Electromagnetic Theory

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Lecture 37
Magnetic force, torque & dipole

So in this module we will discuss torque when a magnetic dipole because of the magnetic force. So we will first begin by discussing what a magnetic dipole is, what is the physical representation of a dipole and then find the far fields of this dipole and we will actually see why this is called as a dipole, right? So we have of course seen dipole and studied the fields of a dipole in the electric case.

There we had two charges which were separated at a certain distance and we call this as an arrangement of a dipole. This study of dipole was very important for us because we could then imagine the dielectric medium of that any kind that we were considering. Then you could consider the material medium or the dielectric as composed of number of dipoles. The effect of this was that, if you looked at the field from a far away region, then these dipoles.

Actually, I mean the dielectric material, the dipoles inside the dielectric material would generate a field of their own and that would get added to the external field that was present any way. So we introduced a vector called D, the electric flux density to deal with cases of how to obtain electric fields inside a matter. We are going to do the same thing for magnetic materials as well.

So we will begin by calculating the dipole and dipole fields and then assume that the matter is composed or dipoles or model that the matter is composed of dipoles and then we will calculate what would be the magnetic fields in the presence of magnetic matter.

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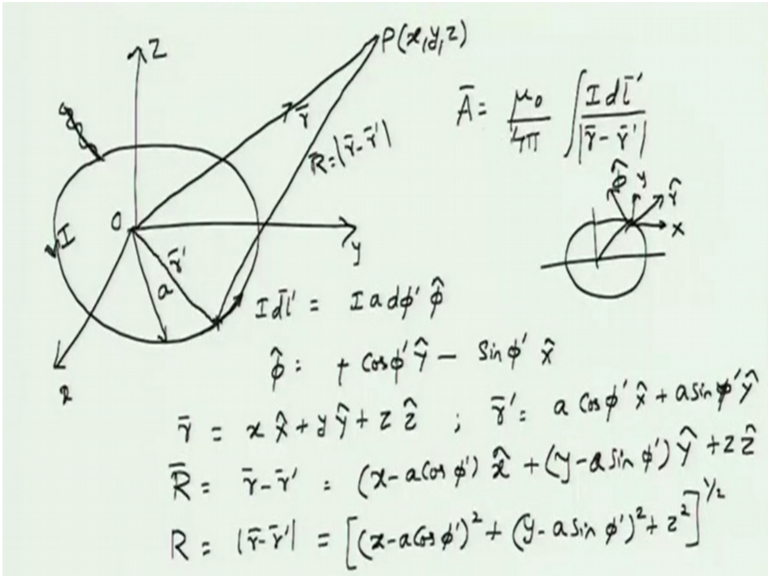

So we begin by considering a dipole. A dipole in our sense actually consists of a current loop of radius a, lying in the xy plane. So this is xy and z axis and the dipole has a radius a and lies in the xy plane and we want to find out the field at a far away point P which is located at a general point of xyz and is described by a vector r, the position vector r with respect to the origin we want to find the fields here.

We have already considered this case. We will also assume that there is an electric field, sorry there is a current I flowing through the loop. We have considered and found out what is the field because of a current carrying loop on the axis of the loop. Now we are looking at the field at a different point. At that time, we remarked that the corresponding integrals will be very difficult to evaluate if you were to go by Biot-Savart's law.

So we will not use Biot-Savart's law here, we will instead use the concept of vector potential that we discussed in the last module, okay? So we want to find the fields at this far away point P which is at a distance r. Let us consider the vector potential A. So what would be the vector potential A. Vector potential is given by at the point P is given by mu zero by 4 pi assuming that the medium here is having a permeability of mu zero.

So this vector potential is given by mu zero by 4 pi integral of I dl prime, divided by r minus r prime magnitude, right? So I dl prime is to be evaluated at the source point. Let us consider this as a source point and then I know that the line element here in cylindrical coordinate ca be represented as I a d phi prime along phi direction. So in the cylindrical co-ordinates this line element is simply circulating increasing the values of phi direction and I can represent that here.

So the line element ideal prime is given by I a d phi prime multiplied by phi. Of course I also need to consider what is capital R which is the distance from field point to the source point. This is the source position vector r prime. It turns out that it is easier for me to work in the Cartesian co-ordinate system.

