Electromagnetic Theory
Prof. Pradeep Kumar K
Department of Electrical Engineering
Indian Institute of Technology – Kanpur

Lecture - 31
Solution of Laplace's equation - III

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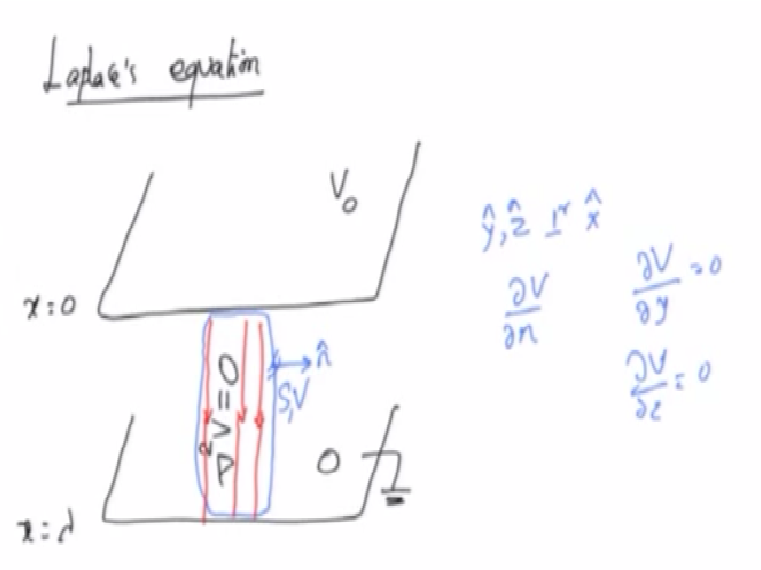

We will get back to solving Laplace's equation, okay. And look at some of the other situations. Of course, what we have done over the past method of images was also solving Laplace's equation, except that we did not have to solve them, because the solutions were already known to us, okay. So, here let us consider a couple of examples of Laplace's equation, we will solve them in one dimension and two dimensions, okay.

Solution of Laplace's equation three dimensions is not normally done, one and two is common, three dimensions is not very common. And we will get back to over favourite parallel plate capacitor problem, okay. So, let us say, keep the capacitors over here and say, at x equal to zero and x equal to d are the top and the bottom plates. The top plate I will keep at a potential V0, the bottom plate is grounded, therefore the potential here is zero.

Now, I consider a close surface in the form of the cylinder, okay, which actually goes through this one, okay. But does not really extent in to the plates, because if they extent into the plates, there are charges. And I do not want to solve Poisson's equation, I want to solve

Laplace's equation. So, I chose my surface just below the top plate, just above the bottom plate, but I chose a closed surface, okay.

So this is a closed cylindrical surface, which also encloses certain volume if you would like. Inside this, I want to apply Laplace's equation, right. Now, because the plates are extending towards infinity on both y and z directions, the electric field lines will be completely uniform, and they will be downwards, right. So the electric fields will be uniform and they would be downwards, okay.

So, I can specify the potential on the top as V0 and the bottom plate as zero, and for the curved surface that I have, right. The potential is directed along x, but the normal to this curved surface is actually directed along the radialed direction y and z, right. The normal to the surface would be directed along the coordinates y and z, but those coordinates will be perpendicular to x.

The simple fact is that the field is going vertically downward. There are no tangential or the horizontal components. So the inner product of the tangential or the horizontal vector, and the normal downward vector will be equal to zero. So what have done is we have specified potential values at the top and the bottom, for the curved surface, we were actually specifying this del V by del n, right.

So if n is along y or z, then del V by del y is equal to zero, del V by del z is equal t zero. So these were the two conditions that we were specifying on the curved surface, okay, alright. So let us go back and solve Laplace's equation inside. The idea being that V is the function only of x, because the plates are extending infinitely on the y, which allows me to write down the electric field to be uniform inside, or think about that has uniform electric field inside, okay.

