Electromagnetic Theory
Prof. Pradeep Kumar K
Department of Electrical Engineering
Indian Institute of Technology – Kanpur

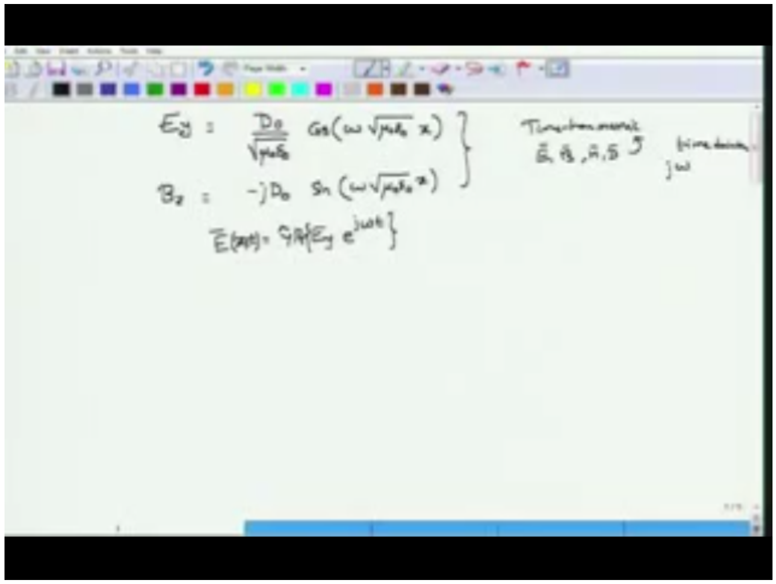
Lecture No - 47
Maxwell's Equation

So in this module, we will continue to study Maxwell's equation. We will first begin by looking at the solution of Maxwell's equation. We will first assume that a certain solution has been given and then check whether that solution is valid or not, whether the expression for electric and magnetic fields satisfy Maxwell's equation. If you remember in the previous module, we had actually seen a case where such an expression was not valid for Maxwell's equation that is it was alright for quasistatic case.

But it was not valid for as solution of Maxwell's equation because it did not satisfy all of the Maxwell's equation. So, to see the counter example for that, consider an electric field which is given, having only the y component. So, the y component of the electric field is given as some constant D 0, okay divided by square root of mu 0 and epsilon 0. We are considering the medium which is free space.

For free space mu is mu 0 and epsilon is epsilon 0. So, I have the y component of the electric field being some cos omega square root mu 0 epsilon 0 times x. So, this is the electric field that I have. It is a cosine function in x, okay. Now, someone also has calculated magnetic field or magnetic flux density component B z and they have given this as minus j D 0 sin not cos, but sin of omega square root mu 0 epsilon 0 into x, okay.

(Refer Time Slide: 00:59)



You might actually question what happened to the time variation, I mean after all we made such a hue and cry about time variation, but time variation seems to have disappear. The key to remember this is the time variation is not really disappeared, but time harmonic solutions that are those E and B fields are all the other fields, E, B, H and D fields. When they are time harmonic, then the time derivatives or the time functions can be replaced by omega or rather j omega.

Therefore, time actually sort of seems to have gone away, okay. But there is a time, see if you want to really obtain an electric field value, correct electric field value what you have to do is, you have to first convert this scalar E y into a vector component. So, here E is along y direction and then to this E y, you need to multiply by E power j omega t and then take a real part of it, okay.

If you do that, you will actually get E which is function of both space as well as time variables, okay. We are not going to do this when at this point, we are going to still assume that we are in the time harmonic form and then we will use the time harmonic Maxwell's equations to see whether these pair of solutions with someone has worked out is consistent with Maxwell's equation or not.

They are solutions of Maxwell's equations or they are not solutions of Maxwell's equation, okay. Now, to perform this solution check, I have individually checked for all the Maxwell's equation.

So, if I start with Faraday's law, I have curl of electric field being given by minus j omega B. I do know that B field is only allowing Z direction. So, this can be written as minus j omega B z and you are interested only in the z component of the curl of E and what is the z component of curl of E?

