Electromagnetic Theory
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Lecture 22
Boundary Condition

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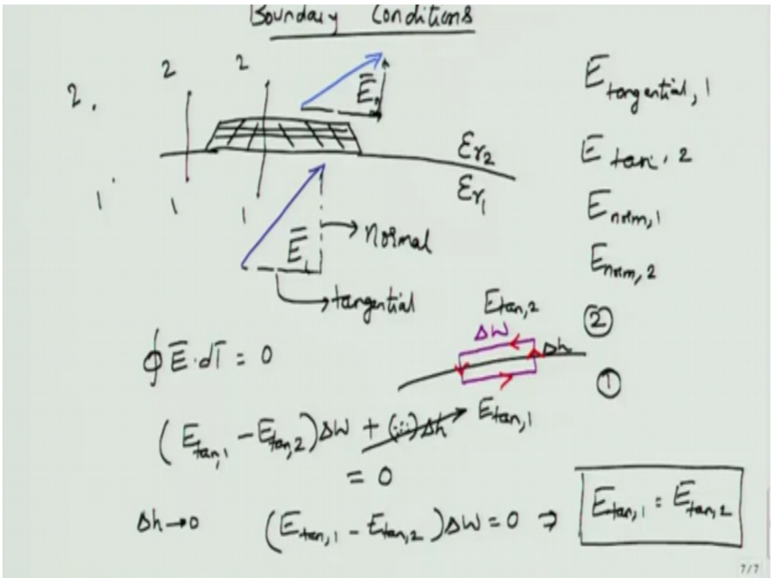

So you understand why we need boundary conditions, right? We need boundary conditions in order to connect electric fields outside of a dielectric to the fields inside of a dielectric. It is not just one dielectric, you could imagine there are tens and fifteens of slabs of different dielectric material that we could be putting in and we want to find out the electric field relationship between medium in one dielectric to medium in another dielectric, air being the very special type of dielectric medium.

We can also consider with air as one special dielectric material. We are looking at air to dielectric, dielectric to air and dielectric to dielectric medium relationships and that can be obtained by looking at boundary conditions. Now, here is a point about boundary condition. Most boundaries would not change abruptly. We will be considering boundary conditions to be changing abruptly.

We say that, well, for x less than A, let us assume it is all free space and for x greater than or equal to A, just at x is equal to A and onwards there will be a certain dielectric material. Now clearly that cannot be true because there has to be some transitional layer between the air to dielectric that we are considering and we cannot consider this transition layer to be zero in practice, I mean, zero in theory.

However, in practice this transition layer is only a few atomic width. Few atoms layer, the material property will change and therefore in practice we are able to consider this transitional region to be so small that we can eliminate it and consider an abrupt boundary change. Values of the material properties would change abruptly from one point to another point although we do know that they cannot really change that fast.

But this is a very good approximation. We will use that approximation and we will look at what is the fields inside and outside of the materials, assuming this abrupt boundary conditions. Sometimes these are called as jump boundary conditions. You are jumping from one value to another value. What do we mean by boundary conditions? Let us place two points and we do not worry about what these points are?

Let us place these two points one and two. You take the point one to be inside one dielectric, point two to be inside on another dielectric. Now this is a fundamental geometry which says that between any two points you have a line, right? I can consider any two points and draw lines. This is not a straight line but that should essentially be a straight line. It is also a geometric fact, that for any line that I consider between two points I can actually have a plane, correct?

There would be a plane. As simple as that, right? So you take two points, for a line. A plane would be bisecting the line in half. So this is my plane, okay? So on this plane I have considered two points, point two and point one which the plane is actually separating. So this plane if I now consider to be the boundary between the two media then I can denote the properties of the second medium by giving its relative permittivity as epsilon r2 and the relative permittivity of medium one as epsilon r1.

Now have an electric field inside the first dielectric medium. I have the electric field inside the second dielectric medium. These two are not the same, please note that. The question is what should be the relationship between the two. Can they be simply be equal to each other? Well, you will see very soon that they cannot be exactly equal and also I have not specified what fields I am considering, these two are electric fields that I am considering.

