Electromagnetic Theory
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Lecture - 32
Solution of Laplace's equation-IV

In this module, we will discuss Laplace's equation, solution of Laplace's equation in two dimension plus we will consider Poisson's equation, solution of that in one dimension.

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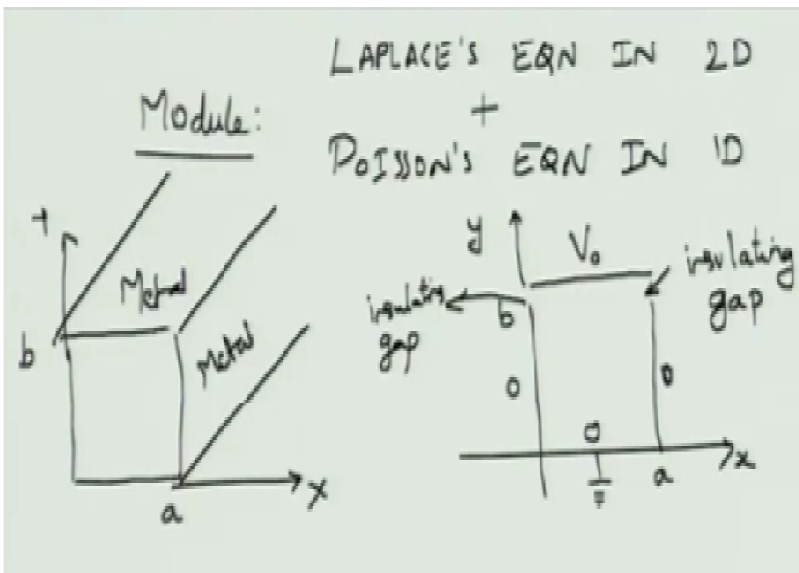

So, let us begin by looking at a very typical problem in two dimension Laplace's equation. Consider a metallic trough or a rectangular cube if you would will. So, we will consider that the tube of this rectangular piece is actually along, it has this dimensions of 'a' along x and dimension of b along Y. So let me mark down the corresponding axis x and y. And the tube actually extends all the way towards z direction.

We are not concerned with what happens along the z direction. We are only concerned with what happens with x and y direction. Therefore, for us, we could have a simpler equation, which would be valid in two dimensions x and y. And the functions that we are going to calculate will be independent of z. We will assume that in z, nothing is really happening. So, we have this metal here. So, this all four sides are actually metal.

But between this, the top surface and the other surfaces, there is some amount of insulating gap. So, we fill this one with an insulating gap, so that when we hold one of the surfaces at a particular potential, then the potential will be different from the other surfaces. So, you have an insulating gap here. The rectangular trough itself goes, has a dimension of ‘a’ cross ‘b’. So it has dimension of ‘a’ along x that is length along x and b along y.

This is my y axis and this is the x axis. So, we have 4 surfaces of the metal to consider and the top surface of the metal, which is actually, I am showing it with an exaggerated gap here. You can imagine the gap to be very very small that we can consider this distance, the top surface to be from the bottom surface to be at b itself. So, this figure of showing you an insulating gap is just to show that there is actually a gap out there.

So, we hold the top surface at a particular potential, so let us call it as some V 0. The top surface is held at a potential V 0, while the bottom 3 surfaces are grounded. So, these are all at 0 potential, 0 potential at 0 potential. What we want now is, what is the potential inside this trough? So, clearly to calculate the potential inside this trough, we need to solve Laplace’s equation. We need to solve Laplace’s equation in 2 dimensions.

This particular problem that we are going to discuss is important because this illustrates a very important technique of solving 2 dimensional equation and 3 dimensional equations, later which we will be using them when we solve for wave guide modes. Moreover, this is a sort of a canonical problem that will be considered when we want to discuss the numerical solutions of Laplace’s equation.

So, for these two reasons, this problem that we are going to discuss in this module is actually very important.

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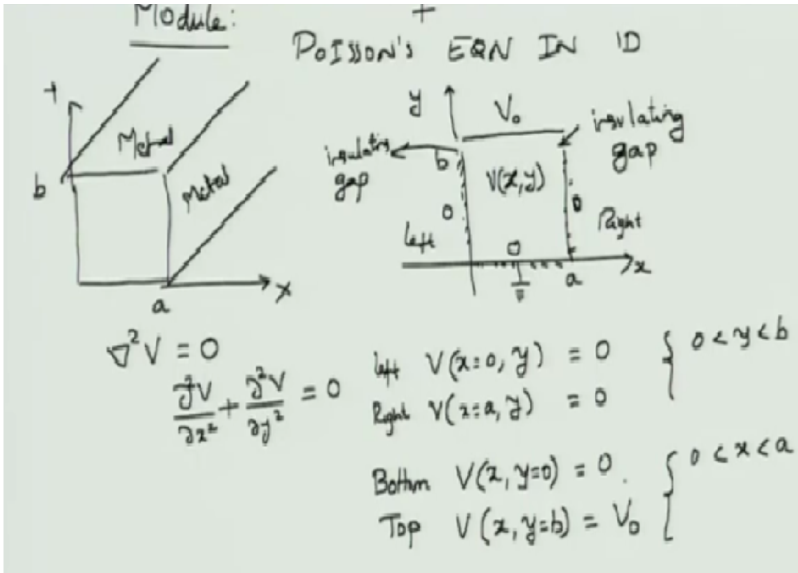

Now, this is our problem. To reiterate what the problem is, we have a metallic trough. So, you could identify the metallic trough over here. It has 4 surfaces. Three of those surfaces are held at a particular potential of 0 volts, the top surface is held at a potential of V 0. It is a constant potential. This is the potential that is held and what we want is potential inside, which will be a function of both x as well as y co-ordinates.

To solve that we need to recall what is Laplacian equation. Laplace's equation for the potential V is Del square V equals 0 as long as you are within the interior of this trough. And Del square V in Laplacian Cartesian co-ordinate system can be expressed as Del square V / Del x square + Del square V / Del y square =0. There is another term Del square V / Del z square. But that is not important for us because our potentials are independent of that z co-ordinate.

So we have this two dimensional Laplace's equation that we need to solve, subject to certain boundary conditions. So what are the boundary conditions that we have here? We have four boundaries and we can write down four boundary conditions over here. So the first boundary condition is what happens at x=0 and at x=a walls? So, we can think of these four surfaces as sort of four walls.