Therefore, I will convert this vector phi into Cartesian co-ordinate system and if you remember what was that conversion, right, on the two dimensional case that we considered, this vector was r hat and perpendicular to that was the vector phi hat and in terms of x and y, this was given by the unit vector phi in terms of the unit vectors along the x and y direction is given by minus cos phi along y, right?

So it is actually positive along y, but negative along x. So this would be cos phi prime y minus sin phi prime x, okay? So this is my vector phi hat. You can see this, so phi hat along y is cos phi prime and phi hat along x is minus sin phi and this would be along minus x direction. So I can substitute this into ideal prime. I should also be writing down what is the position vector r.

The position vector r is given by x x hat plus y y hat, z z hat and the source position vector r prime is given by a cos phi prime x hat plus a sin y prime y hat, there is no z component to the source vector because the source vector lies in the x and y plane. So what will be the distance vector R, which is r minus r prime. This will be equal to x minus a cos phi prime, x hat plus y minus a sin phi prime along y hat plus z, z prime.

So the magnitude of the vector r which is the magnitude of this vector r minus r prime is equal to this fellow, so you can write down this, okay? So this is the magnitude of the vector R. We now have all the relations that are required in order to evaluate this integral, okay It turns out that instead of using Biot-Savart's law in this case it is actually easier to use the vector potential A and this is one of the reasons why we introduced vector potential as well to evaluate the fields when Biot-Savart law becomes very difficult to handle.

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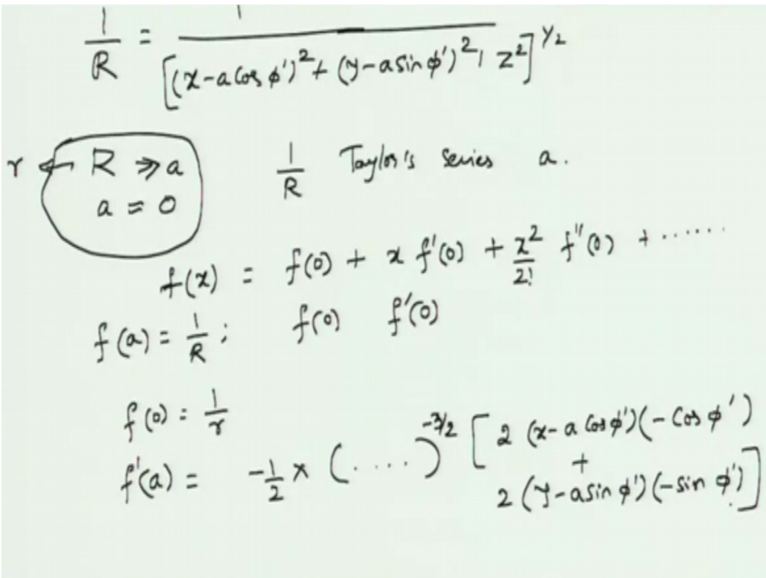

Now before we actually write down what the integral is, we can see that the integral actually has this fellow, 1 by R, where R is r minus r prime, the magnitude of that vector. This is the distance vector from source to the field point or field to the source point and it is in the integrand appearing as 1 by R. So what would be 1 by R. I already know what is R, so if I change the power from half to minus half then I will get what is 1 by R.

So 1 by R will be the same thing except that there is a power of minus half, okay? So if you want we could just write it down like this. You have cos phi prime square plus y minus a sin phi prime square, plus z square, to the power half. Now I don't want to just work with 1 by R like this, what I want to work with is based on two assumptions. One, I am going to assume that the distance from the source point to the field point is much larger than a and A itself is very close to zero.

In other words, I am going to assume that the loop radius itself is quite small and this loop radius is at, because it is quite small and my observation point p is very far away from the dipole or far away from the current loop. So because of these two assumptions, I can actually expand this 1 by R in terms of Taylor series around a. Okay? So I will assume that a is a variable and then I will apply Taylor series to expand or approximate this 1 by R.

I'm justified in this assumption because I am observing this loop at a very far away distance which means that my vector R is much larger than a and of course this also implies that the magnitude of the vector R itself is much larger than the loop radius a itself is very small. So if the loop radius is small and I am observing it from a far away distance, it makes a to be a very small value.