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$$\frac{d^2V}{dx^2} = 0 \Rightarrow V(x) = Ax + B$$

Top: V_0 ; $x = 0$ $\quad V(0): \boxed{V_0 = B}$

Bottom: 0 ; $x = d$ $\quad V(d): 0 = Ad + V_0 \Rightarrow \boxed{A = -V_0/d}$

$$V(x) = \frac{-V_0}{d}x + V_0 \qquad \bar{E} = \hat{x}\left(-\frac{\partial V}{\partial x}\right) = \hat{x}\frac{V_0}{d}$$

And solution of this one will be very simple the Laplacian in Cartesian coordinates is del square V by del x square plus del square V by del y square plus del square V by del z square. But those two terms anyway cancel out, I mean anyway become equal to zero, I have only left with the first term, which is dependent on x, and partials can be replaced by full derivatives d, okay.

So d square V by d x square will be equal to zero, integrating this one twice, I get V of x equals, Ax plus B, where A and B are constants. Here is where I want to put the constants, right. I use the values of the potentials at the top and bottom surfaces and find out what are the constants A and B, okay. Top surface, the potential is V0, the top surface is given by x equal to zero.

So this implies V of zero equals, V0 equals A into zero plus B, so this is B. So I am actually able to find one constant B. For the bottom plate, right, for the bottom plate where I have potential is zero bottom plate is given by x equal to d. So V of d is equal to zero, is equal to A times d plus value of B, B we have already seem to be equal to V0. So this gives me A as minus V0 by d, okay, alright.

So, I have found out both the constants, so the potential V of x will be given by, A is minus V0 by d x plus V0, okay. Now do not worry about the minus sign, that happens simply, because I put the top plate at x equal to zero, and bottom plate at x equal to d. And I know that the field line will start from top plate and go on to the bottom plate, okay. You can actually see that one.

If you find out what is the electric field, electric field will be directed only along x axis, because V is only function of x axis. So differentiate this one, e is minus gradient of V, so if you do minus del V by del x, okay, you will see that differentiating this one, the constant vanishes, and x becomes one. So, I am left with minus V0 by d, a multiplication of minus V0 by d with a minus a sign will give me V0 by d.

So the electric field is directed vertically in the x direction, and it is completely uniform, okay. So this was the parallel plate capacitor that we wanted to look at. Now this is something that we have already seen, so there is not much of an interest in this one, or not much of interesting thing that happen.

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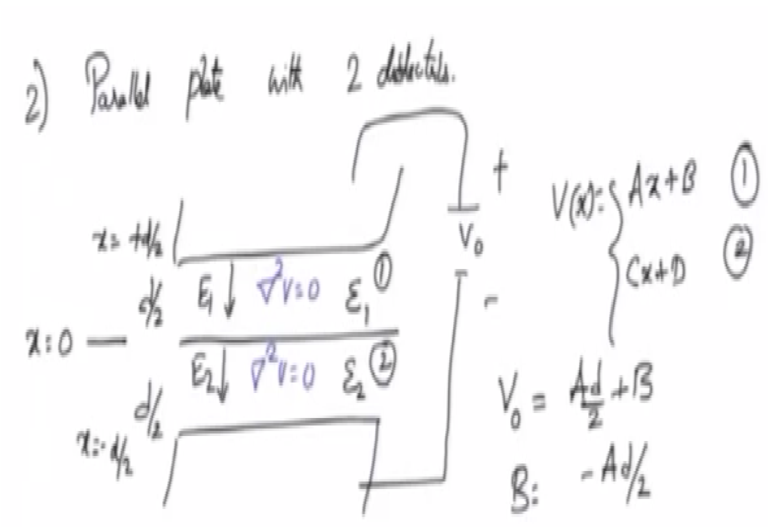

So let us ramp up the problem slightly. Let us look at parallel plate capacitor, but filled with two dielectrics. So I have the top plate here, I have the bottom plate here. At some distance d by two, okay, I fill this plate with material of permittivity epsilon one. And for the distance d by two, I fill this with permittivity epsilon by two, okay. Clearly, there will be electric field, which is still uniform, because the plates are extending towards infinity in both directions.