And what happens to the left wall and the right wall is that, the potential there at x=0 and for all values of y, the potential is equal to 0. This is the left wall. So this is the left wall here. Of course, y needs to be between 0 to b. So it is in this region, we are applying Laplacian equation. Therefore, there is a limit for y. But at x=0, the left wall no matter what point you pick on the wall, the potential is going to be 0.

Similarly, what would happen to the right wall? On the right wall, you have the same boundary condition that at x='a' for all values of y, this will be equal to 0 as long as y is between 0 to b. So consider now, what are the boundary conditions for the bottom and the top walls. For the bottom wall the potential is still equal to 0. See that the bottom wall is grounded, the potential there is described by whatever value of x between 0 to 'a', but y=0 on the bottom surface.

So this potential is also equal to 0 as long as x is between 0 to 'a'. The limit for x is between 0 to 'a'. So there remains only one additional boundary condition that we can apply, that is the potential of the upper surface or the upper wall. And the boundary condition there is for all values of x but at y = b, remember we said that insulating gap is going to be very small. So at y=b the upper surface is located but the potential at all the points along x is equal to V 0.

So this is again the same condition that x has to be between 0 to a. Clearly if V 0 = 0, then the potential everywhere will be equal to 0 and there is nothing there to solve the problem. So our problem, which is the physical problem that of having a metallic tube or in the two dimensions a metallic trough having the dimensions 'a' cross b with the appropriate boundary conditions of 3 walls being grounded.

And one wall being held at V 0 potential reduces to solving a mathematical equation known as the second order partial differential equation subject to four boundary condition that we have here. The left and the right as well as the bottom walls have the potential 0 and the top wall has the potential V 0 or V not. So how to solve this type of partial differential equation? There are many ways of solving partial differential equation.

Unfortunately, there is no general way of solving these equations.

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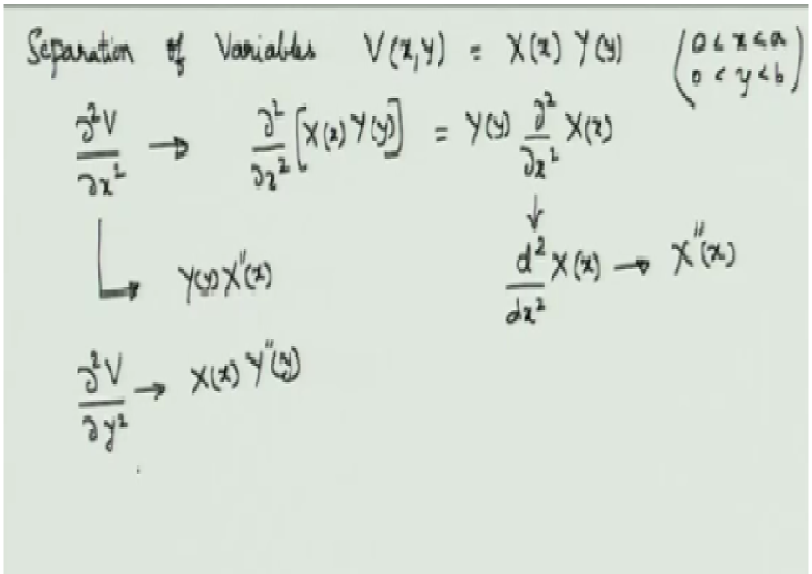

So one solution actually is what is called as separation of variables. What are the variables that we have? We have two variables here, x as well as y. So, the separation of variables technique starts like this. It assumes that the potential V (x, y) which is what we want to solve for is given by a function of x, which is traditionally represented as X (x) and a function of y multiplied together.

So, wherever x as well as the region for y are there, we are going to assume that the potential can actually be represented as a product of two functions each of which by themselves are functions of individual variables alone. So, you have X(x) and Y(y). You could have of course written this as f(x) and g(y) it would not have mattered. What matters here is that this X(x) is purely function of x. And Y(y) is purely a function of y.

Now, with this assumed solution, we are going to put this solution, the assumed solution into this partial differential equation. So, first consider what happens to this Del square V/ Del x square. So, this becomes Del square / Del x square (X(x) Y(y)). Now, Del square/Del x square is actually partial derivative with respective to x co-ordinate and we know that Y(y) is constant as for as x is concerned. Because, Y(y) is independent of x.

So, it is just a constant, so you could move out this Y(y) out of the derivative. So, this becomes Y(y) and then differentiating partially X(x) twice. Now clearly this differential is not, is no longer a partial derivative because X is just a function of ‘x’ alone. It is not a function of ‘y’. So I can replace this partial derivative by total derivative and I get d square / dx square X(x). Of course, I don't know what is this d square / d x square of X(x)?

So, I am just going to leave it at this point. But there is a shorthand notation that we are going to employ for this and that is X double Prime(x). This double Prime indicates 2 times you have differentiated the function X(x). Now so what happens to this Del square V / Del x square. What you got here is Y(y) times X double Prime(x). Similarly it is (()) (11:04) not any more difficult to show that Del square V / Del y square becomes X(x) but Y differentiated twice.

So this is what it happens. Now you go back to the full partial differential equation that we were solving. The partial differential equation is the sum of Del square V / Del x square and Del square V / Del y square is equal to 0. Inside that region, where we are considering the Laplace's Equation, this sum of these two terms must be equal to 0.

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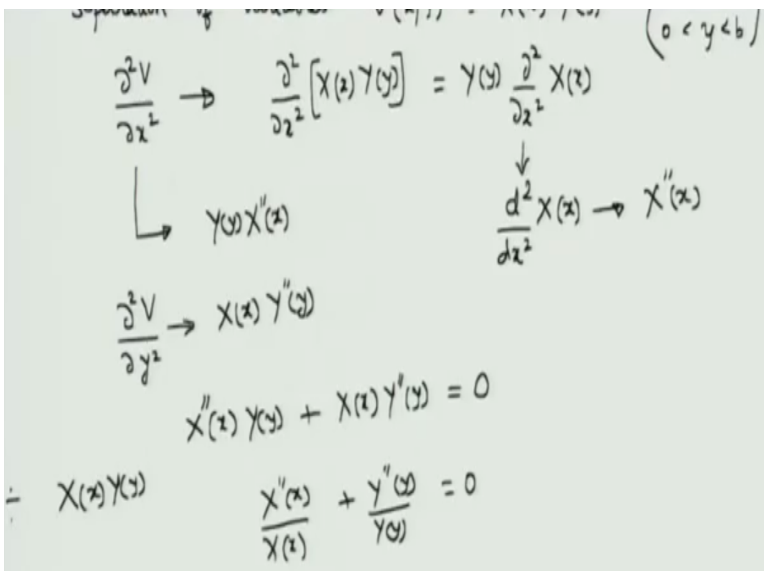

So you have X(x), you can write this as X double Prime(x) Y(y), which is Del square V / Del x square + X(x) Y double prime(y) is equal to 0. Now we are going to do an important operation which our mathematical friends would certainly not agree upon. Without really going in to the details of why we are doing this? What we will do is we will simply divide this equation by X(x) in to Y(y).