This would be del E y by del x minus del E x by del y, okay. This is the z component of the curl of E. This should be equal to minus j omega B z. I hope that this change you guys understood, although this is not the proper way to write this expression because this is the vector and this is the scalar, but the point I was trying to make here is that the B field has only z component. Therefore, I can write down this as minus j omega B z becomes a scalar, right.

Similarly, for the curl, I can have x and y components, but those components do not have any counterpart in the right hand side, so which means that those components must be 0. We do not want to bother about those. All where interested now is only the z component of the curl of E, right and this z component is actually given by del E y by del x minus del E x by del y. What are the values of electric fields that we have been given?

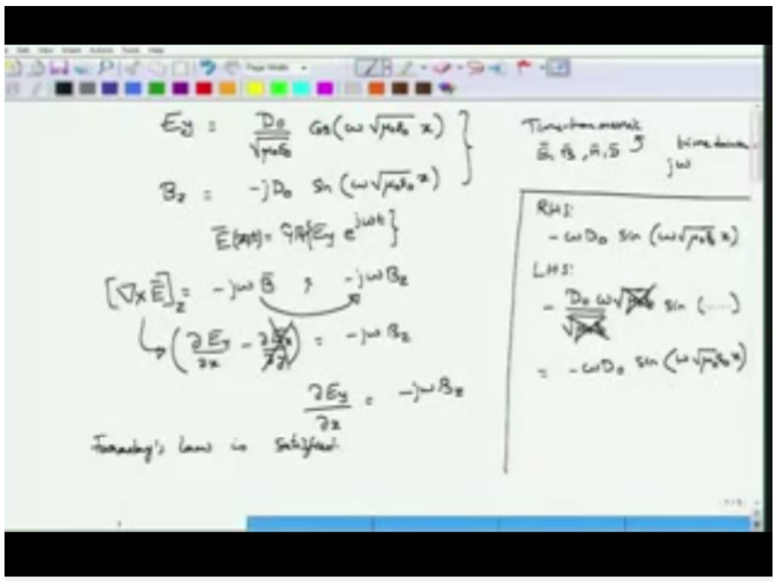
We have been only given E y component. E x component is 0 and therefore, this term goes away, right. So, now that the term has gone, we are reduced to a simple equation del E y by del x is equal minus j omega B z. What is the next step? We substitute for the corresponding values, right. Let us evaluate them over here, so that I can see the expressions while they are on the board. So, what is minus j omega B z.?

So, first consider the right hand side term. So, minus j omega B z will be minus j omega into minus j d 0. Minus and minus will become plus. j and j becomes minus. So, I will get minus omega D 0 and nothing happens to the sin term. So, it is sin of omega square root mu 0 epsilon 0 x. So, if I actually get the left hand side to be exactly this one, then Faraday's law would be satisfied that is just one law that we are looking at.

So, what is left hand side? To get the left hand side, I need to differentiate this E y with respect to x. So if I differentiate E y with respect to x because this is cos of something into x. Cos of something into x when you differentiate with respect to x becomes sin of something into x. Then, that something will come out of the argument, right. So, the left hand side is basically D 0 divided by square root of mu 0 epsilon 0 omega square root of mu 0 epsilon 0 is something that are actually come out.

Cos has become minus sin. Therefore, there is a minus sin here, minus sin of omega square root of mu 0 epsilon 0 into x.

(Refer Time Slide: 03:31)



Clearly, square root mu 0 on numerator and denominator will cancel. Leaving behind, minus omega D 0 sin of omega square root mu 0 epsilon 0 x. This is precisely what is equal to the right hand side. So, Faraday's law is satisfied. So, we come to the conclusion that since left hand side and right hand side are equal, Faraday's law is satisfied. So, at least out of 4 equations, we have satisfied one equation.

Now, let us try and find out whether the divergence equations are satisfied. So, first let us look at del dot D equal to 0 and D is epsilon into E and epsilon is a constant. So, this equation simply becomes del dot E is equal to 0. Is this del dot E equal to 0. Is this correct? I mean I am assuming 0 because I am assuming that the medium has no charges, right. If this turns out to be nonzero, then it means that there are some charges which we are trying to find out.

So, if I try to find the divergence of this, remember divergence will involve components like del E x by del x del E y by del y del E z by del z. E x and E z are 0 already and E y unfortunately is not even a function of y. E y is function of x. Therefore, this would also be equal to 0. So, we have seen that in free space del dot E is equal to 0. Well, you might say that there is no big deal out here, we already knew that, but it is gratifying in some sense to show that Gauss law for electric field is also satisfied.