Of course the actual value of 1 by R or R depends on what value of a is because of this assumption, any small change in a does not really affect significantly the value of R. So because of these two observations I can actually expand this 1 by R in terms of a Taylor series and I am going to do this thing for Taylor series about a equal to zero.

So if you take any function and you want to expand this about zero using Taylor series, this expansion will be f (0) the value of the function at x equal to zero plus x multiplied by f prime zero where f prime zero is the value of the derivative, the first derivative of the function of f of x evaluated at x equal to zero plus x square by 2 factorial, f double prime of zero and so on. It goes all the way up to infinity.

So applying this here and recognizing that instead of f of x we are dealing with f of a which is equal to 1 by R, I need to evaluate what is f of zero and I need to evaluate what is f prime of zero. F of zero is easy to evaluate. Go back to this 1 by R expression and put a is equal to zero in this expression. So if I put a equal to zero, I get x square, y square and z square and to the power of half which is simply the distance r. right?

So f of zero is equal to 1 by R. What about f prime of a and then set a is equal to zero, well you differentiate this one with respect to a and you can see that if you differentiate this one, it is probably easier to consider this 1 by R as this expression to the power minus half, so that when you differentiate with respect to a you will get minus half here and then this term will be to the power minus half minus one which will be minus 3 by 2, so that would of course go into the denominator and then differentiate this argument which is x minus cos phi prime square plus y minus a sin phi prime square.

So when you differentiate that one you are going to get two times x minus a cos phi prime and when you differentiate this minus cos phi prime itself, a minus a cos phi prime itself with respect to a, you are getting minus cos phi prime plus two times y minus a sin phi prime, because of this term, y minus a sin phi prime whole square, multiplied by minus. So let me write down this as a product. So that there is no confusion here. So this is minus sign phi prime, okay?

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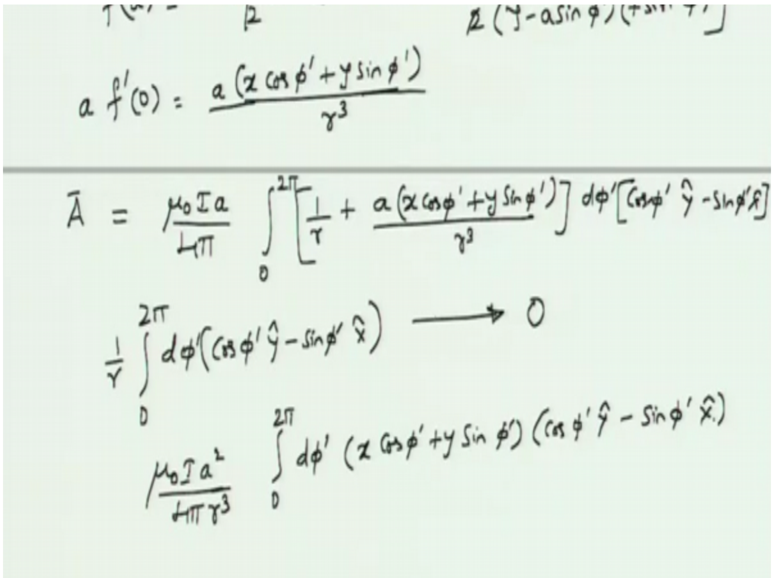

So clearly this minus half goes and cancels with this two here in the numerator and so this is all gone, now if I substitute a equal to zero, this term becomes x cos phi prime. This becomes y sin phi prime and in the denominator this becomes r to the power cube. So f prime of zero is given by x cos phi prime, plus y sin phi prime divided by r cube, if I multiply this one by a, I will get a multiplied by f cos phi prime plus y sin phi prime divided by r cube.

I could of course evaluate the next term and the next term but you get the idea that if the next term is evaluated that would be going as r to the power phi and if r is very large then I can actually neglect this r power phi in comparison with the first two terms. If I try doing neglecting this term r cube then I will actually be left with nothing as you can see very soon, okay?

So let us go back what is the expression for A with assumption in mind or with this Taylor series substitution in place. So I will actually get A as mu zero I and there is already an a sitting here from the top, so I can actually pull that down out. There was an a inside here. This is dl prime and dl prime is I a d phi prime and for phi hat you have to replace that one with the x and y unit vectors over here.