Now clearly this is not at the beginning it does not any seem like a good idea to do this because it might just so happen that X(x) at a particular point x and Y(y) at a particular point y could be equal to 0, which means that we are dividing 0 / 0 and that would be really absurd. But without really bothering about that small detail, let's go ahead and divide this and see what we get. So we divide on both sides by this quantity X(x) Y(y).

So when I divide this one from the first term Y(y) vanishes, from the second term X(x) cancels out. When I said vanishes, I meant cancels out and what you get is X double prime (x) divided by X(x) + Y double prime (y) divided by Y(y) is equal to 0.

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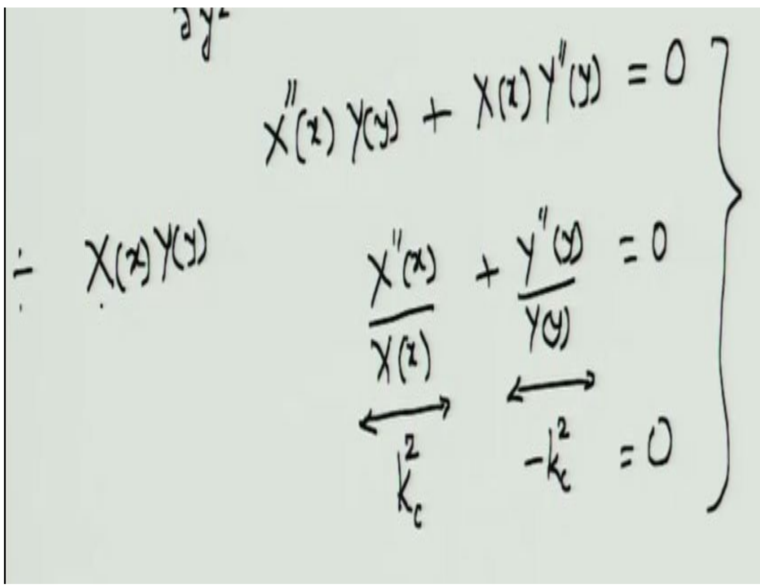

Now here is where, our second important observation that needs to be made. What can you say about the nature of this term X double prime (x) / X (x). Clearly no matter what value of x you choose since the sum of these 2 must be equal to 0. And moreover, x will be independent of y. In essence or in effect this X double prime (x) / X (x) must be a constant. It should be a constant let us call this as K c square.

So let us call this as K c square and this is essentially a constant. What will happen to the second term here? No matter what value of y you take, the sum of these two must be equal to 0. And we have just seen that X double prime / X is actually a constant. So, Y double prime / Y must be another constant. And that constant will be equal to - K c Square. Why should it be - K c Square? Because only then the sum of these two will be equal to 0.

So this is very important, please note what we have done over here. The first instant we reduce the partial differential equation by the separable variable method. And then divided that by the assumed solution X (x) Y (y) to eventually arrive an equation in which both the terms that form the sum are actually constants. So, since these are constants and these are second order ordinary differential equations there are solutions for this.

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$$\frac{X''(x)}{X(x)} = k_c^2 \Rightarrow \frac{d^2X}{dx^2} = k_c^2 X \qquad k_c^2 > 0 \qquad k_c^2 < 0$$

$$k_c^2 > 0 \qquad A \cosh k_c x + B \sinh k_c x \rightarrow A \exp(k_c x) + B \exp(-k_c x)$$

$$\frac{Y''(y)}{Y(y)} = -k_c^2 \Rightarrow \frac{d^2Y}{dy^2} = -k_c^2 Y$$

$$C \cos k_c y + D \sin k_c y \rightarrow C \exp(+jk_c y) + D e^{-jk_c y}$$

The equation that you want to first consider is X double prime (x) divided by X (x) is equal to K c square which simply implies that d square X / d x square is equal to K c square X and we know how to solve this equation. Now, it actually of course depends on whether K c square is positive or K c square is negative. If K c square is positive, the solutions for this would be in the form of hyperbolic Cos x + hyperbolic Sin x or equivalently it would be in the form of e to the power or exponential + or - x with some constant K c of course in between.

So, the solutions will be in the form of either hyperbolic Cosine and Sin functions or it would be in the form of increasing and decreasing exponentials, the super positions of these with appropriate constants that are multiplied. So let me just write down the constants also here. So, you have hyperbolic Cos of k c x, hyperbolic Sin of k c x or the solution here could be exponential of k c x + exponential of - k c x with certain constants A , B.

Similarly, you have A and B over here. So this would be the condition when k c square is a positive number. So, if you assume k c square to be a positive number, these are the solutions for X (x). Now, if k c square is positive the second equation becomes Y double prime / Y. This was equal to - k c square and therefore this term on the right hand side will be negative. Because k c square is positive, - k c square is negative.

This implies that the solution of this equation, which can be written as second order ordinary differential equation in y equals - k c square y has a solution of some Cos k c y + Sin k c y. Or you could also have the solutions in the form of say some C exponential - j k c y and let's put a + over here and then D exponential of - j k c y. So you could have these two solutions in the form of exponential functions or you can have them in the form of hyperbolic Cosine and Sin and Cosine functions.

If k c square is negative to begin with, well then you just interchange the hyperbolic Cosine with Cos and hyperbolic Sin with Sin and then you will be good to go. So, to just briefly remind you what we were doing. We had this metallic tube. The top surface of the metallic tube was held at a potential V 0. The other three surfaces were held at ground potential. The Laplace's Equation in two dimensions is this, Del square V / Del x square + Del square V / Del y square is equal to 0 subject to boundary conditions and we applied the method of separation of variables.

We assumed a solution in the form of X (x) Y (y) and then we substituted that, divided the resulting expression by X (x) into Y (y). And then we found that each of those terms which are functions of “x” alone, functions of “y” alone. If they have to be equal to 0, the sum of them have to be equal to 0. Then better both the terms be equal to constants. And if one of them is a constant k c square then the other term must be equal to - k c square.