That is the solutions that someone has worked out satisfy Faraday's law. It also satisfies Gauss' law. Can it be situation where del dot B is nonzero, well we do know that del dot B cannot be 0 and to test that one, you look at what is del B z by del z. B z is the function only of x, it is not a function of z. So, del B z by del z should also be equal to 0. So, clearly Gauss' law for both magnetic fields as well as for electric field is also satisfied, okay.

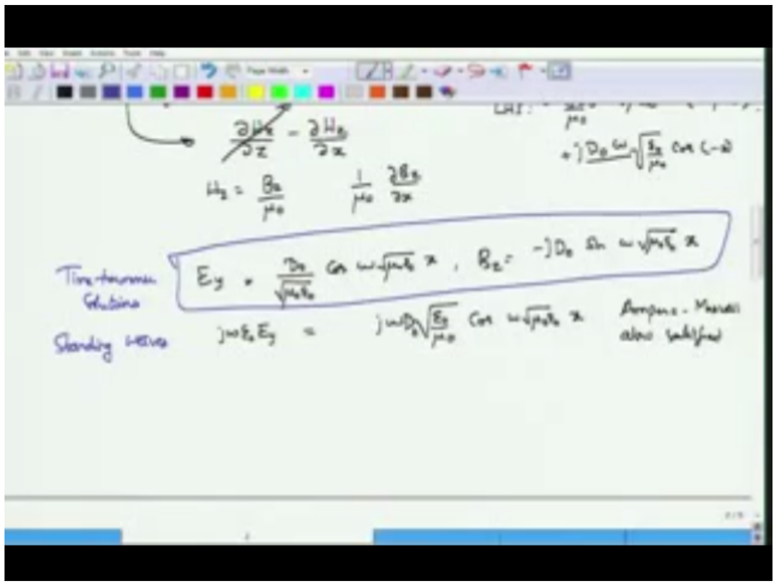
Then, remains only one other equation which is curl of H is equal to there is no j fields here because this is perfectly dielectric medium. There are no free conducting charges. There is no conduction current. So, what you have is, curl of H is equal of plus j omega epsilon E, correct. Because j omega is del by del t. Epsilon E is D. So, this should be j omega epsilon E. Again going back to the scalar form, this is j omega epsilon and since this is free space, we have epsilon 0 and electric field is E y.

So, I need to find out only the y component of this curl of H and what is the y component of curl of H, the y component is del H x by del z minus del H z by del x. Again H x is 0 because B x is 0 and H z is given by B z divided by mu 0, right. So del H z by del x will be 1 by mu 0 del B z by del x. So, now I just need to find out what is del B z by del x. So, go back to expression over here, there is a sin of something into x. So that sin of something becomes cos of something, right.

So, this becomes cos of something, right and that something will come out of the integral. So, what you will get here is for the left hand side, 1 by mu 0, sorry there is a minus j D 0, omega square root mu 0 epsilon 0, which is that something that has come out and sin has become cos and the argument of this one is omega 0 mu 0 into square root of mu 0 epsilon 0 into omega into x. So, there is a mu 0 here, there is a square root of mu 0 in the numerator.

So, you can rewrite this as minus j D 0 omega divided by or omega into square root of epsilon 0 by mu 0 cos of this into x, right. This is what the left hand side would suggest to you. Is this the same as the right hand side? Well, what is E y? E y is given as D 0 by square root mu 0 epsilon 0 and cos of omega square root mu 0 epsilon 0 into x, right. This was E y. Now, if I multiply this E y by j omega epsilon 0 into E y that will be equal to j omega.

(Refer Time Slide: 07:29)



So, there is a minus sin here because this curl of H for y component is minus del H z by del x. Therefore, the left hand side will actually have a minus of minus that becomes plus, okay. Now, you have J omega epsilon 0. So, multiplying by epsilon, I have 1 epsilon in the numerator. Square root of epsilon in the denominator. So, I can replace that and say this is epsilon 0 by mu 0. There is a constant D 0 here and cos of omega square root mu 0 epsilon 0 into x, okay.

