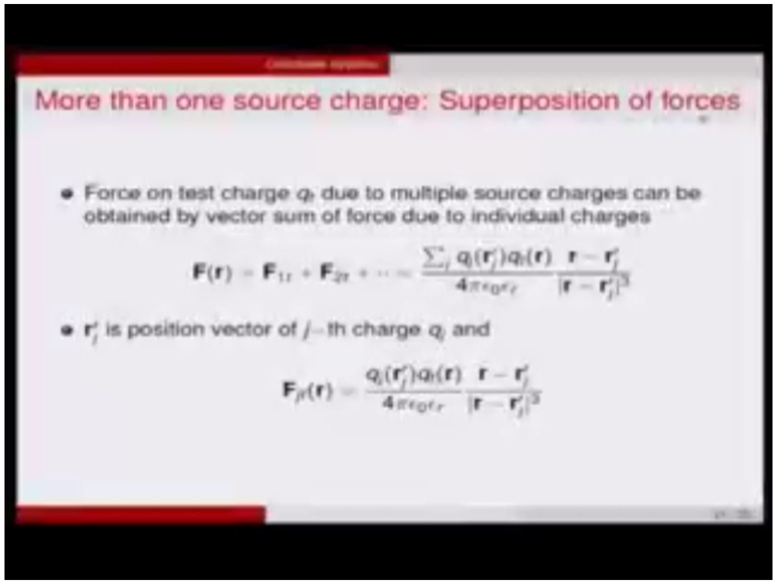

What is this superposition of the forces? Fix the test charge location and do not change that one. So let the test charge be located at the position r and we will call S q t, just that subscript of t indicates to you that this is a test charge. So I have q t of r, r standing for the position vector of the field point, where I am keeping the test charge. Now, the source charges could be placed anywhere.

This placing of the charges anywhere, I am denoting this by r j prime, where j corresponds to the j th charge of magnitude q j or sign charge magnitude q j and what I want to find out is the force experience by the test charge and this force will be given by the vector sum of, this is vector sum, this is not a scalar addition, this is the vector sum, because individual forces are vectors. What is this F 1 t mean?

It is the force because of the first charge on the test charge, force F 2 t is the force of the second source charge on the test charge and so on and how are these forces obtained? You simply break up the charge or you simply find out the force because of each of these source charges and then sum together and summing is happening only on the source charge distribution, okay, that is why the summation sign with the j is applied to the source charge not to the field point.

So in fact, you can pull this field point outside and whatever that you are left with when you multiplied by q t of r is going to give you the force by that j th charge on the test charge q t. okay.

So this is the force of the j th charge on the test charge and if you sum all these forces, you are going to get this particular expression, okay. We will not solve any problem with superposition of forces, we will in fact solve at slightly different problem of continuous charge distribution later.

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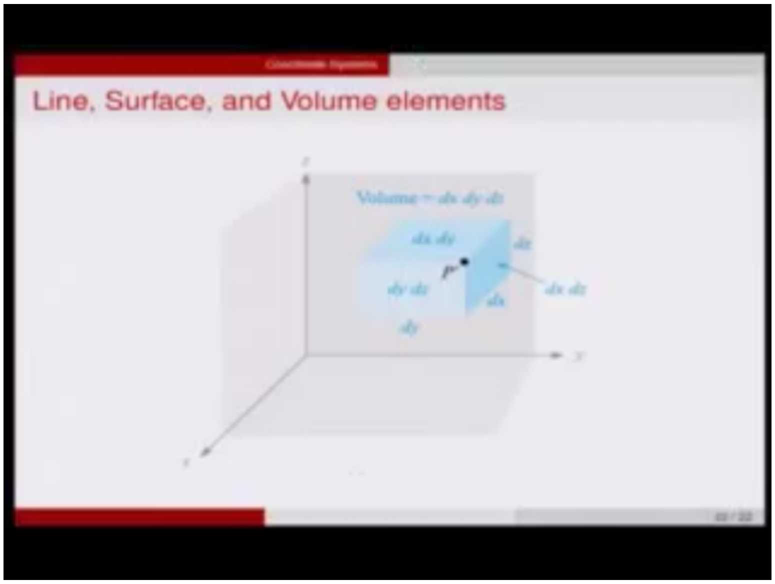