So, that the sum is actually equal to 0. This reduces our second order partial differential equation into two second order ordinary differential equations in “x” and "y". And the corresponding solutions depend whether you have assumed constant k c square as positive or negative. //ok//. So, these are our solutions. We will assume the same thing.

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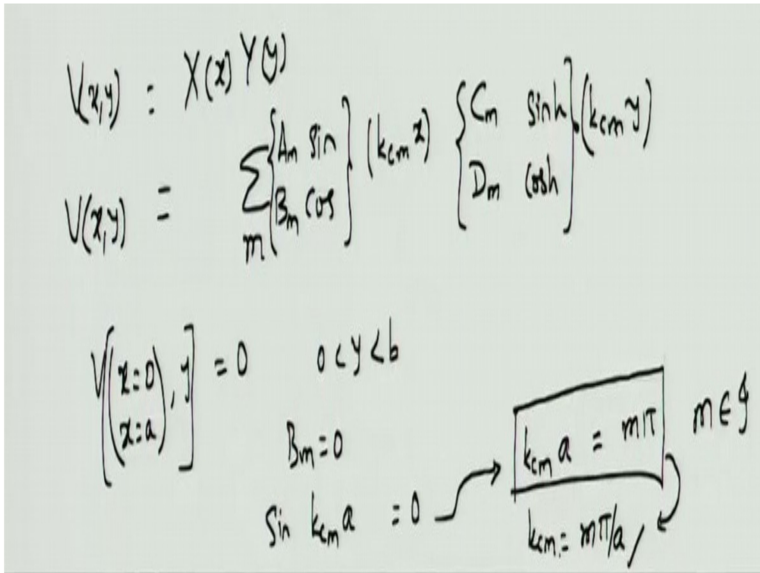

So, if you now want to reconstruct what the original function V (x, y) was we know that this must be equal to X (x) and Y (y). So, that would mean multiplying each of these assumed solutions. So, let me write down that and you can actually have this solutions, not just you know, one solution there could be multiple values for this argument. So the most general way that I can write down the solution for this is to write this as Summation over m hyperbolic Sin hyperbolic Cos and then k c m x.

With an appropriate constants that would be A m and B m multiplied by C m and D m Sin of k c m y Cos. So, this is a shorthand way for me to represent the fact that, the solutions are hyperbolic Sin and hyperbolic Cos, Sin and Cos our functions of k c m x and k c m y. So, if as I said k c square is negative, you simply have to interchange the Sin and Cosine orders. So, this is a solution that we have.

Of course, we have not really solved anything here we have although obtained a close form solution, it seems. We still need to evaluate the constants A, B, C and D. To evaluate these constants, we have to make use of boundary conditions. So let us first consider, the boundary conditions which we had at the constant walls, you know, that the left wall as well as the right side wall.

So, you have V of x equal to 0 or x equal to ‘a’, at for all values of y the potential function was 0 as long as y was 0 to b. So you could actually now substitute what happens to this. When you substitute “x” equal to 0 in this expression, you will immediately see that Sin, well this term doesn't really change anything because you are substituting “x” equal to 0. So what you get here is hyperbolic Sin of h of 0 and hyperbolic Cos of h of 0 are actually 0.

Let's just do one thing so we have assumed k c square is greater than 0. Let's assume k c square is less than 0 that is negative. So, what will happen is these solutions will simply switch. Sorry, about this small confusion it does not really matter which one you assume. But for some reason I would like to take k c square as negative, so I just have to switch Sin and Cos terms. So let me switched here and become Sin Cos, hyperbolic Sin and hyperbolic Cos.

So, these are the solutions that I have. Because I really wanted to bring out this k c m kind of x solution. You will see why I had to switch this around very shortly. So, because the point is that at “x” equal to 0, the left wall at “x” equal to a. The right wall the potential would have to be equal to 0. And if you assume hyperbolic Sin and hyperbolic Cos for functions of “x” It would have been little difficult to get that boundary condition. Rather than it is easy to get the boundary condition when you know that it's a Sin type of function.

So, you can kind of guess that the solution has to be a Sin type of function because the solution for Sin can be made to go to 0 at the boundary points by choosing appropriately the value of k c m. So that’s the reason why I switched between Sin and hyperbolic Sin for “x” and “y” functions. Coming back to boundary condition apply “x” equal to 0, you will see that Sin of 0 will be 0 and at Sin of “a” this term will be something like Sin k c m of a.

But, if Sin of the term is equal to 0, it implies that B m must be equal to 0. Because when you apply x equal to 0, this term A m Sin k c m of x vanishes. Whereas, Cos of 0 is equal to 1. Therefore, what you will get here is B m and that B m must be equal to 0. So, if you apply the boundary condition to the left, you get that B m must be equal to 0. And you still don't know what is A m.

But you have another condition at V of “x” equal to “a”. So, when you substitute “x” equal to “a”, the potential must again go to 0 which means that you have Sin k c m at “x” equal to “a” going to 0. When can this happen? Whenever k c m into a, the argument of the Sin function is some multiple of Pi. So, it whether it is 0 or it is Pi, 2 Pi, 3 Pi and so on as long m is an integer, this k c m times ‘a’ if it is equal to an integer multiple of Pi, then Sin of that term will be equal to 0 and your boundary condition is satisfied.

So, what it has actually given us is an expression for k c m. k c m is actually m Pi / a. So this equation implies this and this equation simply implies that k c m must be equal to m Pi / a. Now you see why we had written down this value of “m”. Because you have an infinite number of solutions available to you if you change the values of “m”. So that dependence on “m” is what I have captured here by instead of writing this as k c I have written this as k c m.

Now that we have k c m found out. Now, I still have two more boundary conditions that I need to apply. I can apply the boundary conditions now.