So, we have ampere Maxwell or modified ampere's law also satisfied. So, we have satisfied all Maxwell's equation and therefore, this E y and B z which is equal to minus j D 0 sin of omega square root mu 0 epsilon 0 into x is a perfectly valid solution for Maxwell's equation, okay. These are the perfectly valid time harmonic solutions.

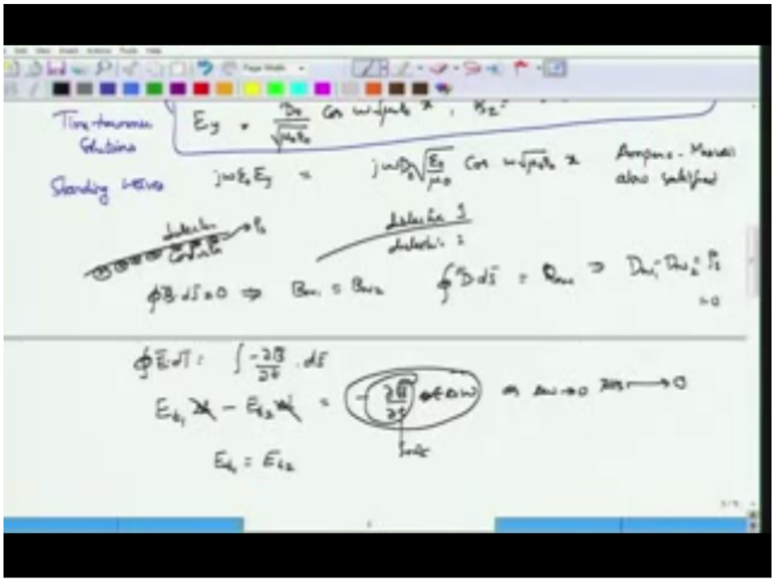
We will later see that this actually come up, this type of cos of something into x and sin of something into x would actually come up when you consider wave guides and these are known as standing waves, okay. Whenever you have 2 conductors or region bounded by 2 conductors, then you will actually have these standing waves. So, you will see this later. There is one other aspect which I want to talk about and that is the boundary conditions.

So, we will be looking at 2 types of boundary conditions. One is for dielectric and conductor and other one is for dielectric and dielectric, okay. So, you have dielectric 1 and dielectric 2, okay. So, what are the equations that would change. The equations for integral of B dot d is equal to 0 does not really change whether you have a dielectric or a conductor, right. So, this will continue to give you a normal component B N 1 must be equal to B N 2.

Similarly, integral of D dot d s over that closed surface would actually give rise to is equal to the total charge and closed would actually give rise to D N 1 minus D N 2 must be equal to surface charge density, right. When will the surface charge density be present, only when I have a conductor, right. So, for a conductor, the surface charges, sorry, the surface charges are slightly in the inner region.

So, there will be some surface charge density on the small thin layer, there will be some surface charge density. However, between dielectric, dielectric, surface charge density is zero. So, this would be either equal to rho s or equal to 0 depending on whether you are in the dielectric to conductor or dielectric to dielectric boundary, okay. For the other 2 equations, you have integral of E dot d l being equal to integral of minus del B by del t dot d s, right.

(Refer Time Slide: 13:49)



Now, for this I know that if I apply the loop onto the left hand side, I end up having E tangential 1, you know with some delta l minus E tangential 2 with some delta l that must be equal to minus del B by del t into delta l into delta w, right. Now, here is where we actually have a slight problem. Previously, you could simply have this term been equal to 0 and delta l, we could have removed everywhere. We could still remove delta l from all sides, we will remove that one.

But what would happen to minus del B by del t into delta w as delta w goes to 0. Now, if del B by del t remains finite, right. If this del B by del t remains finite, then term will actually go to 0, right. As long as, this term is finite, then the product will go to 0, right. As delta omega is 0, the right hand side term will go to 0 and your left with tangential continuity of the electric fields, okay. This is certainly true for dielectric to dielectric, okay.