And there will be two fields E1 and E2. The top plate is kept V0, okay, and the bottom plate is kept at zero, okay, or grounded. And now, because of the symmetry in the problem, let me locate the x equal to zero line in the middle of the plates, and x equal to d by two will be on the top, x equal to minus d by two will be the bottom plate location, okay. Nothing has changed.

Inside here, I have to apply the same Laplace's equation, del square V equal to zero, del square V equal to zero. There are of course now three boundaries involved, so you have to consider three boundary conditions, right. So there is one boundary condition at the top, boundary at the middle, and boundary at the bottom, okay. And we have to also see that the potentials have to be continuous and the derivatives of the potentials also have to be continuous.

We will see that one, so if I solve the equations I am going to get V of x is equal to Ax plus B in the region between zero to d by two, right. So in the region one, if call this as region one and region two. So in region one the potential is Ax plus B, and in the region two, the potential is some Cx plus D, where these are the constants, right. I can apply two boundary conditions, one boundary at the top, one boundary at the bottom.

So apply the boundary condition at x equal to zero, which the top plate, sorry, at x equal to d by two, which is the top plate, the potential is V0. So V0 must be equal to Ad by two plus B, okay. This does not give you anything, except writing B in terms of A. So I can write B is equal to minus Ad by two, okay. Now I can apply the boundary condition at the bottom plate, which is at minus d by two.

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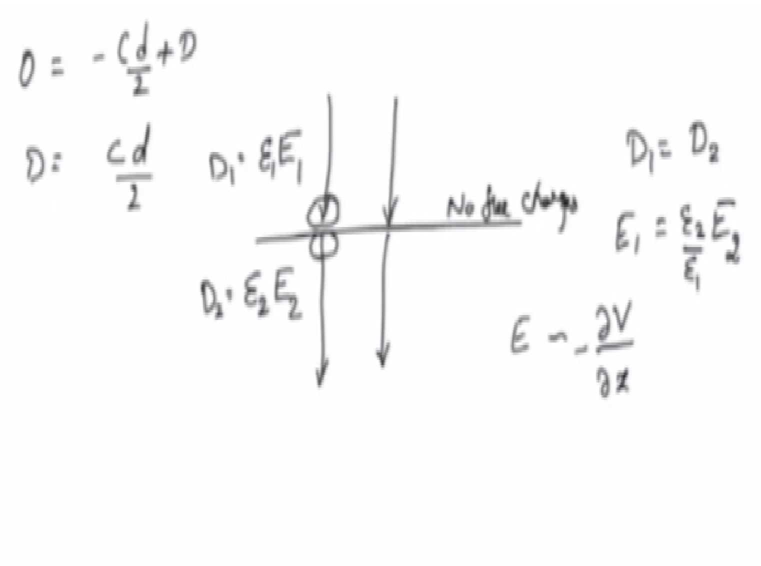

So I get zero, which is the potential of the bottom plate, equals minus Cd by two plus D. Again, I can write D in terms of C, by taking Cd by two on to the left hand side, okay. So, I have removed two, out of four constants, I have removed two constants. By writing, B in terms of A, and writing D in terms of C. Now here is where I have to use the third boundary, right. So at the third boundary, what is happening.

I know that the field is coming vertically, uniformly downwards. The plane is here, okay. The field lines are all coming out here, okay. So if I multiply the field in region one, by epsilon one, and multiply the field in region two by epsilon two. This epsilon one even becomes D1 and epsilon two E2 becomes D2, where D1 and D2 are flex densities, right. So these are the flex densities.

And I know that the normal component of the flex density must be continuous across the boundary. Thankfully, there are no horizontal components of the electric field, therefore we do not have to consider the horizontal component. So the normal component of the D must be continuous, and because this is a perfect dielectric, there are no free charges, right. There are no free charges here, therefore D1 must be equal to D2.

This simply implies that E1 must be equal to epsilon two by epsilon one times E2. But what are E1 and E2? E1 and E2 are the values of the electric field at the boundary just above and E2 is the value of electric field just below the third boundary, which is the boundary at x equal to 0. And I also know that electric fields are related to the corresponding potential gradients.