If you do that there is an a here which I can pull it out and the integration limits will be from 0 to 2 phi. So I will actually get mu zero I a square divided by 4 pi, integral from zero to 2 pi. So let us substitute for 1 by R. 1 by R is one by small r plus this a multiplied by f prime of zero, which would actually be, hold on, let us not write down a square over here, let us still keep it a here, because that a square will come now.

So this will be a x cos phi prime plus y sin phi prime divided by r cube. This needs to be multiplied by d phi prime integrating from o to 2 pi but then you have cos phi prime, y hat minus sin phi prime x hat. So this is the integration for a that I need to perform. I can separate this out into two integrals.

One integral involving 1 by R, d phi prime and the term in the numerator, this is cos phi y hat minus sin y prime x hat. And the second term which involves r cube clearly if you look at this term 0 to 2 pi 1 by R, I can push this 1 by R outside the integral and what I am left out here is integration with respect to d phi prime, what quantities cause phi prime y hat minus sin phi prime x hat, so clearly cos phi prime and sin phi prime if you integrate over 0 to 2 pi will give you zero.

So essentially you are going to get a big zero over here. So the term involving 1 by R, if you had kept only that term it would have told us that the fields at the point where we considering was actually zero. So this is something that we would not expect and therefore this is the reason why we took the next order term. The next order term would give you the correct fields to the first order.

So this integral 1 by R has gone out. So what is the integral that is left, that integral is this quantity. So if you would look at what is the integral that is left, now I can pull a outside the integral, this becomes I a square divided by 4 pi, right, and there is r cube which can also be pulled out. So to leave behind, zero to 2 pi d phi prime that is this fellow and x cos phi prime plus y sin y prime, multiplied by cos phi prime y hat which is the phi hat vector minus sin phi prime x hat, so this integral you need to evaluate and you will see that there are 4 terms now.

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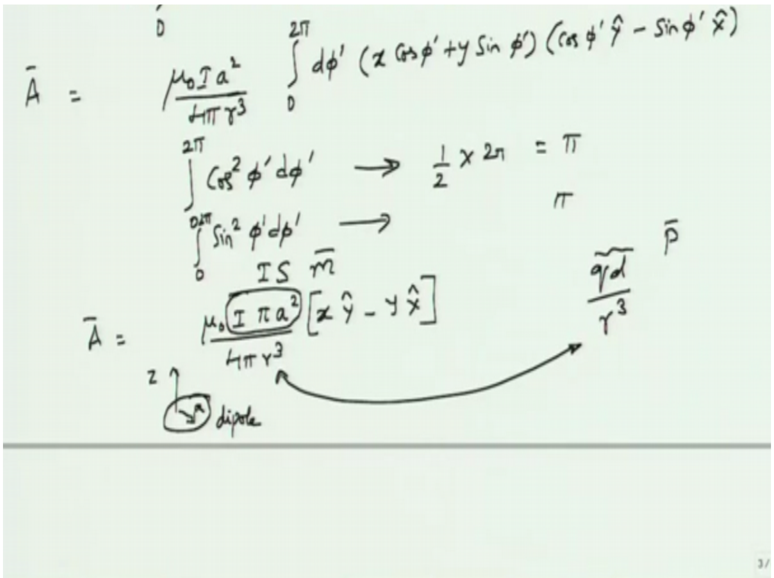

There is x cos phi prime, cos phi prime, this becomes x cos square phi and then similarly you will get y sin phi prime square. So minus sin phi prime square you are going to get and the additional to terms are going to be cos phi prime multiplied by sin phi prime, sin phi prime multiplied by cos phi prime. When you integrate over zero to 2 pi because these are or functions they will go to zero, okay?

So all I am not left with is integrals of the form cos square phi prime d phi prime integrate from zero to 2 pi. This will give me half multiplied by 2 prime which is equal to pi, similarly the integration of sin square phi prime, d phi prime over the same limits zero to 2 pi also gives me pi, okay? So I can now simplify this expression for the vector potential A becomes, vector potential becomes mu zero I pi a square divided by 4 pi r cube and the terms corresponding to x and y are still remaining.

So this becomes x y hat minus y x hat, okay? This is the expression for vector potential A. You have to recall the vector potential for electric dipole and you will see that the potential there would have gone as 1 by r cube. See, if you remember the potential calculation for the dipole you would have seen that the dipole was actually. So on the numerator there was something like q d but on the denominator it was varying as r to the power 3, r being the distance from the observation point to the midway of the dipole.