Here is where I want to talk about line surface and volume elements and kind of finish this Cartesian coordinate system. For now, we will be requiring only Cartesian coordinate system. Suppose, I have a two dimensional plane, a two dimensional coordinate system, this is the x axis, think of the green board here or a black board in your class as an example of a plane. We have x axis and perpendicular line which is y axis to point where these two meet is the origin o.

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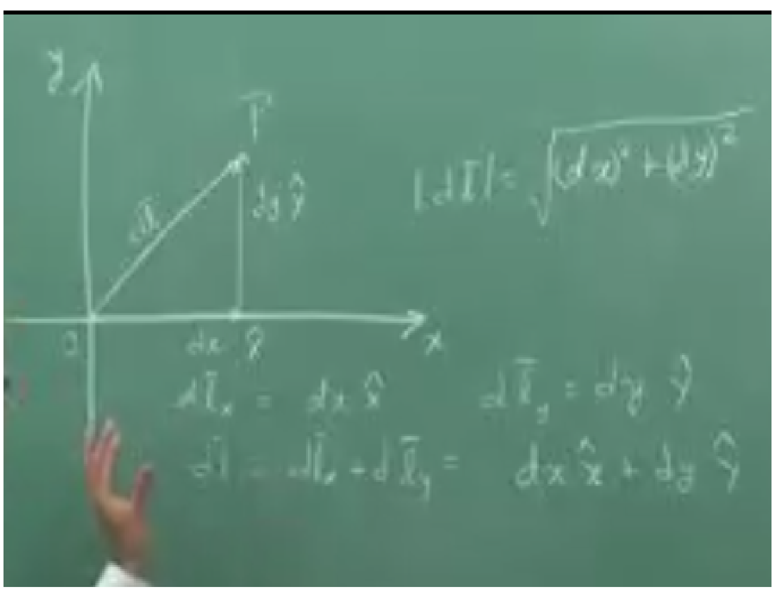

Now, if I move a certain distance say d x along the x axis, I will actually be generating a vector whose head or whose tail or the origin lies at 0 and whose head lies at d x that is this particular vector that I have generated by moving along x direction is d x x hat. I could now move vertically to generate another vector which is d y y hat. Of course, this d y y hat is actually parallelly translated from this particular vector, right, so you need to remember that one, okay.

These vectors that I have generated are called as the line elements. This is called as the line element along x direction and this line element along x direction is given by the distance that I have moved along x axis. Similarly, I have the line element along y which is the distance I have moved along the y axis. This is d y y hat, okay. If I now consider a general direction of movement, I can write down this as some d l, okay.

And this vector d l which represents a vector from the origin to this particular point, call this as point P, okay. Can be now decomposed into two components d l x plus d l y which is nothing but d x x hat plus d y y hat. What is the magnitude of this d l vector? The magnitude is d x square plus d y square under root. The magnitude of the d l vector is d x square plus d y square under root, okay. These are called as line elements.

You will be seeing line elements and line integrals very soon. So, this is the simplest line element that we have considered, okay. Now, we will consider surface elements. For this, I will be going from two-dimension to three-dimension because with two-dimensions, I can only define one surface area, with three-dimensions, I can actually define three surface areas, okay. So in accordance with the right hand rule.

I have this as the x axis, this is as the y axis and this becomes my z axis.

