Electromagnetic Theory
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Lecture No 35
Biot Savart law & its application II

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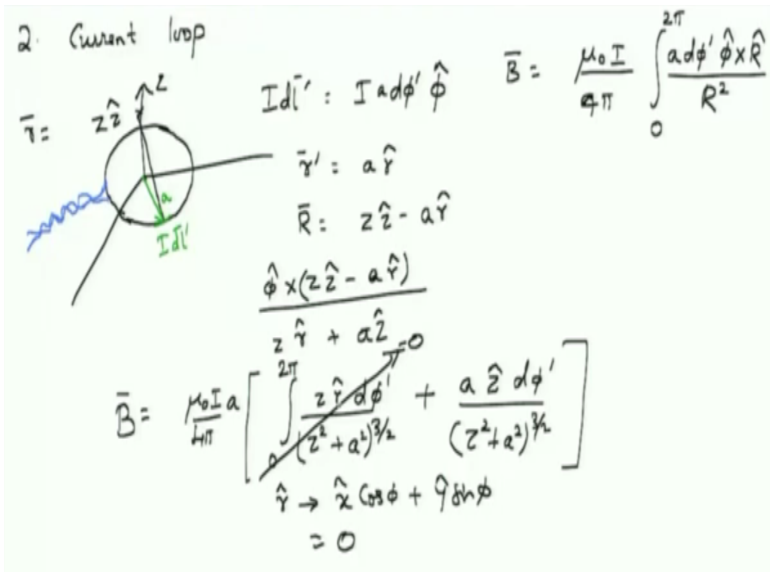

So we were discussing in the last module about finding the magnetic field or the magnetic flux density of a current loop and here is this half thing that we have done. So we have a current in the form of a loop of radius a. And then we wanted to find the field on the axis of this loop at a distance of Z from the loop itself. And we wrote down this integrals I dl prime was I a d Phi prime along the Phi axis.

And then the unit vector R or the vector R directed from the source point to the field point had both components of Z as well as r. And then you did this integrant and then you found that there are 2 integrals to consider. One integral involves the unit vector r and the other integral involves the unit vector Z with respect to the Phi being the variable of integration. And we were discussing that where there is an r hat but its radial term is actually zero.

We call you can one way of thinking about this is to see that r hat is equal to x hat Cos Phi + y hat Sin Phi. And Cos Phi and Sin Phi when you integrate from zero to 2 Pi. The area under those will be equal to 0. Symmetrically that would be equal to 0. So, the only integration that is left, which is not 0 is the second one, which is a Z hat d Phi prime / Z square + a square 3 / 2. Again you can actually do this integral.

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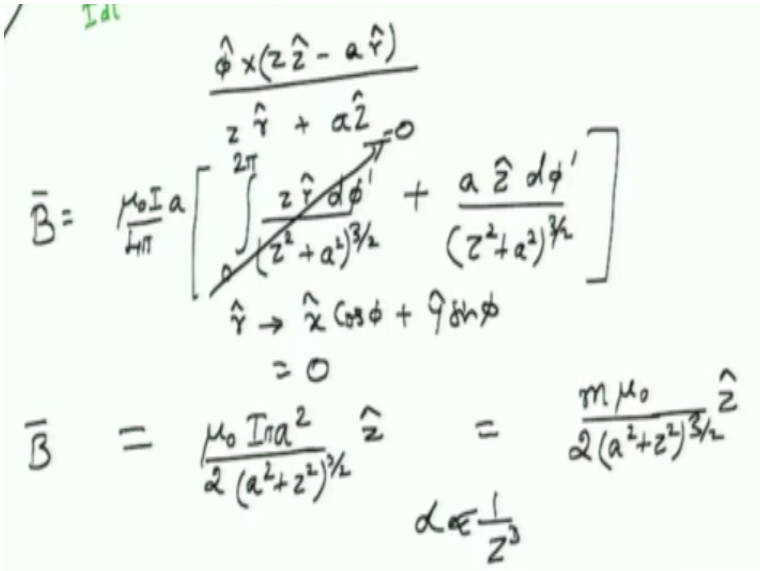

And simplify the resulting expression you are going to get I am writing the final expression for this one, you will get Mu naught I a square. Let me also write down this as Pi a square. I don't want to cancel the Pi. So you will actually see that there will be a Pi in the numerator and the denominator. But I don't want to cancel the Pi there. So, I get Mu naught I Pi a square divided by 2 a square + z square to the power 3/2 and the field will be directed along z axis.

Now, this type of an expression you have already seen earlier. You will soon see where you have seen that expression but before that let me write down this I Pi a square and Z hat as m Mu 0 / 2 a square + z square to the power 3 / 2, along Z hat. Where m is by definition in this case is I into Pi a square and what is Pi a square. Pi a square is the area of the loop. So area of the loop multiplied by the current carried by the loop is what is called as magnetic moment.