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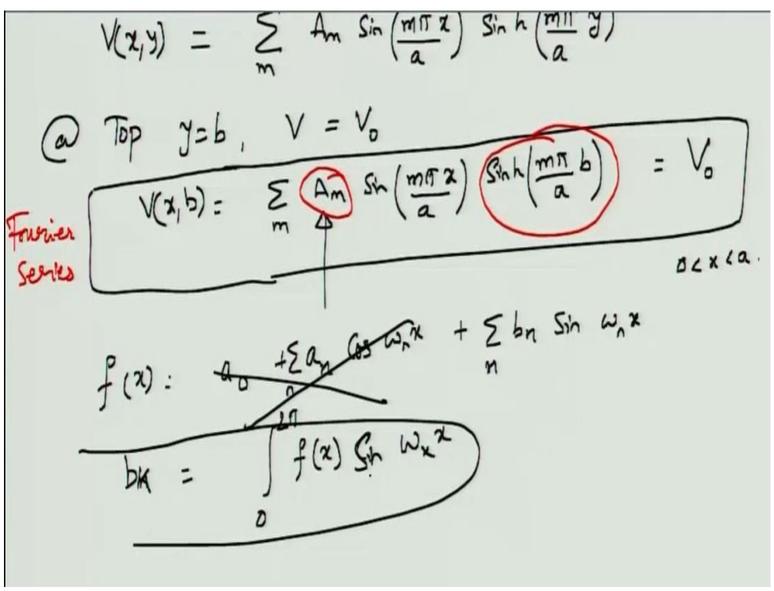

Before I apply let me just write down what has happened to V (x, y). Because we have shown that V is equal to 0. The solution simply becomes A m Sin, now I know what is k c m which is m Pi / a. So this becomes Sin m Pi x / a, and as for the C m and D m are concerned. I can apply the boundary condition for the bottom surface and what I get here is that this Cosine term will go to 0.

So, by applying boundary conditions at the bottom surface this Cosine function goes to 0. So, with that the solution becomes Sin hyperbolic m Pi / a, that is the k c m there and you still have the value of 'y' here. So this is the solution that what we now have after applying 3 boundary conditions. The boundary condition to the left and right gives you, eliminates B m, and gives you a value of k c m.

And the boundary condition at 'y' equal to zero 0, when you apply you will see that the Cosine term goes to 0 and then you get that the solution has form of Sin m Pi / a x hyperbolic Sin m Pi / a into y. I still have to find out one boundary condition here. I mean one constant here which is A m. To obtain that let me apply the last boundary condition that is remaining which is at the top wall y is equal to b. The potential there will be equal to V 0.

So, if you apply that boundary condition substitute for "y" equal to "b" and you get A m Sin m Pi x / a. Remember you are applying "y" equal to "b" and not "x" equal to "b". For hyperbolic Sin you are going to write this as m Pi / a into b. So this must be equal to V 0. Now, this solution that we have written over here is valid for all values of 'x' as long as 'x' is between 0 to 'a'. //Right//. So on the top surface 'x' is between 0 to 'a'.

And this solution is valid. To find out what is the value of A m, I just have to recognize something about this solution, something about this box solution is should actually remind you of a Fourier series. With the Fourier expansion terms A m Sin hyperbolic m Pi / a into b. So, if you remember what the trigonometric form of the Fourier series, expression for some f (x) was it had some constant a 0, it had some constants a n Cos some Omega n x summed over all values of, integer values of n + you had a constant b n Sin Omega n x for a function which has a periodicity of certain period in the function f (x).

So this is the form of the, trigonometric form of the Fourier series wherein you have f (x), a periodic function being represented by this series. So clearly if you compare the box solution with this Fourier Series you will quickly recognize that there no ‘a’ terms here, because a term sitting here is Sin m Pi / a into x. So if you think of this Pi / a, as the period or the fundamental period, then you have Sin of some Omega m x.

Where Omega m is m Pi / a, forms the m th harmonic or the m th multiple of the fundamental period and then this Sin hyperbolic m Pi / a into b is a constant. So because “b” is constant, “a” is constant for different values of m, this Sin hyperbolic term is also constant. So, this is really a Fourier series. So, let me just highlight this one and say this is actually a Fourier series.

This Fourier series for a function Sin m Pi / a x with the expansion coefficients b n as A m Sin hyperbolic m Pi / a into b and this Fourier Series sum should be equal to V 0. Now, these are the expansion coefficients. If you want to find what is b n? You would actually find that one multiplying on both sides by a function Sin Omega k into x. So you would multiply this one by Sin Omega k into x to obtain what b k is and then integrate this one over the appropriate period.

So, maybe 0 to 2 Pi if f (x) is periodic with 2 Pi. So, you integrate this one and what it would bring out is that since Sin Omega k x and Sin Omega n x will be orthogonal to each other, then the integration will be valid only when ‘n’ is equal to ‘k’, which then gives you the value of b k. So I am not going into too much detail here. But I do hope that if you recall from your signal scores, then you know how to actually extract this constant ‘A m’.

So I am going to multiply on both sides by sum sin n pi /a into x. Integrate both sides to obtain ‘A n’. So, let me do that one.

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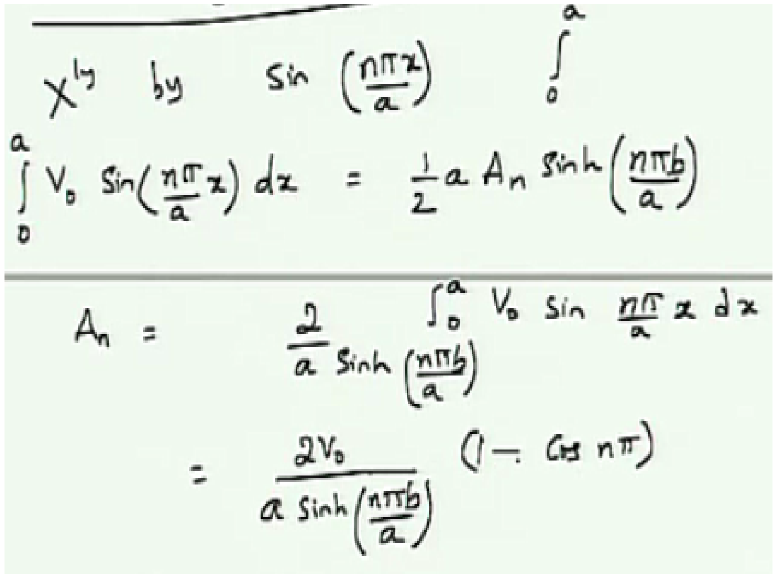

Multiplying both sides of this boxed equation over here by Sin multiplied by sin (npix/a) and integrate over zero to ‘a’, because ‘a’ is the fundamental period. So, integrate this one over this. So when you do that, you are going to get integral of zero to ‘a’ V zero sin (npi/a into x) integrate with respect to ‘dx’. So when you integrate this, after multiplying by sin (npi/a into x) and integrate it over 0 to ‘a’, what you will get is half ‘a A n hyperbolic sign (npib/a)’.