So all I am saying now is that because electric field is derivative of V, not only the potential is continuous, minus del v by del x is continuous across that x equal to 0 third boundary, okay. So if I differentiate the potential V, I get for the top potential right, so let us go back to that V of x, so if I differentiate this one with respect to x, I get A, differentiate the second expression with respect to x, I get C.

Minus signs on both sides will cancel with each other out, so I do not have to worry about that.

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$$\varepsilon_1 A = \varepsilon_2 C$$

$$C = \frac{\varepsilon_1}{\varepsilon_2} A$$

$$V(x) = \begin{cases} A\left(x - \frac{d}{2}\right) + V_0 & 0 < x < d/2 \quad \text{Region } 1 \\ \frac{\varepsilon_1}{\varepsilon_2} A\left(x + \frac{d}{2}\right) & -\frac{d}{2} < x < 0 \quad 2 \end{cases}$$

$$-\frac{d}{2} A + V_0 = \frac{\varepsilon_1}{\varepsilon_2} A \frac{d}{2} \Rightarrow A = \frac{2\varepsilon_2 V_0}{(\varepsilon_1 + \varepsilon_2) d}$$

And multiplying by epsilon, I get epsilon 1 times A equals epsilon 2 times C, right. The electric field was all uniform. Differentiate that one with respect to x in the region 1 and region 2 and equate the two after multiplying by epsilon 1. When I have done this thing, I can write down C as epsilon 1 by epsilon 2 times A, good. We have gone from four constants of integration to three constants of integration, okay.

So let us write down V of x all in terms of the constant A. In the upper region, that is in the region 1, the potential V of x is A into x minus d by 2, plus V0. This is in the region 1, right. The potential is also equal to epsilon 1 by epsilon 2 A x plus d by 2, when you consider this in the region 2. Okay, this is in the region 2. Check that these two actually satisfies the boundary condition.

At the upper plate, x equal to d by 2, top plate of the capacitor, so this terms drops out and the potential will be equal to V0. In the bottom plate, x is equal to minus d by 2 that is where I have kept the bottom plate, so this term in bracket will also be equal to zero, and the potential will be equal to zero, right. Now you can actually apply the condition for x equal to zero and you will be able to stitch the potential to be equal.

So if you that one, you will get minus d by 2 A plus V0 must be equal to epsilon 1 by epsilon 2 A at x equal to zero, that will be d by 2, right. Now this equation allows me to find out what is A. If you solve this equation, you will see that A is equal to 2 epsilon 2 V0 divided by epsilon 1 plus epsilon 2 into d, okay. Now, I have found out all four constants and my solution is now complete.

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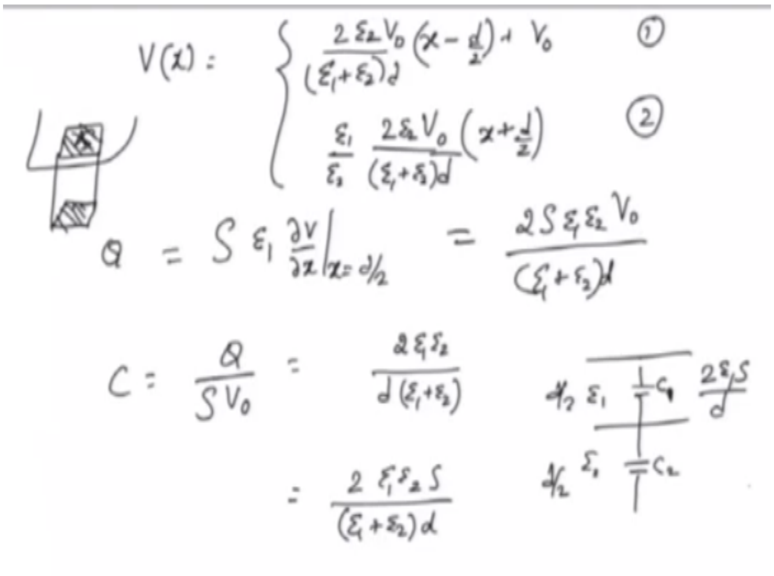