It is actually a magnetic moment and we will be discussing the magnetic moment, the force, the torque, all those things later. But this is something that I wanted to point out right here, because of a loop. When you look at the field at a far distance, you know at a very far away distance from the loop, you will see that the field actually goes as 1 / Z cube. So, as you go up and up along the

Z-axis you would see that the field goes as 1 / Z cube.

There was a situation where you had two charges. One charge +q and another charge -q and if you looked at far far away from the charge configuration, you found that the electric field was actually decaying as 1 / R cube. So, 1 / distance cube and similar thing you will see over here. So, the field on the axis is actually decaying as 1 / Z cube. And you can visualise the reason for this one by looking at the current carried in the loop and considering two pieces, which are opposite to each other.

Along one piece I have the current in the clockwise direction or the anti-clockwise direction and along the other piece of the conductor I have the current in the anticlockwise direction. So, you can think of these two as two short line segments are the current elements which are carrying current in the opposite direction and when you look at from the far far distance this situation is quite similar to the two charges of opposite polarity and you are looking at from the far distance.

So that is the physical idea as to why the field goes as 1 / Z cube or 1 / R cube in the dipole case. So this would actually be a magnetic dipoles type of a situation. So, you have the fields which are very similar to electric fields. One final example that I would like to consider here before we move on to next law is something called a strip of a conductor.

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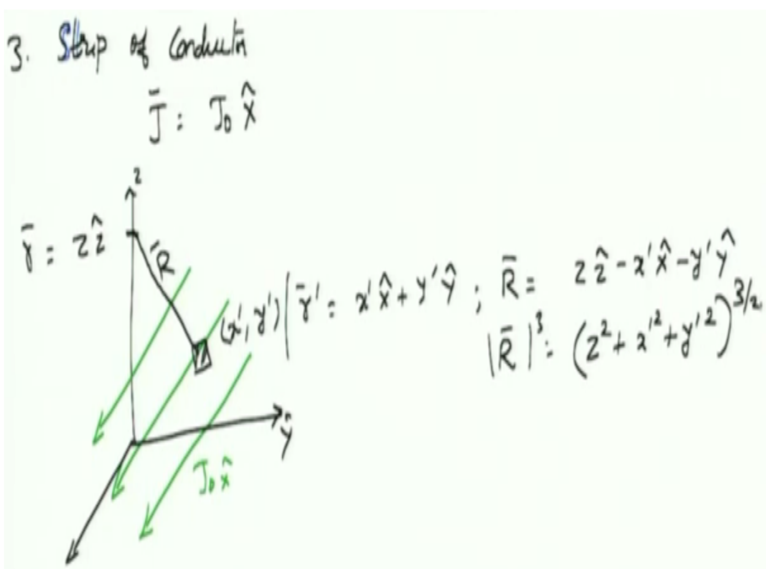

This strip of conductor comes up in various places. It could be the eddy currents carried by a machine or it could be the wall currents on a waveguide. And we want to find out the magnetic field because of a strip of a conductor that is carrying current or current density. A constant current density of J 0 along the x axis. So I want to find out the field at the axis point that is along the Z at a height of Z.

So, this would be my r vector just like the previous case, where I found out the field on the Z axis. I wanted to find the field here also on the Z axis. The strip is carrying current J 0 along the X axis and we want to calculate the field at this point. So let us select any particular point here in the XY plane. This point in the XY plane will be at a coordinates of x prime and y prime. Therefore, r prime will actually be equal to x prime X hat + y prime Y hat.

And from there you look at the vector that joins the source and the field points. That vector R will be given as Z z hat - x prime X hat - y prime Y hat. And the corresponding magnitude vector R the magnitude vector to the cube, which is what you want is given by Z square + x prime square + y prime square to the power 3 / 2.

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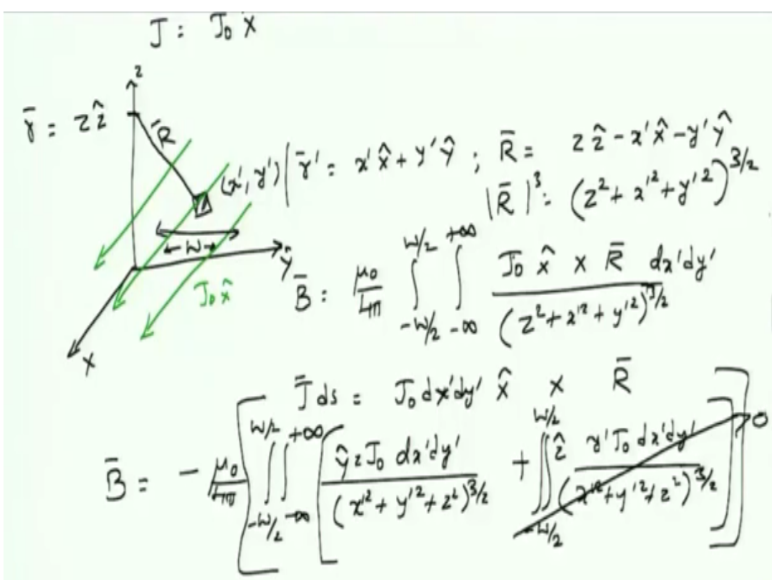