So this allows me to write down ‘An’. The expansion coefficient for the solution of ‘V of x’ as ‘2/a sin hyperbolic (npib/a)’ and integral from ‘zero to a V 0 sin npi/a xdx’. But what is this integral of ‘zero to a sin npi/a xdx’? This we know. ‘V zero’ is a constant of integration, therefore this comes out. So this becomes ‘2V zero/ a sin hyperbolic (npib/a)’. Now integral of ‘sin of x’ is, ‘sin of constant x’ is cos.

And then if you apply the appropriate limits over here ‘zero to a’, what you get over here is (one minus cos npi). Now I know what is cos npi. It will be equal to minus one when ‘n’ is equal to odd and zero when ‘n’ is equal to even. So, ‘cos npi’ will be equal to minus one when ‘n’ is equal to odd, otherwise it will be equal to zero.

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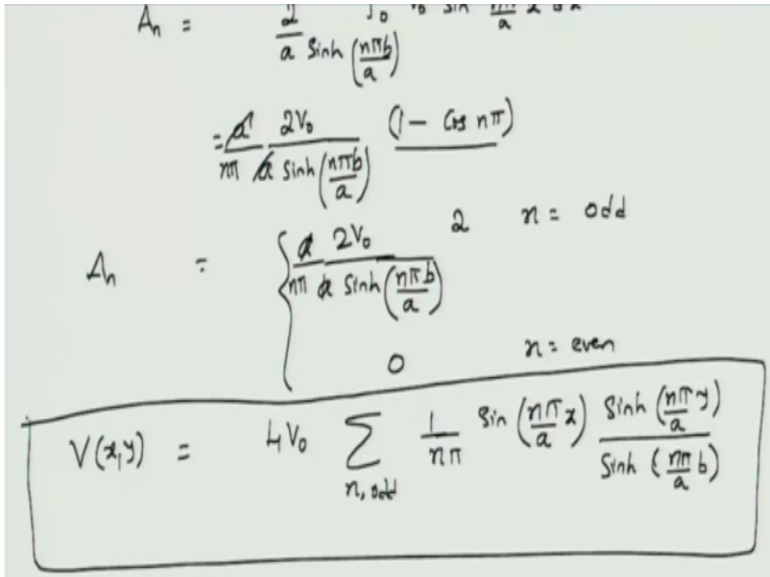

So, what will happen to this one is that you are going to get ‘2V zero’ divided by ‘a sin hyperbolic (npib/a) and ‘one minus cos npi’ is actually equal to 2, when ‘n’ is equal to odd and zero when ‘n’ is equal to even. So, this 2 gets multiplied with this one. So, I am going to get and this will be equal to zero when ‘n’ is equal to even.

So, what you have here is ‘A n’ is equal to ‘four times V zero by ‘a’ into sin hyperbolic (npib/a)’ when ‘n’ is equal to odd and ‘n’ is an integer. And it will be equal to zero because cos of 2pi is equal to one and one minus one is equal to zero or cos of 4pi is equal to one and therefore that could be equal to zero, when ‘n’ is equal to even number. So, this is the expression for ‘A n’ that you wanted. Now you can complete your solution by plugging in the value of ‘A n’.

Then you have to understand that ‘n’ can only take on odd value. So you have ‘4V 0 divided by a’ so you have summation over ‘n’ or ‘m’, does not matter, but ‘n’ must be odd. So you have ‘one by this one minus cos npi’, so somewhere I did not do the integration properly I think. So when you do the integration here, I have to ‘npi/a’ as a constant. So that would have made it ‘a multiplied divided by npi’.

So, sorry about that. So this when it comes out, it would be ‘one by npi of a’. So it becomes a/npi, a will cancel with each other. And what you get here is ‘a/npi’, so now I am done. So I have ‘one by n pi’, if you would like, so I have ‘1e / npi’ and for ‘sin (npi / a into x)’ and ‘hyperbolic sin (npi / a into y)’ divided by ‘hyperbolic sin (npi / a into b). So this is my full solution.

So, ‘a’ of course has already gone out because I did not really ‘a’ after I cancelled that out here. So this is my solution. Everything about this solution is now known to me. So everything about this solution is known because I know what is ‘V 0’? This is the potential that I have applied. I know what is this? ‘sin npi by ax’ because I know ‘a’ and I can pick the value of ‘n’ that I am interested in. So if I start picking different values of ‘n’, I am going to get different solutions.

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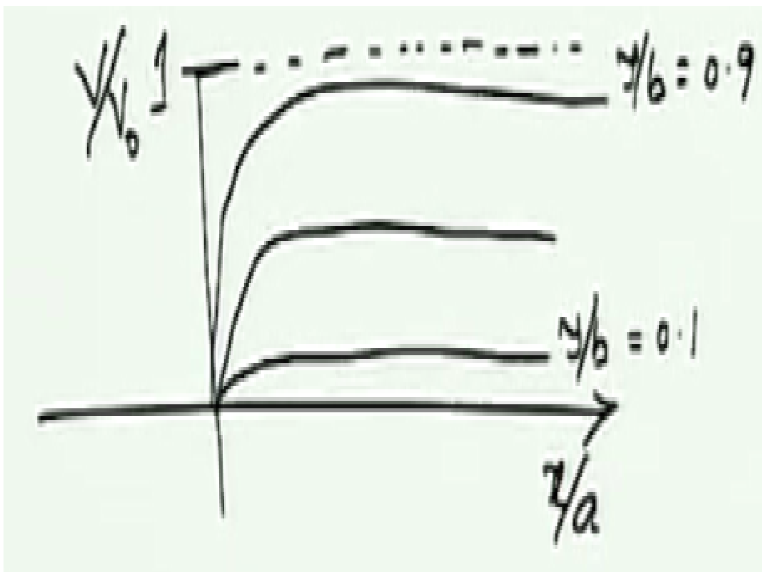

And if you have to plot this function, potential function, you will see that if you plot ‘V / V 0’ that is normalize a potential to ‘V 0’, you will see that for different values of ‘y’ by ‘b’, if you plot it as a function of normalized values of ‘x’, you will see that the solution actually looks something like this. So, you have to imagine the solutions. So, this is for y/ b equal to 0.9. And this is for the solution y / b equal to 0.1.

So, this is how the potential would look for different values of ‘y / b’. So the potential is actually function of both ‘x’ as well as ‘y’. And this is only half part, which we have written. So, if you could complete the solution in two dimensional case, you will see that the potential goes to zero at the three ends, whereas it will retain its value of ‘V 0’, it approaches the value of ‘V 0’ at the top surface. So you can see here, at y is equal to b, the potential will actually be equal to 1.

