Electromagnetic Theory
Prof. Pradeep Kumar K
Department of Electrical Engineering
Indian Institute of Technology – Kanpur

Lecture - 20
Polarization - II

In the last class we were discussing about dielectric materials and we were considering what happens when an external field is applied to a dielectric material. We saw that, when the external field is applied to the dielectric, we can model the dielectric as being consisting of dipoles, that is dipoles are introduced or induced inside the dielectric material and these dipoles in turn will generate an electric field of their own.

We saw that, we could actually relate this response of the material right in terms of its polarization to the electric field. If you recall, the dielectric material when it gets polarized the degree of polarization depends on its polarizability.

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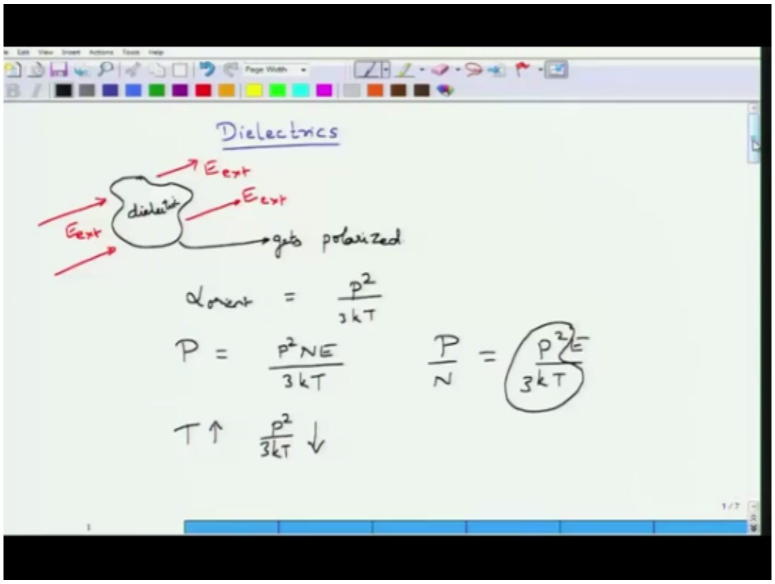

Dielectrics

$$\alpha_{orient} = \frac{p^2}{3kT}$$

$$P = \frac{p^2 N E}{3kT} \qquad \frac{P}{N} = \boxed{\frac{p^2 E}{3kT}}$$

$$T \uparrow \quad \frac{p^2}{3kT} \downarrow$$

If the material is made up of a single type of atoms then, we calculated what is the polarizability of this particular atom, if the atom has the radius R okay, then the polarizability is proportional to R Q and it is although most importantly it is proportional to the linear electrics, the linear of the electric field.

That is, the polarizability is linearly proportional to the electric field. Similar to this atomic polarization, there is ionic polarization in which the ions inside the material gets polarized. There R needs to be replaced from the atomic radius to the ion radius right. After this, we started talking about oriental polarization. Oriental polarization, we actually started describing the expressions for that one.

We found that oriental polarization actually is inversely proportional to temperature and for small values of a certain quantity called A which was E by K T. For small values of P E by K T, we found that the Langevin function through which we had defined the oriental polarization can be replaced by a simple expression of A by 3 and thus write down the expression for alpha orientation or alpha oriental polarization as P square by 3 K T right.

So, this was the oriental polarization and now if you assume that there are lot of these molecules each of them getting polarized according to this polarizability function. Then, you can write down the total polarization P has that of P square N E divided by 3 K T where E is the electric field right, N is the dipole density that is the number of dipoles per unit volume and P is the dipole moment. Therefore, P into N would actually become the net dipole moment density.

Of course, we are interested in polarization per molecule which turns out to be simply P square E by 3 K T. Again notice the linear dependence of the polarization on the electric field. Well, we have discussed 3 kinds of polarizations or this polarizability expressions 1 was atomic polarization, the other one was ionic polarization and the third one was oriental polarization. Amongst these three, atomic and ionic polarizations are temperature independent okay.

However, oriental polarization is actually dependent on the temperature and in fact, it is inversely dependent on the temperature. Meaning, that as temperature T increases this corresponding quantity P square by 3 K T actually decreases. Which means simply that as temperature has increased the kinetic energy that this atoms or molecules inside dielectric material that the kinetic energy of these molecules would actually increase and therefore it becomes extremely difficult to align them to a particular direction right.

Oriental polarization was how you could align into a particular direction and that alignment actually becomes a problem as you increase the temperature okay. So, we can actually combine all these 3 polarizations and write down the expression for polarization. Remember this polarization vector is denoting the amount of dipole moments per unit volume.