Now apply Biot-Savart law and write this as integral from some width over the Y direction. And for the X it would be from - infinity to + infinity. The current directions are all constant and they are all going from - infinity to + infinity along X axis. Whereas for Y, I am considering a strip of width W. And integrating over that W. And this Mu 0 / 4 Pi is going to be a constant.

So, I can push this outside and inside what I have is the current element itself will be along J 0 X hat and X hat * this R vector or the R bar vector, divided by and this is the integration with respect to X and Y. Because this is the surface integration with respect to X and Y plane and this is given by Z square + x prime square + y prime square to the power 3 / 2. I hope this integral is alright with you.

Because the current density J ds is actually given by J 0 dx prime dy prime, this is the surface element and it would be pointing in the direction of X because J is pointing in the direction of X. This is getting cross product with respect to the unit vector R or the cross product with respect to vector that joins the source and the field points. This is the numerator in the integral and I hope that you can identify that one clearly. So, when you calculate the cross products, you would actually see that this is going to be slightly tricky, not tricky it could be slightly tedious.

We will see that X crossed with respect to Z will be non-zero and it would be a component along Y. X with respect to X will be zero. X with respect to Y will be along Z axis. Therefore, what you get here is that B field is given by that Mu 0 / 4 Pi can be removed, outside the integral. This integration limits are still -W / 2 to W / 2 and - infinity to + infinity. So, I have two integrals, 1 integral pointing in the Y direction, the other one pointing in the Z direction.

So what are the integrands here? This is Y hat J 0, there is still dx prime, dy prime up here, in the numerator I have Z divided by x prime square, y prime square + Z square to the power 3 / 2 is as it is and then I have another integral which is along the Z axis. And that is y prime J 0 dx prime dy prime is the same divided by x prime square is the same thing. So this is, this denominator is actually the same. And there are two integrands up there.

So, if you look at these two you will see that y Prime is going from - W / 2 to + W / 2 and within that the sign of y prime is actually changing. The denominator is always a positive quantity but the numerator is actually changing sign which means that it is an odd function, over this y prime integral this is an odd function. Therefore, this integral will give you 0, there won’t be any contribution of that one and therefore that need not be evaluated.

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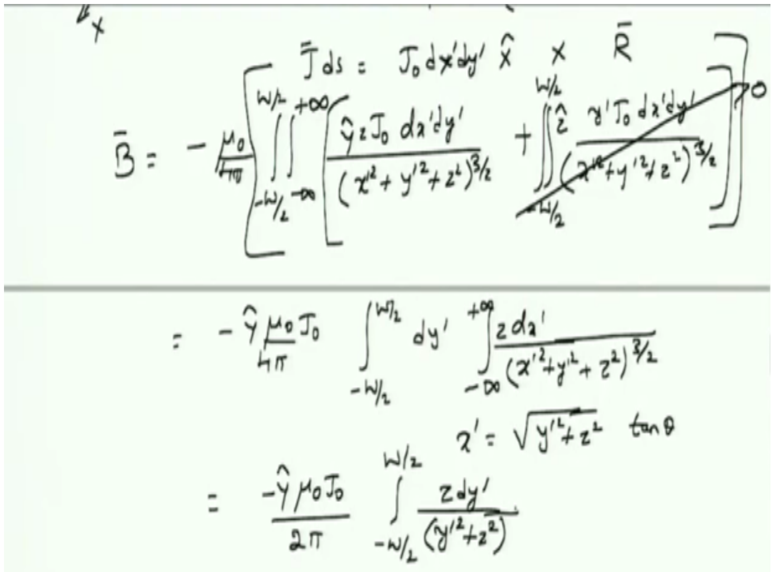

So, you are left with evaluation along the y hat direction and then there will be an integral, which you will see that is given by – Y hat (mu 0 by 4 pi) J 0 integral from – w/2 to w/2 dy prime integral from – infinity to + infinity dx prime and there is a z inside there divided by (x square prime + y square prime + z square) to the power 3/2. So, you could evaluate this integral. So, to evaluate this inside integral you will be substituting something is equal to tan theta.