So, this is the boundary for 1. So, this completes our solution for Laplace's equation in two dimensions.

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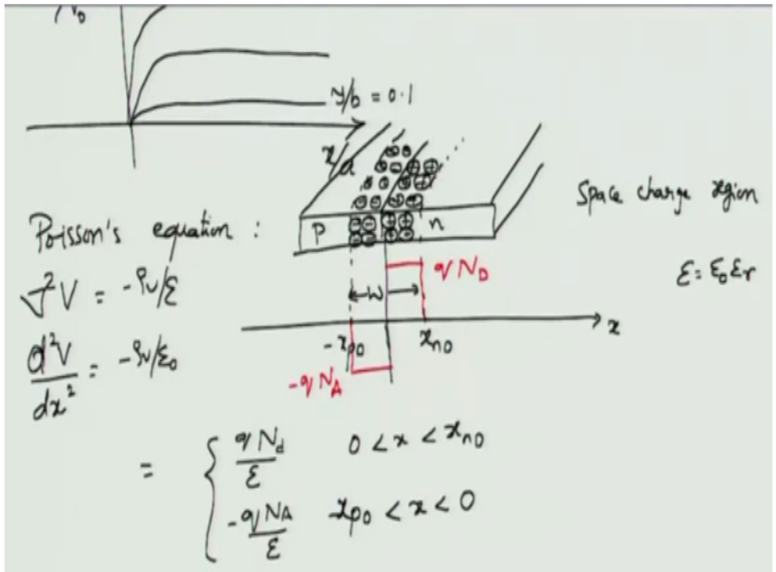

Now we will look at solution of Poisson's equation in one dimension. We will consider case known as space charge region of a diode. So, if you recall the first lecture, I showed you how to calculate the total charge given the charge density and for a PN junction that I am considering over here. So, I have a 'P' junction, I have an 'N' junction. You can see that after the equilibrium is attained, depending on how much we have doped there will be a space charge region.

Because what happens is, if you bring 'P' type region and 'N' type region of a semiconductor together, the 'P' type region has a lot of holes, 'N' type region has a lot of electrons. These electrons will then diffuse towards the 'P' side region leaving behind positive ions, which are immobile. And the holes, which are in excess in the 'P' region will diffuse towards the 'N' region leaving behind immobile electrons.

So, if you look at the region in between this, you know the contact point being at 'x' equal to zero, if we can take that at 'x' equal to zero, then there exist a certain region, let us call this as 'x n zero' and 'minus x p zero', so there is certain region of width 'w', over which you do not really have any free charge carriers. But there are immobile ions in this region. So, these immobile ions are positive in the ‘N’ side region and negative in the ‘P’ side region.

And this region of width ‘w’ is called space charge region. This is a terminology coming from vacuum tubes or sometimes called as space charge layer. Now our question is, I know this physical situation, can I obtain the potential as well as the electric field. And in some way, can I determine what is this ‘x n zero’ and ‘x p zero’ values are? In order to do that one, I need to apply Poisson’s equation in that region that I have shown here as having width ‘w’.

So in the space charge region, you apply Poisson’s equation. And what is Poisson’s equation? For the potential ‘V’, this would be equal to ‘minus rho V divided by epsilon zero’. So if you assume that the junction on the bar, actually is having a similar for a uniform ion density, so one can actually forget the fact that this is actually a two dimensional layer and work with only one dimension.

Because no matter where I take a cut here in the one dimensional region, I am going to get the same charge density on the ‘P’ side as well as same charge density on the ‘N’ side. And where will be an electric field between? Electric field will originate from the positive charges ions in the ‘N’ side region and go to the negative ions in the ‘P’ side region. Because of this uniformness assumption, I can consider Poisson’s equation only as a function of ‘x’.

Now to apply Poisson’s equation, I need to know what is the charge density? And the charge density here is positive. So, let me write down what is the charge density here. And if ‘q’ represents my charge, then the charge density here is ‘q time N D’. ‘q time N D’ standing for, ‘N D’ standing for doping, donor concentration, which actually gives you an excess of electrons. And the charge here on the ‘P’ side region is negative. And it has a value of ‘minus q N A’.

So, these are the charge densities that I have, positive charge density here and negative charge density over here. In case, you do not want to use ‘q’, you can use electron charge density ‘e’ and ‘e’ will be equal to ‘minus q’. Now these are the charge densities in this region I know. Now I need to apply Poisson’s equation. Thankfully this ‘del square V’ becomes only ‘d square V by dx square'. This will be equal to 'minus rho V by epsilon zero'. And 'rho V' is 'q times N D'. 'q' is the charge and 'N D' is the dopant concentration and 'N A' is the acceptor concentration.

So this equation actually becomes 'N d q divided by epsilon'. Let us not write this epsilon zero because the silicon has a different epsilon assuming that this was made out of silicon, it has a different value of epsilon, so let me just write down this as epsilon. Epsilon, if you remember, must be equal to 'epsilon zero times epsilon r'. 'Epsilon r' is the relative permittivity. So you have q N D by epsilon, well between 'zero to x less than x n zero'.

This is on the 'N' side region. And you have 'minus q N A by epsilon' for 'minus x p zero to x less than zero'. So, this is my problem now. The physical situation of having immobile ions on the 'P' side and on the 'N' side and creating a space charge layer or a space charge region, is now captured by Poisson's equation in one dimension. And this is the equation that we need to solve.

This is an ordinary second order differential equation with the right hand side being a constant. So the solution of this is fairly simple. So, the solution of this is very simple.