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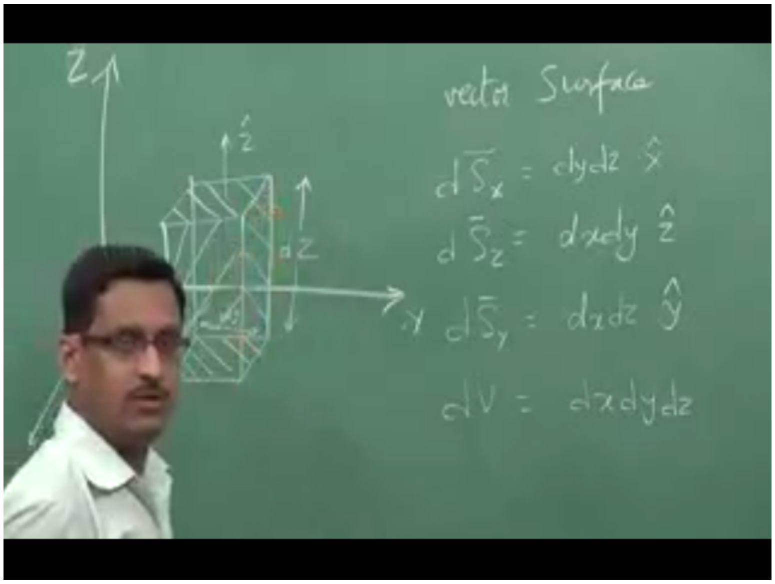

You can imagine that this is the object that I am considering the line that you are seeing along this one will be the x axis or maybe we can consider this way. The line that you are seeing here along this is the x axis, this would be the y axis and if you move from x to y, you are going to see the direction along z axis, okay. Now if I ask you what is the area of this particular phase? The phase that you are seeing, what is the area of this phase?

To get the area of a phase, I need to know its length and width. What is this length? Some length let us say, but this if I am assuming that all these lengths are some d y, d x, and d z. So the phase has an area of d y, so as you move along y, you have d y and then d z, right. So if I move a distance d z in this direction along z and if I move a distance d y, I will get the area d y d z that would be the area d y d z.

So I have an area which is d S x, I will call this as d y d z, this is the front surface area. In which, I have moved d y and I moved d z. Now, I want to associate a vector to this phase, how do I associate a vector to this phase? I will simply multiply this by a vector x. How is vector x pointing? Vector x is pointing away from the surface, right. In fact, this vector is perpendicular to this surface area. So this is how we associate a vector surface area, okay.

We associate a vector surface area by giving the area and then considering a direction that is perpendicular to that area or in mathematical language, we called this as normal to this surface area. So this vector x is normal to y z, the surface area d y d z, so this is the vector surface area along x. Similarly, if I go d x and then move d y along this one. I am going to get the area of this top surface or the bottom surface, okay. I will call that as d S z and say this is d x d y and z direction.

Of course, I could I have chosen the direction of x for the front surface to be going inside which would be opposite to the x direction, but we always use the surface areas again as similar to the right-handed rule. The right-handed rule tells you that you go from x to y, you will move along z. Similarly, d S x will be d y d z and move along positive x direction. So d S z will be a vector surface area of this top surface or the bottom surface pointing along the z direction, okay.

So this particular bottom surface or the top surface is the d S z. Here, it would be pointing along z direction, here it would be pointing along x direction. Can you figure out what would be the vector area for the sides? Yes, you can easily figure this out. This would be d S y and d S y will be d x. So you move now along like this d x and then you move d z. So it would be d x d z along y direction. So these are the 3 surface elements or surface areas that you are going to see.

This is the magnitude of the surface area and this is the vector that would be perpendicular to the plane that contains this 2 elements, okay, d y is the direction along y, d z is the direction along z, in which you have moved or distances that you have moved and a vector surface area will be something that will be perpendicular to the 2. What would be the volume element? That is what is the differential volume of this box while you have length, you have width.

And then you have height, so this is nothing but d x d y d z. Is there area associated with that? Thankfully, no. So the volume element is the scalar which is d x d y and d z and gives you the area of this particular cuboid.