So I have V of x is equal to two epsilon 2 by epsilon 1 by epsilon 2 into d, so there is V0, x minus d by 2 plus V0 in region 1 and the solution is epsilon 1 by epsilon 2 times A. A is two epsilon 2 by epsilon 1 by epsilon 2 d, there is a V0 here times x plus d by 2. So this is a solution in region 2. Now let us go ahead and compute the capacitance of this structure. To get the capacitance, I need to know what is the charge stored in the surface area, right.

So if I have a top plate here, so let me pick out a uniform surface element A and I want to find out what is the charge stored in this region, okay and I find that one and I can actually find out the capacitance. Because capacitance is the charge on one plate divided by the potential difference between the two. The charge stored is of course, the charge density times the surface area. I have already found out the charge density.

The charge density should be epsilon 1 del v by del x evaluated at x equal to d by 2. This is the charge density because d must be equal to rho s on the conducting plate multiplied by the area s will give me the charge enclosed in this region A and we will see that charge per area will be the surface charge density and that can also be used if you are really interested in that one. This will be equal to 2S epsilon 1 epsilon 2 V0 by epsilon 1 by epsilon 2 into d.

You can show this one by differentiating this expression and then substituting x equal to d by 2. So you differentiate this top expression with respect to x, V0 cancels out and this x minus by 2 will be equal to 1. The derivative of that one will be equal to 1 and this is what I have.

Multiply this one by the surface area S, and you will get the charge enclosed in this surface area s, okay.

Now divide the charge enclosed in that surface area divided by V0, you are going to get the capacitance of this parallel plate. Since this is uniform parallel plate capacitor, you can also find out the charge over the surface s and call this as capacitance per unit surface, okay. You will have Q by S V0 that is given by 2 epsilon 1 epsilon 2 divided by d epsilon 1 plus epsilon 2 or if you include the surface area S.

You can actually re-write this one as 2 epsilon 1 epsilon 2 S divided by epsilon 1 plus epsilon 2 into d. In fact, you can show that since this plate from the top to bottom can be considered as some capacitor, okay and the bottom to other capacitor can be considered as another capacitor, so you have C1 and C2 as two capacitors. These capacitors are now in series, okay and for the capacitance of each structure can be thought of as 2 epsilon 1 A by d.

Because epsilon 1 is the permittivity of this fellow and epsilon 2 is the permittivity of region 2 and d by 2 is the height of each of these capacitors, okay. So this will be epsilon 1 surface area A, so surface area let us make it S, okay and d by 2 and there will be one more 2 epsilon 2 S by d and if you add them together, but capacitors in series, must add according to 1 by C is equal to 1 by C1 plus 1 by C2, so if you add them, you will get.

To see that this is actually two capacitors in series, okay. So this is the capacitance of this parallel plate capacitor with two dielectrics obtained by solving the Laplace's equation and applying the appropriate boundary conditions, okay. We now want to solve Laplace's equation in two dimensions, okay. Before doing that one, let us look at one more solution of Laplace's equation in one coordinate.

Since there is nothing much we can do by putting the plates in the y or z direction, let us put plates in a different coordinate, okay.

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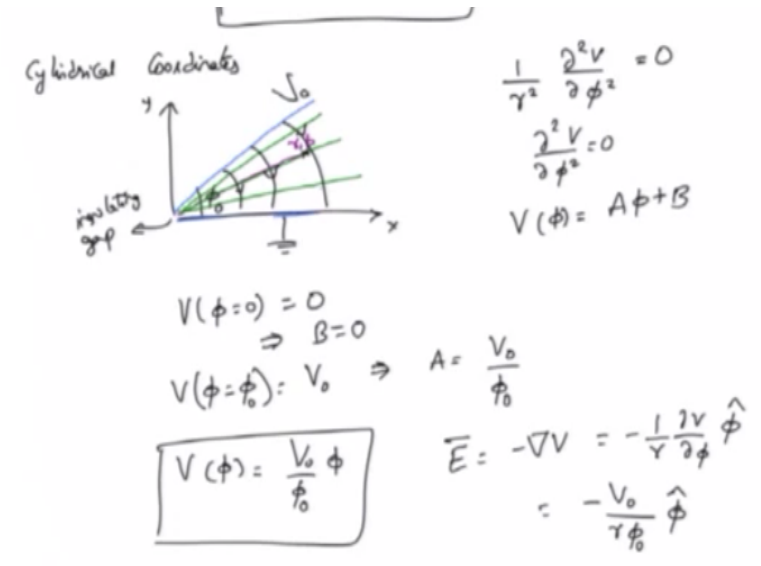