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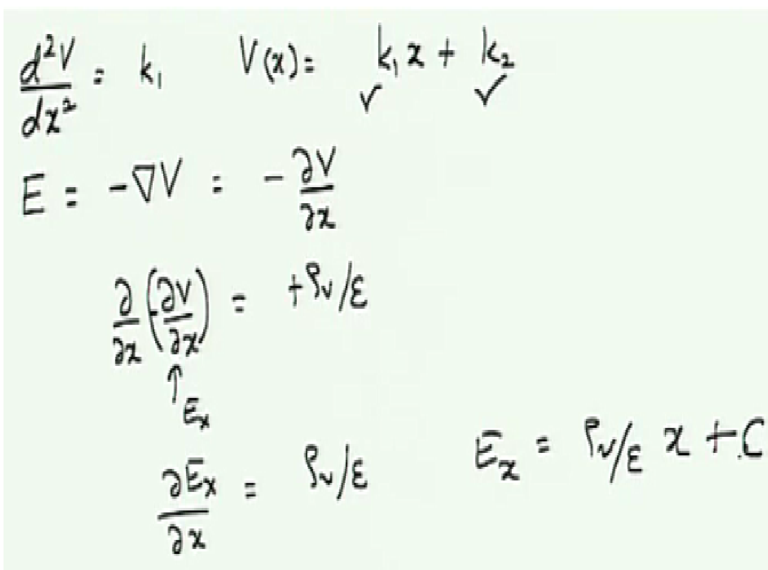

So if 'd square V of x by dx square' is constant, some constant 'k 1', then 'V (x)' is equal to k 1 x plus k 2', where 'k 1' and 'k 2' are further constants that need to be evaluated. So, if you do that and if you also recognize that electric field is 'minus gradient of V', in this case, it would be 'minus del V by del x'. So, I can directly write down the expression for electric field. Have 'minus del by del x (del V by del x)' for Poisson's equation.

So 'del by del x of (del V by del x)' is equal to the charge density 'minus rho V by epsilon'. So if I put minus sign on both sides and a minus sign this becomes plus and this is nothing but electric field. So, if I write this as electric field 'del E x by del x', this would be equal to 'plus rho V by epsilon'. And this equation can be integrated to obtain 'rho V by epsilon times x plus some constant'. Outside the space charge region, there should only be a constant. Inside, the constants could be different.

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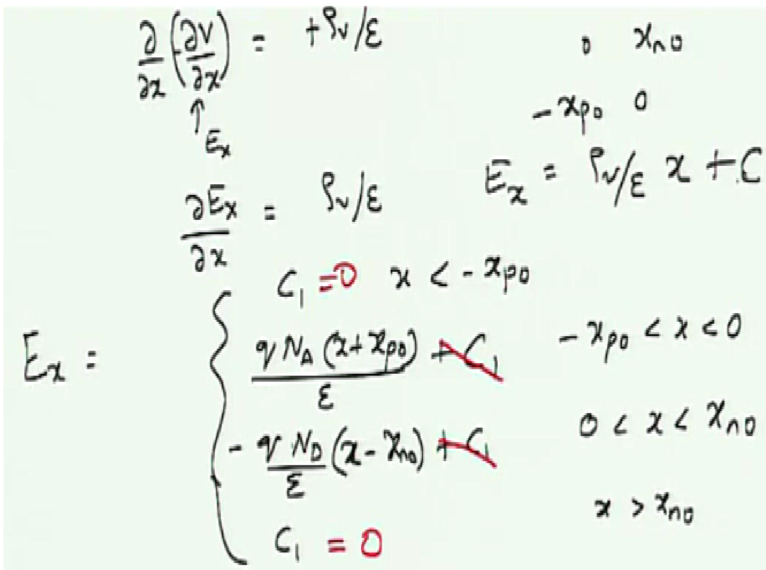

So if you write down what is 'E x', 'E x' will be equal to outside the region some constant. Outside 'x less than minus x p 0', it could be a constant C 1. Inside, it would be 'q N A (x plus x p 0) divided by epsilon', this is the charge density and integrating this one over and applying the appropriate limits for the integration. So, this integration limits must be 'minus x p 0 to zero' for the 'N' side region, sorry 'P' side region.

And 'zero to x n 0' for the 'N' side region. So, if you apply that you will get a constant. So, here you have 'minus x p 0 less than zero'. This is the 'P' side region. And you will have 'minus q N D by epsilon (x minus x n 0) plus the constant C 1, for 'zero less than x less than x n zero'. And finally outside of this region, you again have a constant 'x 1', so for 'x greater than x n 0'. So you can evaluate all these constants, because you know that the field outside must be equal to zero. So, all these even constants will become equal to zero and the field essentially becomes equal to zero outside.

So, the field is actually equal to zero outside. All these constants disappear and this is the electric field that you have.

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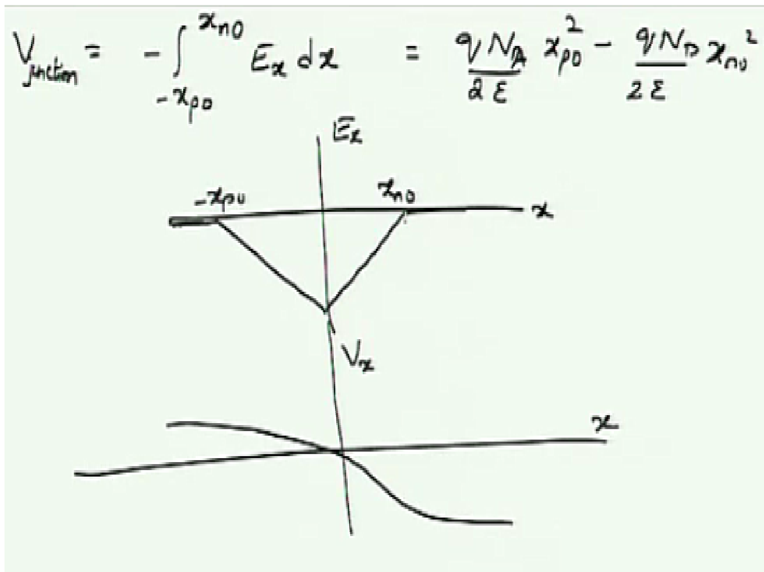

You can also calculate the junction voltage by integrating the electric field from 'x p 0 to x n 0'. So you can calculate the junction voltage 'V' as 'minus x p 0', which is from the 'P' side region to 'x n 0' with a minus sign and integrating this expression for electric field will become equal to 'two epsilon x p 0 square minus q N for the x p 0 and this is q N D by two epsilon (x n zero square). So this is the junction potential.

If you sketch the potential as well as the electric field, you will see that the electric field must go to zero. It actually starts like this. And it goes to zero, beyond this 'x n 0' on the 'N' side and 'minus x p 0' on the 'P' side. So, this is your field E x. You can see that this is the electric field E x. And if you sketch the potential, which would be an integral of this fellow, the potential would actually be something like this.

I hope that I have got the sign, I mean sign of the potential correct. Otherwise you need to properly integrate and then check. So, this completes our module for Laplace's equation and Poisson's equation. We covered Laplace's equation in two dimension, with the canonical problem of this metallic tube and we considered Poisson's equation for one equation, which was the equation for diode, where we evaluated the junction potential and the electric field that could exist between the junction.