So call this E1 and call this as E2. E1 cannot be equal to E2. If they are equal, then there is essentially no medium in between, right? So you cannot have E1 equal to E2. The relationship needs to be defined. Now E1 can be broken up into two parts, so I can break it up into two parts, one component will be parallel to the plane. The other component will be perpendicular to the plane.

This parallel component is called as the tangential component, tangential to the plane that is separating the two medium and the component of this perpendicular is some sort of point which is coming out of the plane is the normal component. You can do the same thing for both electric fields inside medium one and medium two, so you will have tangential electric fields and normal electric fields.

So you will actually have E tangential in medium one. You will have E tangential in medium two. You will have E normal in medium one. E normal in medium two and we want to relate all these. Now we can relate these fields by using the line integral E dot d l equal to zero, okay? And to do that let us imagine the line to be that of width, some delta w and a height delta h, the path that I am considering over here and let me locate the path to be in this direction.

The path is in this direction. Now inside the material with dielectric one, okay outside, I mean the second medium is dielectric two, the electric field, tangential electric field that would be there along this path will be say E tan 1 and outside here will be E tan 2, okay? Along delta h here would be half of E normal and half of E normal two, sorry, E normal 1, E normal 2, they will be multiplied by respective delta h by 2, delta h by 2, okay?

So to this path if I apply the line integral expression what I get is, E tan 1 minus E tan 2, okay? Multiplied by delta w plus some terms that contain delta h. This would be equal to zero. The terms that contain delta h will be the normal electric fields. This is E tan 1 minus E tan 2 because delta w is, sorry, the path is along one direction and the path in the other medium is directed upwards to the path in the first medium.

Therefore, E tan 1 minus E tan 2, delta w equal to zero. Now if I let delta h go to zero, the terms corresponding to delta h would any way go to zero and what I get is E tan 1 minus E tan 2 multiplied by delta w is equal to zero. If I keep delta w as non zero this would indicate that the tangential electric fields across the two dielectric media must be continuous and they would be equal.

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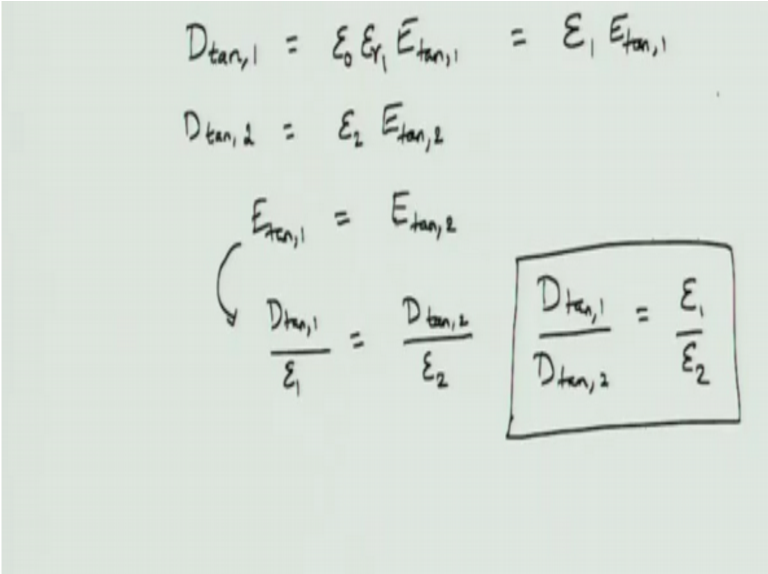

Now we have found out the relation for the tangential electric field that allows us now to write down the expression for the d field. The tangential d field can be related to the tangential electric field in medium 1 by the permittivity of that medium. So I have epsilon zero, epsilon r1 times E tan 1. I can write down this in a more short form as combining epsilon zero and epsilon r multiplied by epsilon.