So in both cases in the vector potential which is going as r cube and the scalar potential for the electric dipole was going as r cube and this r cube similarity is what we call a loop of radius as a dipole. Although there are no poles or charges on the loop of radius a, you won't see the charges here. So this is an equivalent way of saying. This equal comes because the field expressions are or the field potentials are going to be exactly the same, same for the fact that there is a numerator, in the numerator there is q d, which eventually became our dipole moment P.

Similarly, I can identify a dipole moment here as I multiplied by S, where S is a surface area of the circle. The area of the circle being pi a square, this areas of the circle of radius a being pi a square. So if I identify this I S as the moment, similarly the dipole moment P which I identified as q multiplied by d, if I identify this I S as the magnetic moment, then the field expressions are very similarly at least with respect to 1 by r cube term they are looking very similar to each other.

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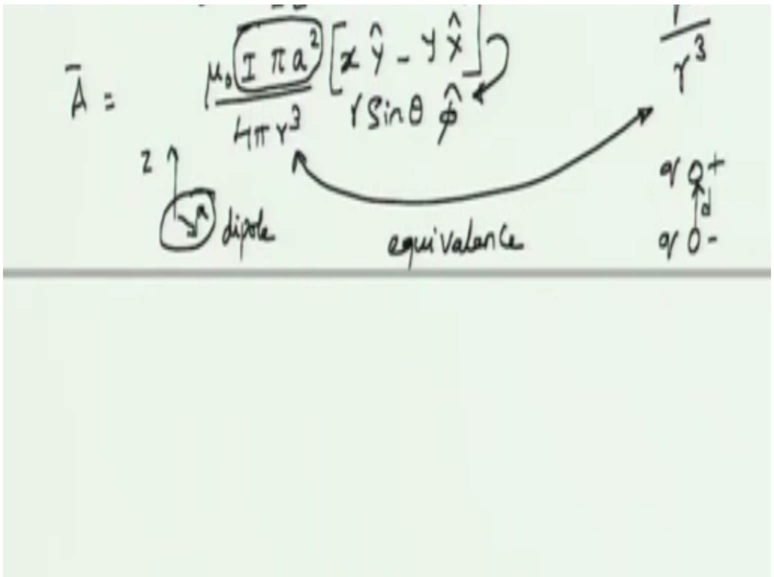

So this similarity is exploited in saying that, the loop of radius a which is producing fields at a far field is actually equivalent of electric dipole of equal charges but opposite polarity located at a certain distance d. So this is the equivalence between electric and magnetic dipole. Why should one care for such an equivalence?

Well, technically you don't have to care for an equivalence, you could do whatever that you want with magnetic statics that whatever you want to do you can do with electrostatics. If you recognize the equivalence though, then you can actually exploit the physical ideas that we employed in electro statics and carry over those physical ideas to magnetic matter. So that is the reason why we want to establish some amount of equivalence.

Okay, coming back to the vector potential A, we have shown that this is the vector potential, in the spherical co-ordinate system this term x y hat minus y x hat simply becomes sin theta, r sin theta in fact and it would be going in the direction phi hat. You can do this as an exercise of conversion from Cartesian to spherical co-ordinates, I would not really want to do that but if you are practicing some co-ordinate system calculation.

This is a good exercise for you to do. x y hat minus y x hat can be written as r sin theta phi hat, okay?

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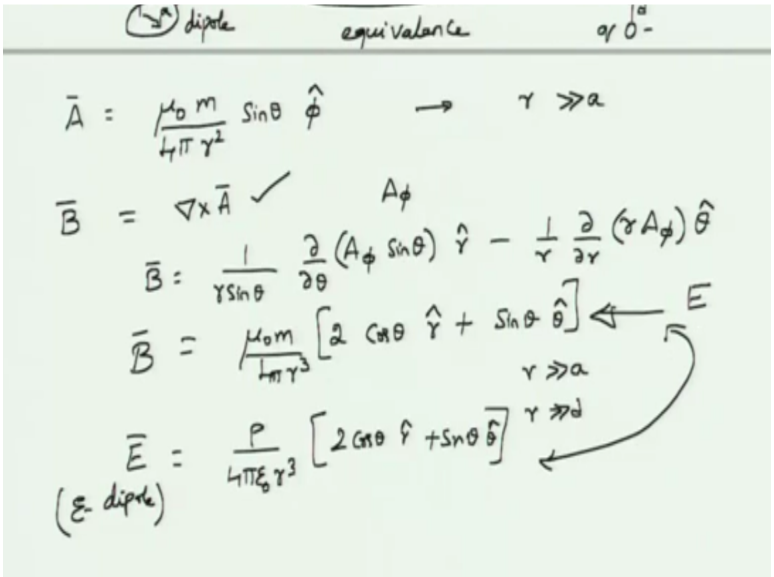