So, you will be substituting x prime is equal to square root of (y prime square + z square) tan theta. So, you can find out and change the integration limits and then do all these things. So, effectively or eventually what you are going to get is this fellow - Y hat mu 0 J 0 divided by 2 pi integral from –w/2 to w/2, z dy prime divided by y prime square + z square.

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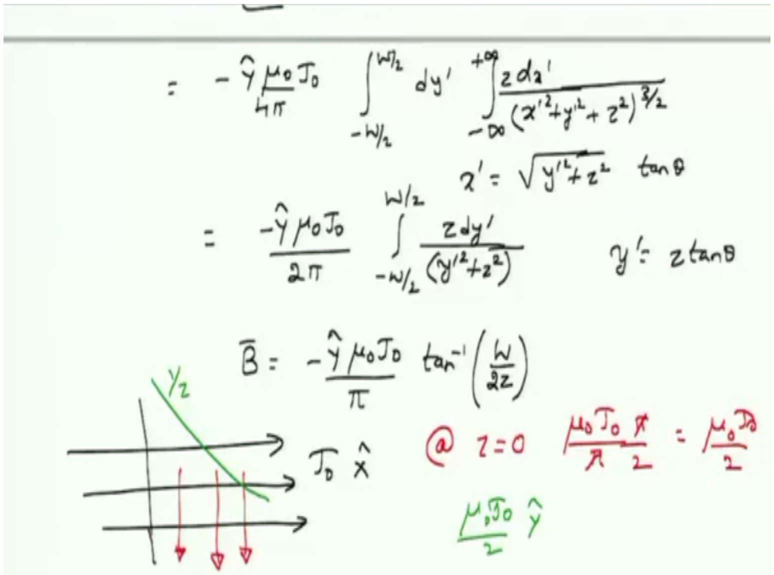

So, thankfully that integration with respect to x has vanished the x component here and you just have this second integral. Here again you can put y prime is equal to z tan theta. Change the limits of integration. Now the integral limits will be finite because you are integrating from – w/2 to + w/2. Appropriately you change the integrals and you will see that the B field can be written as – Y hat, what you will get is this one, mu 0 J 0 divided by pi tan inverse of (w/2 z).

So you will see that, you have seen that the magnetic field B because of the infinite current density along the X direction is actually directed along – Y direction. So, if you imagine this X plane direction, J 0 X hat as the current density vector, then the magnetic field is actually directed in the – Y direction at all these points. Of course for a given value of z these field lines would be constant but at different planes they would all change.

For example, right on the axis at z is equal to 0 what would be the value of the B field? At z = 0, tan inverse of infinity is pi by 2 and there is a minus sign up there so essentially you are going to get some constant value. So you are going to get the magnitude as mu 0 J 0 by pi * pi by 2 and so pi will cancel and you are going to get mu 0 J 0 by 2. So if you are coming from z = 0 from the top to the bottom at z = 0, you will see that the magnetic field has a magnitude of mu 0 J 0 by 2 and it would be directed along – Y direction.

And if you go up along the Z axis and eventually imagine going to the infinity point that is at z equal to infinity, the tan inverse of w by infinity will be 0 and you will get the field to be progressively going to 0. And while it does it is actually going to follow a 1 by z kind of a dependence on the top. Now from the bottom if you look at z is equal to – infinity, the field would actually be still 0 but if you come with negative values of z, the field lines B will switch their sign because the field lines will become mu 0 J 0 by 2 in the + Y direction.

So, if you come from top to bottom you will see that the field lines start at 0 at infinity and then gradually increase and then there will be constant value as you come to the z = 0 plane but at the bottom they would change the signs. They would actually be directed along the + y direction and they would decay towards 0 in the plane z less than 0.

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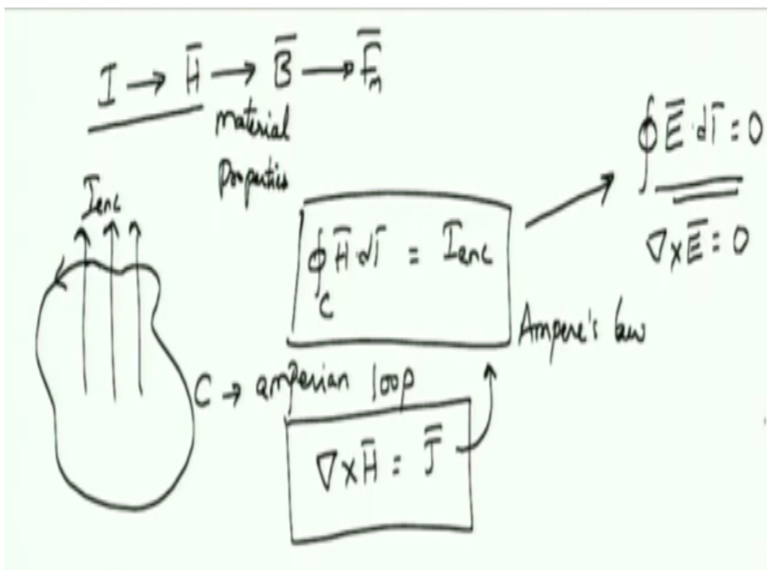