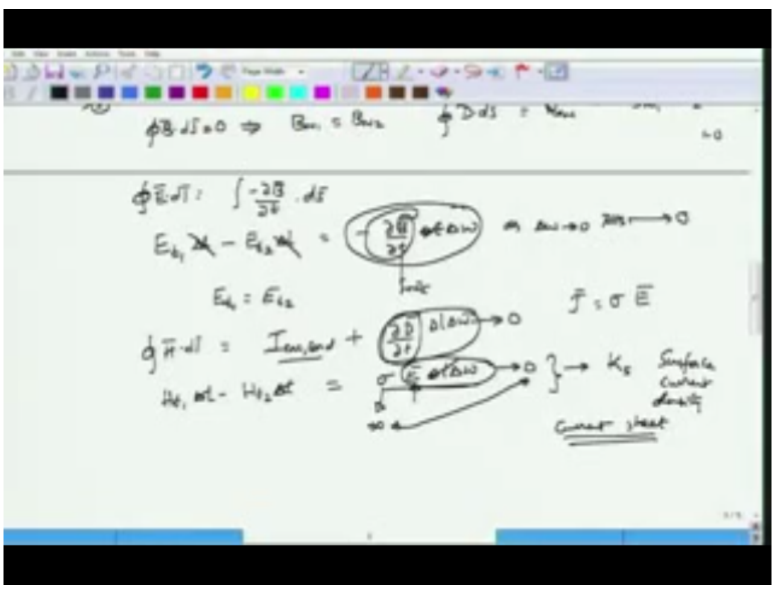
For conductors also, it is true because we do not let our fields go to infinity, okay. We do not let the fields go to infinity because that would imply a lot of other things. Fields are always finite continuous, okay. That is the physical reason behind this one, okay. So, we do not have a situation where del D by del t is actually going of to infinity and because of that reason, E t 1 is still equal to E t 2, okay.

The other condition is that I have integral of H dot d l is equal to conduction current that is enclosed, right. So, let us say I enclose which is conduction current plus a term for del D by del t delta l delta w. Again, writing the left hand side as H t 1 delta l minus H t 2 delta l, where delta l is the path length which we have seen in earlier modules. This seems to be sigma, right.

Conduction current J we have shown that this is equal to sigma into E, sometime in one of the modules that we have already shown this, right. So, this seems to be indicating that this is sigma E delta l delta w as before we do not want to let our del D by del t go to infinity. So, as delta w goes to 0, this term will also go to 0, okay. This is certainly true for the dielectric to dielectric, because in a dielectric to dielectric, this term is anyway 0.

The conduction current density is anyway 0. So, this is actually 0 for dielectric to dielectric and if you assume that the fields are finite, then the right hand side term will also be equal to 0. H t 1 will be equal to H t 2. Tangential H components are continuous, however, when you have an ideal conductor and an ideal dielectric, right. Then, we can have a nonzero value for the right hand side. Even, when E is finite, okay, we assume that as delta l cancels everywhere away.

(Refer Time Slide: 17:11)



We assume that sigma goes off to infinity for an ideal dielectric, but the density E will also be in such a way that the product of this E into delta w goes to 0, okay. So, infinity into 0 that is the large value and a small value can multiply with each other. So, that the resultant term will be equal to some nonzero finite value, okay. So, we let sigma go to infinity for an ideal conductor and we let the electric fields go to 0.

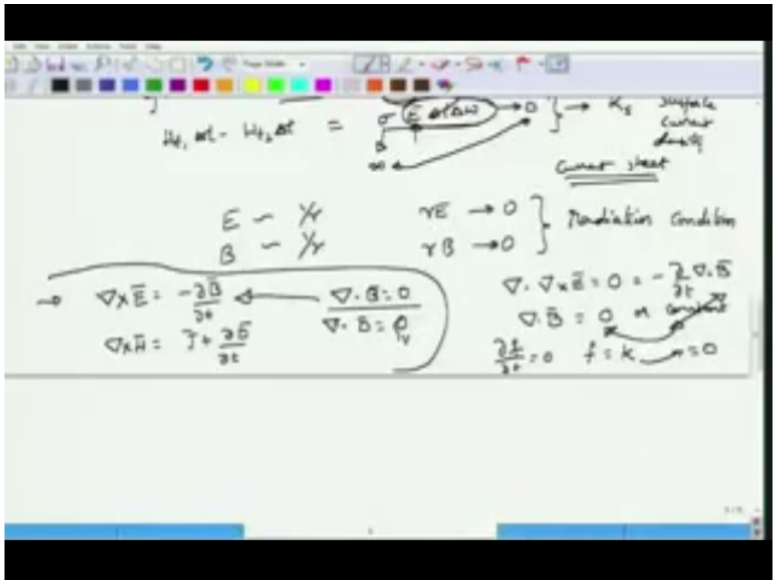
So, you are actually looking at a very small field multiplied by a very large conductor value together they will form what is called as a surface current density, okay, or sometimes called as a current sheet. So, this current sheet is actually spread only over a very small layer inside the conductor and that layer is called a skin depth. It will only penetrate to a small layer inside, but the point here is that in a conductor to dielectric interface, you can have current sheets, okay.