So with that, the expression for electric field becomes mu zero m where m stands for the magnetic dipole moment given by I multiplied by S, the area of the circle multiplied by the current I divided by 4 pi, so now what happens, because of this r, it becomes r square. So you have r square, sin theta phi hat. This expression is valid when r is much larger than a. Now this was the magnetic vector potential A that we obtained.

But I am not really looking for magnetic vector potential A y because I am looking at the magnetic field or the magnetic flux density B. So B is related to A by the curl of A, correct? This is the equation that we used to define A in terms of B or rather B in terms of A. So if we expand this curl of A in the spherical co-ordinate systems, you will see that, B will have, since A has only A phi component.

I have to retain only the terms that correspond to differentials of A phi and that will be 1 by r sin theta, del by del theta of A phi, sin theta, r hat minus, 1 by R, del by del r, r A phi theta hat, okay? So A phi you know which is mu zero m by 4 pi square sign theta and if you substitute for this A phi here and then carry out the differentiation what you will get is, mu zero m by 4 pi r cube, 2 cos theta r hat plus sin theta theta hat.

This is exactly the same expression that we obtained for the electric field of a dipole. When r is much larger than A that is if the observation point is much larger than the group radius or equivalently if the observation point was much larger than the dipole length this was exactly the same expression that we obtained. If you forget about these factors mu zero, here and epsilon zero that comes out in the denominator in the case of electric field.

If you forget those factors, what you get is essentially it is the same field and therefore if you look at the distribution of the field, they should be exactly the same. For reference, let me write down the electric field for dipole. So for the electric field of a dipole, electric dipole, that is not the magnetic dipole, this was equal to P by 4 pi epsilon zero r cube and in the bracket whatever the two cos theta r hat plus sin theta theta hat was exactly the same.

So this is the equivalence which again comes up and that is what we were talking about.

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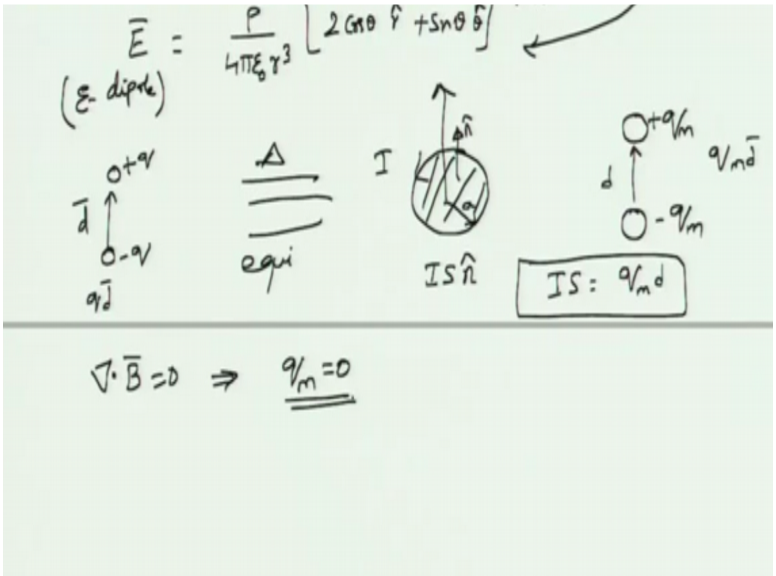

So because of this equivalence, this dipole, right, which is two charges of opposite polarity separated by a distance d is considered equivalent to a loop, so there are two points of view, first point of view is that, this is a loop of radius a and having an area S and carrying a current I. So the dipole moment here is q d and the dipole moment here is I S and if you want to make this q d as a vector then you would have to write down this as q d bar, where d bar was the vector which was directed from minus q to plus q.