All right, this is something that we could use Biot-Savart law to calculate. Now in the spirit of discussion of magnetic materials, which we will be taking up sometime later, we will introduce you to another vector called as H vector. This is sometimes called as magnetic field intensity and it is measured in ampere per meter as we said and the reason to introduce this H field is this. We imagine that current actually generates the H field.

And this H field is related to the B field via the material properties. As opposed to electronic properties or the properties of the electrical materials, magnetic materials are much more exotic.

They have, they are highly nonlinear most of the times and they also have some sort of a memory in between them. This is called as hysteresis. You must have heard about hysteresis. So, this complicated relationship of the magnetic materials to the external magnetic fields that are applied is well captured by calculating the H field.

And then finding a model that you know in the electric case, dielectric case we modeled the dielectric as consisting of dipoles. Similarly, we will be modeling this as consisting of magnetic dipoles and then we will define a magnetization vector that will give us the link between magnetic fields inside the magnetic material as a response to the external magnetic field. So I generates H, H generates B and B applies the force on the charge or the current.

So, this is the sequence that we will be following when we want to find out the magnetic material effects, model magnetic material effects. So, in that spirit we want to introduce H and we want to find out what is this relation between H and I. In a series of experiments, engineer called Ampere showed that if you take a closed path, this path is sometimes called as an amperian loop or amperian path, in honor of Ampere, and he showed that if you take this path which is closed.

This is closed and then if you find out what is the magnetic field H around this, so you calculate the line integral of the magnetic field H around this closed path or the closed curve C you will find that this would be equal to the total current that is enclosed. So, if I enclosed is the total current that is enclosed by this curve and the path should be taken in a direction such that there is this nice right hand rule.

So the path should curve along itself such that the thumb points in the direction of the current that is enclosed. So, if you do that one the amount of current that is enclosed by the path will be related to the line integral of H. You should immediately contrast this one with the line integral of E field over a closed path. This was actually equal to 0 and because this was equal to 0, we also have this relation curl E is equal to 0.

Now here, you can immediately see that the point form of this integral relationship will be curl H is equal to the total current enclosed and because current enclosed cannot be a scalar, it would be current enclosed per surface area because that is what the definition of a curl is if you remember. So, this becomes the vector J or the current density J. So, this is the corresponding point form relation for the ampere's law.

Later we will see that this form of Ampere's law as we have written is actually wrong, because it does not apply in a most general case where time varying fields is considered. This is the case where only time I mean where the fields are not varying with time and in that case, this law is valid or this law is all right. So, this line integral of H dot dl must be equal to the total current enclosed and this is known as Ampere's law.

And this law is useful when you want to calculate the magnetic field H at least in those cases where there is some sort of a symmetry.

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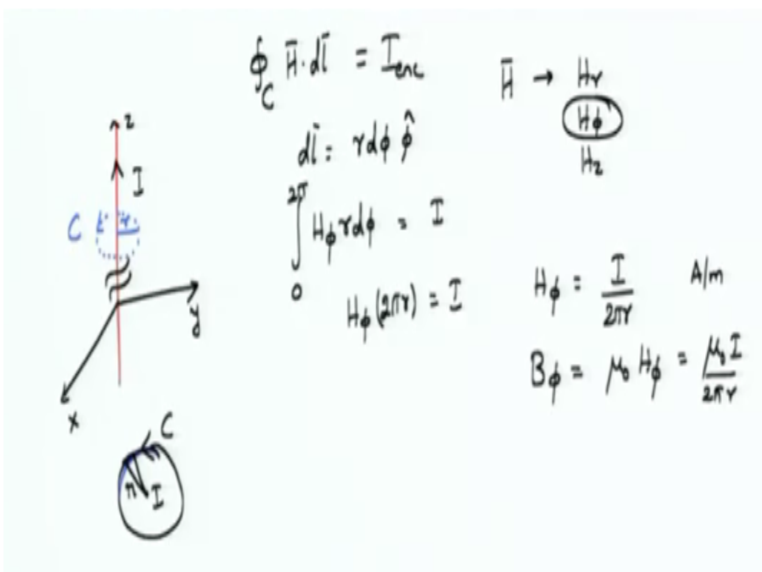

Let us assume that we have a wire which is carrying a current I. This current I is supposed to be uniform and the wire extends all the way from – infinity to + infinity. Let us orient this wire along the z axis of the Cartesian coordinate system. So, I have the z axis along which the wire is extended or the wire is placed and the wire of course goes all the way from z equals – infinity to + infinity.