And I will write this as epsilon one tangential electric field in medium 1, okay? Similarly, I can write down the tangential D field in the second medium as epsilon 2 tangential electric field in medium 2. Since E tan 1 must be equal to E tan 2, I can write down or I can substitute for tangential electric field in terms of D fields, since E tan 1 becomes D tan 1 divided by epsilon 1 that must be equal to D tan 2 divided by epsilon 2, okay?

Or the ratio of the tangential D fields in medium 1 to medium 2 must be equal to the corresponding permittivity ratios, epsilon 1 by epsilon 2. Okay, so out of the vectors that we have, we have obtained E tan 1, E tan 2 relationships, so if I know one of the fields then I can find the other field. If I know the permittivity values, I can also find out D tan 1 and D tan 2. So I know the electric field E vector and the D field vector in one region.

I can find the tangential electric fields in the other region. I still have to find out the normal electric field because every vector will have tangential as well as the normal electric fields, okay? So how do I find the normal electric field?

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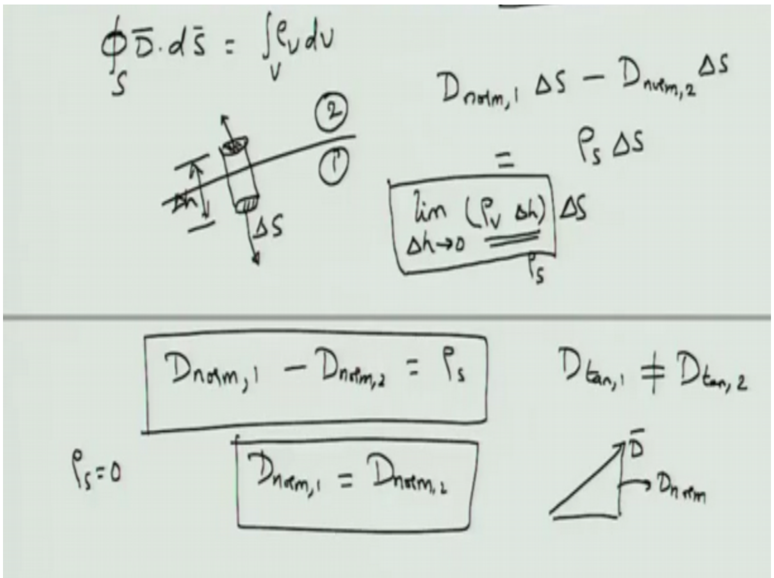

Well, for that I will use the Gauss's law in integral form. Gauss law in integral form states that the closed surface integral of the D field will be equal to the total volume charge that is enclosed in that volume. This is the volume integral, this is the surface integral, not any volume, it is the volume that is enclosed by the surface. So as before I have the two media, okay?

Now I will imagine drawing a small box, whose surface area here is some delta s, the top and bottom surfaces have a surface area delta s and the normal to these surface areas would be pointing along two different directions. They would be oppositely located. This is the medium 2, this is medium 1. Now the tangential electric fields in the top and bottom would not contribute anything.

Only the normal component of D field will contribute to something. So let us just first blindly write on the left hand side. If I write down the left hand side, what I get is D normal one multiplied by delta s minus d norm 2 multiplied by delta s. This minus sign indicates that delta s, the normal to the top end bottom surfaces are in the opposite directions, okay? Which one is minus, which one is plus is not any important.

It is just that these two are opposite in nature that is important. Now let us see what happens to the right hand side. What is there in the right hand side? If these two are perfect dielectrics, then there cannot be any free charges. So a perfect dielectric by definition means that it does not have free charges. What is the material that has free charges? That is a conductor and we do not have a conductor here. So there is no free charge.

So what charge can be there? Can there be polarization charges? Can we have rho s p for one and rho sp for two? For a uniform dielectric that we have considered that will also not be true. So we will not have the polarization charges also. So the only way where we can have this charge on the surface, and hence the charge within this volume will be if someone deliberately takes a charge distribution and places it at the boundary.

Don’t think that this would not happen, this will happen, okay? So someone takes charges and places it deliberately on the surface. Only when there is such a charge distribution, deliberately placed, which will totally unbalanced the charge balance equation, then the right hand side integral can be written as rho s multiplied by delta s. Of course this would be a volume charge, but like the volume charge converted to surface charge, maybe this is a place where you can actually see how the volume charge gets converted to a surface charge.