Here there is nothing like for an area that I can do, for a minus point to another point, so rather than that I will write this as m hat, where m hat is the normal to the loop which ever direction the normal is, that would be the direction of the magnetic moment and the value of the magnetic moment or the magnitude of the magnetic moment will be I multiplied by S, S being the surface area of this loop.

Another equivalent point of view would be to imagine a charge plus qm and a charge minus qm and a distance d such that numerically I S is equal to qm multiplied by d. And call this as the dipole, magnetic dipole with qm d bar. This equivalence was probably appealing to you because there are charges plus q and minus q and similarly charges plus qm and minus qm.

Of course in the last module, one of the modules we showed that del dot B is equal to zero, implies that qm must be equal to zero, that is there are no magnetic charges or magnetic poles that you can isolate separately and find that out. However, this point of view as I said in that last module is extremely important in that sense that it gives you an idea of modeling the magnetic matter.

So this is one point of view that is widely considered. This is another point of view in which a dipole is actually a, dipole is that of a loop. And if you look at the fields of an electric dipole went something like this.

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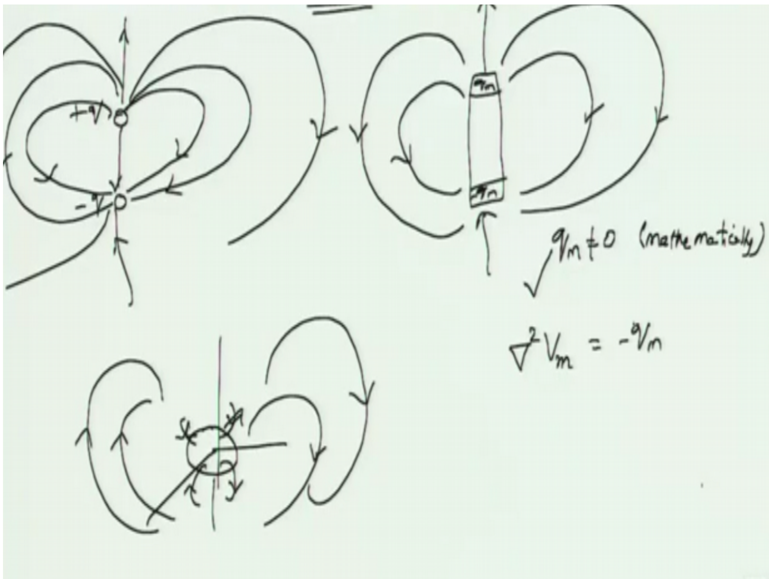

I am going to draw only the fields for the far away regions, so I mean I am looking at the fields of the far away region, the field was actually like this, right? It was symmetric around here. Sorry, these were supposed to be circles, I am slightly having difficulty in drawing the circles, this is the fields of an electric dipole. Similarly, if you consider the magnetic dipole as two charges plus qm and minus qm, physically you could think of this as a permanent magnet with North Pole and a South Pole.

The fields of this far away will be exactly equal to the fields of the electric dipole. So these are the fields of the electric dipole and this would look exactly the same in the far away region. The other one was the loop, the loop would also look the same way, right? So you have the loops here and if you look at the fields at the far away region, they will all look like this.

So if I have the current carrying like this, then applying the right hand rule, so here the fields would be, say if I consider the current in this way, the field lines will actually be going clockwise and here it would be going anti clockwise. So here you have fields which are going like this and fields which are coming out this way and if you interchange the current direction, then they would actually be the same way, okay, I think maybe we should actually interchange the current direction.

So the point here was that, right, if you were looking for a field at a far away region, then you would actually see that the fields would look exactly the same. So if you look at these two. There is absolutely no difference between the way the fields are present. So because of this we consider a loop of radius a as an equivalent of a dipole. We understand that physically qm is equal to zero but mathematically one can consider qm as non zero.

This is mathematically, so that we can establish that equivalence between electric dipoles and magnetic dipoles. Now because there is a magnetic charge qm which we have assumed, this is of course fictitious but an equivalent magnetic charge is what we are considering, then the corresponding equation for a scalar magnetic potential if at all that were there at the far away region that would exist, this would be equal to some magnetic charge minus qm or the magnetic charge density.