The wire because of this there would be a magnetic field and what we are interested is to find the magnetic field that surrounds this wire using ampere's law. So, remember ampere's law is consider any particular contour C and the total magnetic field if you integrate the magnetic field, tangential magnetic field that if you integrate over the contour that must be equal to the total current that is enclosed by that loop or the contour.

Now looking at the problem it is quite obvious that we need to involve cylindrical coordinates systems because the wire is extended along the z axis and it is very easy to consider contours, which are all circles of radius r and surrounding this z axis. So for example, one might consider a contour that would look like this; you know this is the contour that I am considering. This contour has a certain radius r and phi of course goes from 0 to 2 pi because you are surrounding that wire.

This is also happening at a constant z plane. So if you cut this z plane and then look at the top view of this one, this is what you would find. So, the top view would show you the contour and then there is a central piece of conductor for the wire which is carrying a current, uniform current of I. Now, for this contour we know that in cylindrical coordinate system the line element at a constant z and a constant r; remember this is a constant r that we are considering.

So, for constant z and r, the line element dl will be equal to r (d phi) phi hat. Of course it is the phase phi I mean it is the angle phi which is changing and your dl line segment will be directed along phi hat. Now, when you put this one into H, remember that now H can consist of H r or H phi or H z component but because you are taking the dot product of the H with respect to dl and dl is pointing along phi, out of these three components only H phi would meaningfully contribute to the integral.

So, that close loop integral in which phi goes from 0 to 2 pi and then H has only the phi component can now be written in this way and what is the current that is enclosed? The current enclosed is simply I. So, now r is constant in this integral. Remember it is phi which is only varying and if you pull this H phi assuming that H phi is constant because of symmetry we can show that H phi is going to be constant.

So, H phi * 2 pi r, 2 pi r being the loop circumference that should be equal to the total current I that is enclosed, which also gives me H phi to be I by 2 pi r. Does this result make sense? If you were to fix r and then look for the magnetic field, the magnetic field we know from right hand rule that it has to curl around the current carrying element. So, if the current is directed along the z axis and you place a constant loop of radius r, then the magnetic field around this must also form circles.

You know magnetic fields must form circles around the current carrying wire and therefore they are given by I by 2 pi r. So for a given value of r this circle amplitude must be constant. Once you know what is H phi? It is a very simple matter to find, what is B phi? B phi will be mu 0 times H phi and this is given by mu 0 I by 2 pi r. Remember that H has units of ampere per meter. So this is what we essentially have used ampere's law in order to obtain the magnetic field.

Let us consider a different example now, something that is slightly more complicated than this. The example consists of circular symmetric distribution that is the current distribution is still circular and cylindrically distributed but it is no longer confined into a thin wire.

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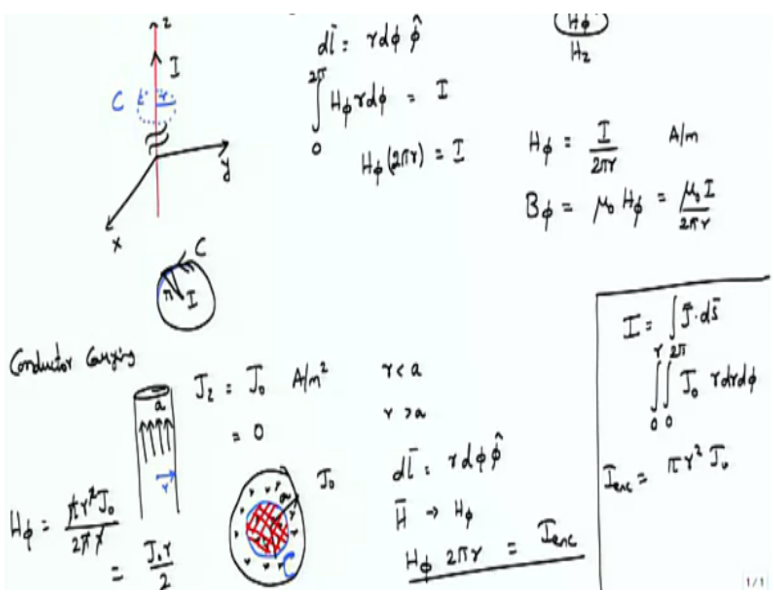

What it actually has is, now a solid wire of radius a. So, we will assume that you have a thick conductor of radius a, but we will also assume that the current is everywhere uniform, that is inside the current everywhere is uniform. So, if you look from the top view for this example, so this is actually the conductor that is carrying current. We will assume that the current itself is uniform or more precisely one can assume that the current density vector J that is going to be uniform.