Let us put plates in cylindrical coordinates, okay. For coaxial cable, we have already looked at capacitance. Therefore, I do not want to put the coaxial cable here, consider an interesting example, okay. I have a conductor horizontally and I have a conductor at an angle phi with respect to the x axis, at an angle phi with respect to the x axis, let us ground the potential, horizontal potential and apply a constant potential of V0 at phi equal to phi0, okay.

Now I want to find out what is the capacitance of this structure. Of course, there must be an insulating gap here, okay. Otherwise, they will all be at the same potential, so I have put an insulating gap, very small one over here. Now, clearly I have to use cylindrical coordinates and the capacitance will be functioning only of phi, right. So the potential will be functioning only of phi and for cylindrical coordinates.

The corresponding term for phi is, the Laplacian for phi is 1 by r square, del square v by del phi square, okay. This must be equal to zero and for a finite value of r that we are considering that cannot be, 1 by r square cannot be zero, therefore I have del square V by del phi square equal to zero leading me to V as a function of the angle phi, okay. So for example, I am considering this particular point, which is given by radius r and point phi.

This is where I am calculating the potential and this potential will be equal to A phi plus B. I have two boundary conditions to apply. I can use that and try to find out the potentials A and B.

If I apply the boundary condition at phi equal to zero, which is the horizontal plate, this is equal to zero, which implies that B must be equal to zero. Because in this expression, if I put phi equal to zero, this A phi will be zero and left hand side is also zero. B must be equal to zero and on to the angled plate, if I put at phi equal to phi 0, the potential is equal to V0. This implies that A must be equal to V0 by phi 0 and the potential V of phi equals V0 by phi zero into phi, okay.

There is no direction associated with the potential, of course this is scalar. What about the electric field? Yes, electric field has a certain direction, which is given by the gradient of V, minus gradient of V and if you look at cylindrical coordinates, this gradient for a function that is depending on phi is given by 1 by r del V by del phi hat and this will be equal to minus V0. If you differentiate this one with respect to phi, you will get V0 by phi 0.

So you get V0 by r into phi 0. It is interesting that the electric field is actually a function of r whereas the potential is completely independent of r, right. So electric field is a function of r, of course this must be directed along the phi axis, right. Why should it be? Well, if you try to plot the equipotential for this capacitor, you will see that the equipotentials all pass through or directed at constant values of phi, correct.

And the electric field must be perpendicular to this angle phi and therefore they will be going curved like this. These are the electric fields and the value of the electric field actually decreases and therefore I am moving them away as I draw them. So the electric field magnitude decreases as you go radially outward, as 1 by r, okay. And you can also see that the electric fields are directed clockwise.

Because the increasing value of phi is along counter clockwise, which is what the positive reference direction for phi is considered whereas for the electric field must be clockwise and therefore they will originate from the top plate and they will terminate on the bottom plate, okay. So this is how we solve Laplace equation for cylindrical coordinates. I have considered only one coordinate.

Now I want to look at Laplace’s equation in two dimensions and I want solve one problem in Poisson's equation, but I do not have enough time, therefore, let me stop this lecture here and continue solution of Laplace’s equation in the next class. After we consider solution of

Laplace equation in two dimensions and one example of Poisson's equation, we will close with electrostatics and take up the subject of magnetostatics, okay. So that is where we will start from the next class.