Let us assume that this is a volume charge distribution rho v. This must be multiplied by the surface area delta s and delta h, delta h being the height of the cylinder. Now if I take delta h tending to zero limit, what will happen to the right hand side? Rho v multiplied by delta h multiplied by delta s, okay? Delta s is not going to zero. Only the height of the box is shrinking, shrinking, shrinking.

As the box height shrinks, delta h goes to zero, but the volume charge density must go to infinity. So rho v goes to infinity and delta h goes to zero. When you multiply a large quantity by a small quantity you will end up with a finite quantity and that is nothing but rho s. Rho s the surface charge density is actually the limit of rho v as the box height shrinks. This is how you convert a surface charge into a line charge.

Now if you have a surface that would look like this, you can reduce the height of the surface and make it into a line charge and you can reduce the line charge length to make it into a point charge, okay. So I can now safely replace this right hand side as rho s delta s. There is nothing to be done more. We have obtained a relationship between normal components of the D fields. D norm 1 minus D norm 2 must be equal to total surface charge that is kept.

In cases where rho s is equal to zero, that would happen if you are not placing any charge deliberately and these are perfect dielectrics, then the normal components of the D fields will also be equal. Please note a very important thing. D tan 1 is not equal to the tangential field in the second region. If you take this as the total D field, only the normal component of the D field, which is this normal component that would be equal.

The tangential components would not be equal, okay? This is very crucial.

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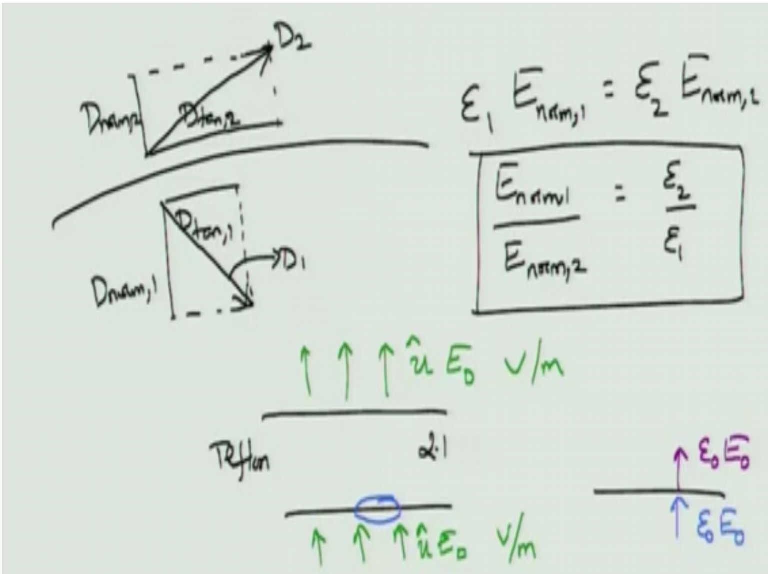

So across the boundary, the tangential components would be different. So let us say this is the tangential components, D tan 1, sorry, this is tan 2, this is D tangential component 1 where as their normal components would be equal, so that actual vector would be obtained by this one, so that vector will be having this tangential component, sorry, this is the normal component D norm 2.

This is normal component one the vector D1 will be this and this is your D1 vector. Similarly, the vector D2 will be, this is completing the parallelogram. So this will be D2. So you can see that, by themselves D2 is not equal to D1. Only that normal components of D1 are equal to each other, whereas their tangential components are not equal to each other. Similar things will happen for electric fields also, why?

Because the normal component of D in medium one is equal to normal component of D in medium two, but how are D and E related, via the permittivity condition, right? So this means, if E norm one denotes the normal electric field in medium one, times epsilon one must be equal to epsilon two E norm 2, which means the ratio of the normal electric fields in medium one and medium two must be equal to epsilon two by epsilon one.