So let us in fact instead of assuming I, let us assume that J inside the material is going to be uniform and it is again oriented along the z axis and is given by J z is equal to some J 0 so where J 0 is the current density measured in ampere per meter square and it would essentially be constant and equal to J 0 for as long as r is less than a. So what is r here? 'r' is simply this, the distance from the center to the conductor itself.

Of course in this way, we have considered 'a' to be the radius of the conductor. So as long as you are inside this conducting wire, the current density is going to be uniform and is given by J 0. Outside of course, this would be equal to 0. So, the current density vector outside will be equal to 0. Of course such drastic current changes are not actually permitted, but we are anyway considering the ideal situation of having a wire that is carrying the current.

We will also assume that the wire goes all the way from – infinity to + infinity. Again the idea would be to try and apply ampere's law in order to find out the magnetic field. Let us do one thing. Let us first find out what the magnetic field would be. If I were to choose a contour that is inside, entirely inside the wire. So going back to the top view for me, this particular would be the contour. So, now I have to consider two contours, that inside the wire and outside the wire.

So, considering first for the inside case, where the wire itself carries, this is the wire, the black colour thing is the wire. And remember this wire is not hollow it is actually solid, it is completely filled and everywhere you have the current density J that is coming out and it could actually be uniform and is given by J zero. So, this radius is a and we are considering inside radius r, in order to apply this ampere's law.

So inside r, first of all we need to find out what would happen to the left hand side of the ampere's law and right hand side of the ampere's law. The left hand side of the ampere's law, if you stretch your imagination slightly, would not really change because the contour is still given by the line element dl going along at a particular radius r, but changing its value of phi.

And H is still going to be H phi because the magnetic field has to be circling around the wire. What would change is the right hand side? So the left hand side does not change. You still have H phi * 2pi r. But this should be equal to the total current that is enclosed. Now, what current is enclosed? In order to find the current enclosed, you need to find out the relationship between current and current density J.

We already know that relationship. So, current is given by integral of J dot ds. What surface should I consider ds being the surface integral? This is the surface that I have to consider, which is bounded by this contour C. So, this is the contour that I have. And binding that contour will be my open hatched area, which I have shown here that would form the surface.

Now from this surface, what would happen to the current density J or to what is the total current enclosed? J is constant, it is given by J zero. But, what is the surface area for this? Remember this contour is being taken at a constant z. But, the surface element in the cylindrical coordinate must point along z itself. So, the surface area element is given by r dr d phi and it would be pointing along z direction.

So, integrate this one over the two limits, r will be from zero to r itself and phi will be from zero to 2pi and you can show that when you evaluate this particular current, you can show that this would be equal to pi r square, which is the area of this particular hatched area times J zero. This would be the total current that is enclosed, if you are inside, your contour is inside here. Now what will happen to this H phi * 2pi r is equal to I enclosed, well.

The magnetic field H phi will be equal to the current enclosed, which is pi r square J zero divided by 2pi r, pi cancels. One of the r in the numerator cancels with the r in the denominator, giving you H phi of J zero r/2. We see that the magnetic field H phi is a linear function or it is actually increasing linearly from the value zero at the center, where r is equal to zero and then it gradually goes up as r keeps on increasing, reaching a value of J zero a/2, as the contour expands to the outer radius a.

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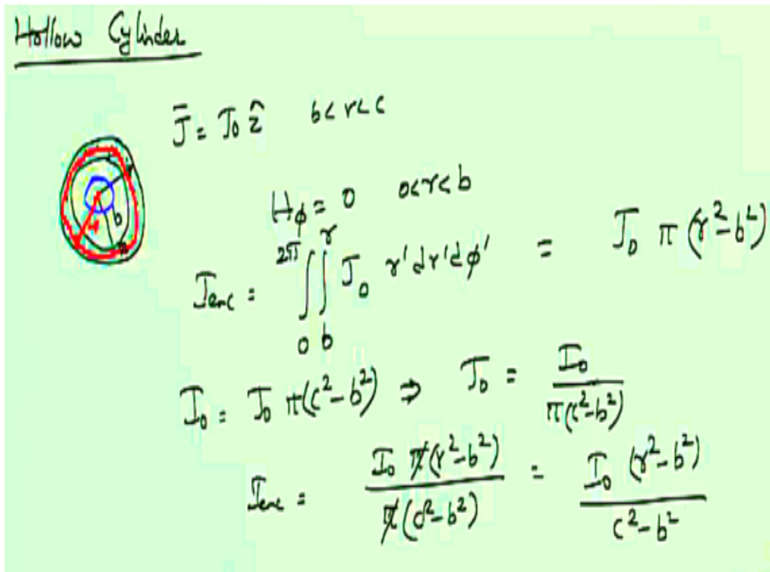