There will be currents on the walls of the conductor, which would be varying with time. So, this is the boundary conditions that I wanted to discuss. Luckily, not much has changed. We could also discuss the uniqueness theorems. If you remember, we discussed uniqueness to show that the potentials of the closed region once you specify the potentials and potentials from the boundary and the charge distribution on certain surfaces, then the electric field was unique, right.

Potentials were, whatever, that you are found was unique. But, you can actually do a similar uniqueness theorem, not for the potentials, but for the electric and magnetic fields and you can show that the same condition would (()) (20:35) once you have found the unique solution for magnetic fields and electric fields that satisfies Maxwell's equations, then those solutions are unique, okay.

You can relax that closed surface and this closed surface going to infinity criteria by making the surface open, right. You can actually extend the conditions to infinity, but provided electric field and magnetic field both decay at least as 1 by r. in the sense that r E must go to 0, r B must go to 0, okay. So, this is the only condition that we have and these conditions are called as radiation conditions.

(Refer Time Slide: 20:57)



A discussion of uniqueness theorem for this is completely beyond the scope of this module. So, we are not going to consider this condition and the corresponding uniqueness theorems, okay. So, once you have solved with Maxwell's equations and found out the electric and magnetic fields, then they are unique, okay, except for some constant which are really not important. So, we have looked at radiation condition and uniqueness theorem.

Before leaving the subject of Maxwell's equation, I would like to point out one commonly asked kind of a solution in Maxwell's equation. We know that written you just write down the Maxwell's equation for you again. Curl of electric field is minus del B by del t, right. Curl of H is equal to J plus del D by del t, del dot B is equal to 0 and del dot D is equal to 0. A frequently asked question is that are these equations completely independent of each other.

Or one of those equations can be derived from the other, right and typically we show this by considering del dot del cross E equal to 0, obviously because divergence of electric field must be equal to 0. So, for this, this must be equal to minus, del by del t of del dot B, correct. So, if you take divergence of Faraday's law on both sides, the left hand side will be equal to 0, the right hand side will be minus del by del t of del dot B, okay.

What this implies? It implies that divergence of B must be equal to 0, right or it must be some constant, right. But if my magnetic fields have been turned on at a particular time, right and before that the magnetic field was 0 that is the initial condition for the magnetic field B was 0 initially, then it cannot change its value, right. It cannot jump to a new value, because del dot B has to remain to that constant, okay and if that was actually nonzero constant.

Then it would mean that there must be some magnetic charges that we should have found since neither of those 2 happen, right. Your constant cannot change from one value to another value without having a time variation in the B field. I mean without having a variation in del dot B or if that was already nonzero to begin with, then we should have found some magnetic charges, leads us to say that this constant cannot be anything, but B equal to 0.

So, it seems that del dot B equal to 0 is not really an independent condition from this curl equation because it can be derived. We have derived del dot B equal to 0 based on del cross E equals minus del B by del t equation. Now, does it actually make sense to you? One school of thought else that this should not make sense, okay. Mathematically, yes we have shown something over here.

But remember we could have said del dot B equal to 0 or we could have said del dot B equal to constant. The reason why we ruled out constants anything but 0 was because we said there are no magnetic charges. Now, how does were know, there are no magnetic charges, is it some rule that was given earlier, no. This is an experimental observation, right. We have so far not been able to detect any kind of magnetic charges.