So I have E norm one by E norm two is equal to epsilon two by epsilon one. So now I have got all the relationships that are necessary in order to solve the previous problem. Remember what the problem was? In the last class we placed a Teflon slab and we said that electric field outside of the slab is uniform upward directed with a value of E zero volts per meter. I did not know what was the field inside, but now it is very easy for me to see what the field inside is.

Well, this is the Teflon slab with permittivity 2.1, susceptibility of 1.1, just to jog your memory, and then electric field outside was uniform with E0 directed along the upward direction, volt per meter, outside of the Teflon slab also it was essentially the same electric field. It was upward directed, uniform electric field with a strength E0 volt per meter. What about fields inside?

Now look at this lower boundary? In this boundary, the electric field is coming in normal. What would happen to the normal D field here? It would be the same, except that it gets multiplied by epsilon zero. So the normal D field at this boundary, the normal electric field here that you are getting will be epsilon zero, E0. What should be the normal D field inside this medium?

That should be equal to epsilon zero, E0, correct? Because Dn1 is equal to Dn2. What would be the tangential component? Unfortunately, there is no tangential component, it is only the normal D component that I have. So epsilon zero, E0 is the normal component of the D field outside that must also be equal to the normal component of the D field inside the medium.

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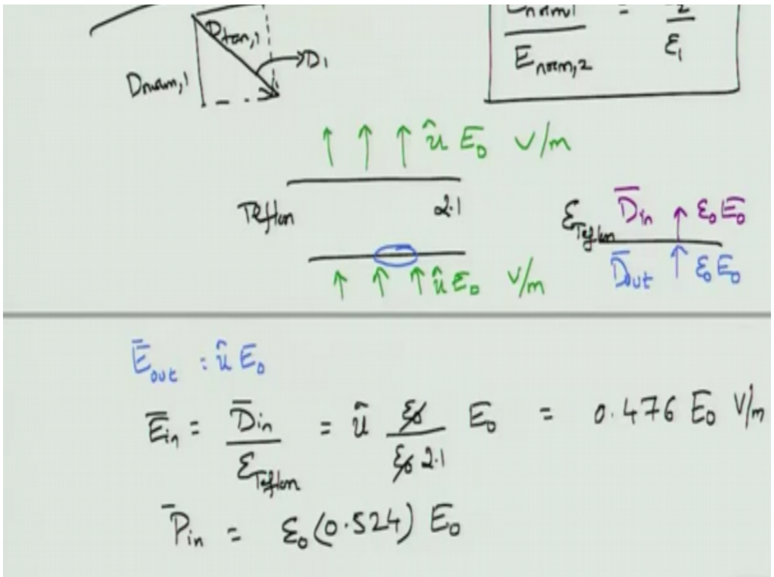

So this fellow is D in and this is D out. Both being normal to each other, they will also be equal to each other. What happens to the electric field? Well, electric field outside is E0, directed upwards. What about the electric field at the boundary? Well, there is no tangential electric field continuity for me to find the relationship. So I have to go back to the relationship between D in and E in, right?

So the relationship between D in and E in is E in will be equal to D in divided by epsilon 2. If I assume that this is epsilon 2, that is, rather let us call this as epsilon Teflon. So this is a Teflon slab, so let us call this epsilon Teflon. So this would be epsilon Teflon, correct? So what is this? Epsilon zero and Teflon epsilon is epsilon zero multiplied by 2.1 times E0, this is along the upward direction.

So epsilon zero will cancel with each other and 1 by 2.1 will be slightly less than 0.5. You can actually solve this equation and you will see that this would be around 0.476 E0, there is an epsilon zero, volt per meter, sorry electric field E in will not have it, right. So 0.476 electric field E0 volt per meter. You can similarly find out what is P in. I will leave this as an exercise to show that this is epsilon zero, 0.524 times E0, okay?

You can find this one by the relationship between D in and E in, okay? So we have looked at the expressions for the electric field and we will now look at another material called conductor. So we will stop here considering dielectrics and we will move on to a new material that we want to consider in little bit of a detail called a conductor.