Let us look at the third and final example of what is called as a hollow cylinder. Two such hollow cylinders will actually give us a coaxial cable. So, we will not discuss coaxial cable now because we want to keep that one for the inductance calculation. But before that, you need to know how the field of a hollow cylinder works. So, what is a hollow cylinder? So, if you look at from the top view, we have a cylinder with effective radius is c minus b, where c is the outer radius and b is the inner radius.

There is a current, uniform current everywhere or current density everywhere, which is along the z direction let us say, so J is equal to some J zero z hat, in this region between b to c. And now you need to find out what is the magnetic field around this hollow cylinder. If you take one loop inside of any radius, as long as this radius is less than b, you are not enclosing any current.

Therefore, H phi will be equal to zero, as long as the radius is less than b. Now at b, you are, that is after b, you are, let us say you are at a distance r, this is the distance r or the radius r, such that r is between b and c. r is greater than b, but r is less than c. What would be the amount of current enclosed here? To find the current enclosed, you need to find out the integral of J.

So, the surface area that you are going to consider will again have r prime, dr prime, d phi prime as a surface element. Over that surface, at all the surfaces J is constant, it is given by J zero. And the appropriate integration limits are zero to 2pi for phi and b to r for r. Why b to r? Because zero to be has no contribution. So, zero to b plus b to r. So you have the integration limits from b to r. So you do this integration, you are going to get J zero * pi r square minus b square.

So, this is the current that you are going to get here. What would be the total current coming out of this surface? The total current is actually from when r is equal to c. That would be total current I zero is equal to J zero into pi c square minus b square, which implies that I can replace J zero as I zero/pi c square minus b square. So I can write down the enclosed current as I zero pi r square minus b square divided by pi c square minus b square. So, pi cancelled from numerator and denominator. The current enclosed is this one.

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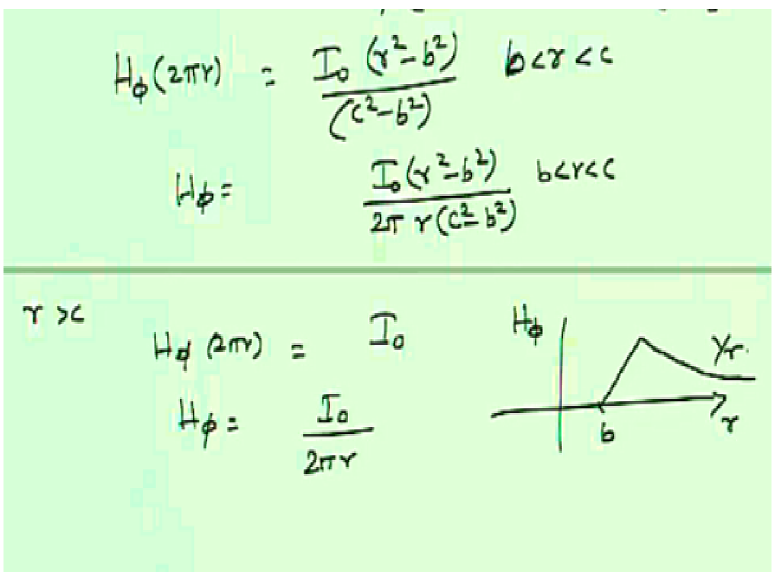

What about the left hand side? Well for the amperian loop that you are considering inside. The amperian loop will be H phi 2pi r. Again H will be directed in the phi direction and the left hand side would be 2pi r. So, this will be equal to I zero r square minus b square divided by c square minus b square, as long as r is between b to c. So, 2pi r can be brought down here in the denominator and what you will get is H phi equals I zero r square minus b square divided by 2pi r c square minus b square, so for b less than r less than c.

Of course, when your amperian loop is having the radius greater than c, then H phi into 2pi r will be the same. But, the total current enclosed will be I zero. So, H phi can be written as I zero/2pi r. So if you sketch, you will see that, until the radius is b, so if you sketch the magnetic field H phi until this radius b the magnetic field will be zero and then it begins to raise, not exactly linearly because the reason r square minus b square divided by r, but it raises with respect to some curve and then it reaches its maximum and then starts to drop as 1/r. So, this is how you would calculate the field of a hollow cylinder.