And therefore, we will say that magnetic charges do not exist in nature, which means that the nature of the B field will have to be continuous. The B field has to continuously close up on itself. This is the physical meaning of del dot B equal 0. So, it is true that mathematically you are getting del dot B equal to 0. But you are getting that 0 only because you are invoking the experimental aspect of saying that magnetic charges cannot exist in nature, okay.

So, if you blindly assume that well mathematics tells me that del by del t of del dot B is equal to 0 and has del dot B equal to 0, it would be wrong because given any function del f by del t equal to 0, all you can conclude is that f must be some constant K. The reason K must actually go to 0 is because we have not found any magnetic charge, right in this case, okay. So, yes, mathematically you can show del dot B equal to 0.

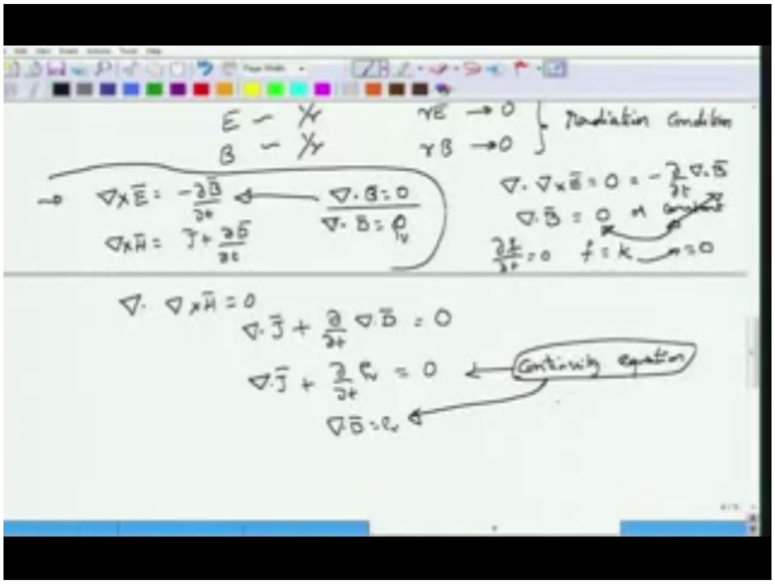
But that is not always follow, I mean that is not follow unless you invoke the experiment value and moreover, these divergence equations are actually not, sorry this del dot D equal to rho v, the divergence equations are not something that is mathematically obtained, okay. They did not depend upon divergence laws or something. Vector analysis was not invented until late 19th century for them to carry out this analysis and derive del dot B equal to 0.

This is an experimentally observed law, okay. So, to my mind you should not say that this curl of E is independent or del dot B equal to 0 is not necessary and there are only 2 Maxwell's equations. There is a place where you are invoking experiment and that can only mean that this cannot be considered as completely independent. Mathematically derivable, only when it supplemented with experimental observation, okay.

A similar thing occurs for del cross H also. Mathematically, I can take del dot del cross H and say this is equal to 0 and what it would mean is that I have del dot j plus del by del t of del dot D equal to 0, right. I know that del dot D is actually rho E. So, I can write this down and say del by del t of rho v. Del dot j plus del by del t of rho v is equal to 0. This is nothing but continuity equation, right.

So, sum actually starts by assuming continuity equation and then say because continuity equation must be true, del dot D must be equal to rho v, okay. You assume this one and then you say that del dot D equal to rho v, but this does not make any sense. This continuity equation was actually something that is postulated. You cannot get this equation.

(Refer Time Slide: 26:43)



So, either you choose between using continuity equation as one of the Maxwell's equation or you use del dot D equal to rho v as one of the Maxwell's equation. The convention is to consider del dot D equal to rho v as Maxwell's equation because that directly talks about the fields and the charge distribution. The continuity equation is simply a conservation equation that talks about the sources themselves.

So, it is kind of the field theory separates itself out into sources which are charge distribution and current distributions and the generator fields, which are electric field, magnetic field, electric field intensity, magnetic field intensity, electric flux density and magnetic flux density. So, that source charges and currents are generating the fields, fields act on other charges and currents to produce the force effects, okay.

So, in that sense these equations are 4 Maxwell's equations and to supplement that you have to introduce continuity equation, okay. Otherwise, you could actually generate charges without any effort and that by way, you would actually have obtained infinite amount of energy in doing so and that would be something that is physics does not accept.