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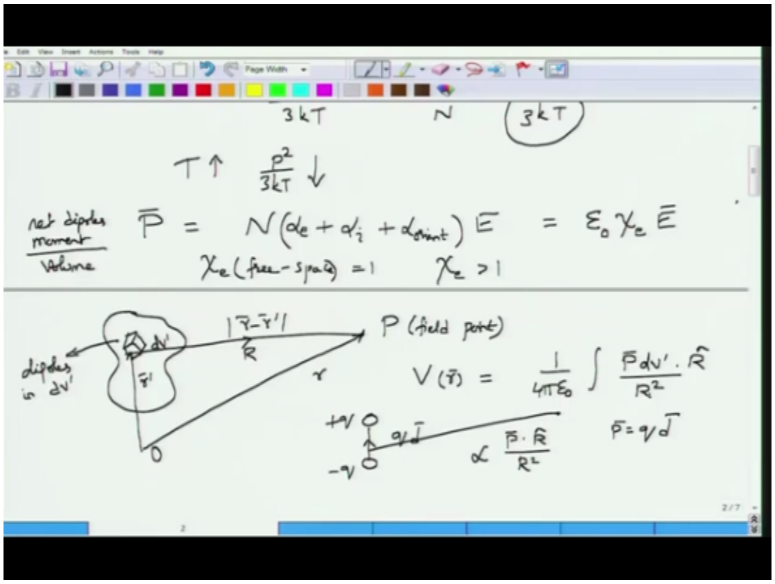

$3kT \qquad N \qquad \boxed{3kT}$

$T\uparrow \quad \frac{p^2}{3kT}\downarrow$

$\frac{\text{net dipoles moment}}{\text{Volume}} \quad \bar{P} = N(\alpha_e + \alpha_i + \alpha_{orient})E = \varepsilon_0 \chi_e \bar{E}$

$\chi_e(\text{free-space}) = 1 \qquad \chi_e > 1$

$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\bar{P}dv' \cdot \hat{R}}{R^2}$

$\propto \frac{\bar{P}\cdot\hat{R}}{R^2} \qquad \bar{P} = q\bar{d}$

It is essentially the volume density of the net dipoles that are present inside the dielectric. So this P e will be equal to the number of elements that you have per unit volume times alpha e standing for electronic or atomic polarization, atomic polarizability, alpha i standing for the ionic polarization. You could replace alpha e or alpha i appropriately when you are considering either electronic polarization or ionic polarization okay plus the oriental polarization that we discussed.

Luckily for all of us the polarization vector, this one, the amount of net dipole moment per unit volume is actually linearly proportional to electric field. In fact, this is typically given by a different name chi e, this chi is the Greek letter and this e just stands for electric field susceptibility. Because we will be seeing a similar parameter when we discuss magnetic fields and magnetization inside a magnetic material.

For now, this subscript e denotes that this material is dielectric and this chi is actually what we call as susceptibility. Susceptibility simply means that how susceptible my dielectric material is to the external electric field okay. That is what susceptibility means and the larger the value of susceptibility, the more easier for us to polarize this particular dielectric. So, e is epsilon 0, epsilon 0 is the permittivity of the free space, this is included so that the expressions become simplified later on.

So this P is epsilon 0 chi e times electric field. Since electric field is a vector, P also becomes a vector. So you have a vectorial relationship of polarization being directly proportional to electric field vector okay. For free space chi e is equal to 1. So, chi e the susceptibility for free space is equal to 1, for all other materials, this chi e is typically greater than 1 indicating that, the polarization vector P would be greater than the free space value of epsilon 0 E okay.

Now, what is this electric field? was it actually the external field that we applied? Actually it turns out that this is not completely the external field because there are these dipoles which have interacted. So, there is some amount of back action that has happened because of the dipoles that are generated and this electric field E should actually be the field that is inside the dielectric, that is inside the dielectric.

There is some back action because of the dipoles because of the molecules, because of the ions that have changed their equilibrium positions as we have applied the external field. Therefore, this electric field is the net total electric field after taking into account all the back action. It just turns out that this back action is so small that for most practical purposes we can neglect that and this electric field will be the field that is generated because of the dipole itself okay.

For dipoles inside the dielectric itself okay. Now, let us imagine that we have a dielectric material okay so this is my dielectric material of arbitrary shape and inside this dielectric material there are lot of dipoles that are induced because of the applied electric field. Now, let us calculate what is the electric field at some point P okay, this is the field point that I am considering at some point far away from the dielectric material okay.

And let us say, we write down this field point vector we can denote this by a particular vector R prime with respect to the origin and we will use first the idea of obtaining the potential and from potential we will try to determine the electric field. That is the easier way of proceeding in this problem because, potential is the scalar function and it is obtaining a vector from the scalar by the gradient operation is actually much simpler than trying to estimate electric field directly okay.

So, let us assume that we had a small volume inside the dielectric that we are looking at and this volume has a differential volume of d v prime and it is kept at a certain position inside the dielectric material. I mean actually we are not putting any volume, we are just assuming that we are actually carving out a small volume for a consideration of finding the electric field okay. So, we just imagine that there is a small volume inside the dielectric.

And then we want to calculate what is the potential because of the dipoles inside this volume on the field point P okay. So, if you look at this their dipoles in this differential volume d v prime and we want to calculate what is the potential at the field point. After we have calculated the potential we will then calculate the electric field okay. So, how do we do that? Well I know this is r prime, this is r so let me get back to the same notation up here.

This is r prime, indicating the source of the charges and in this case charges are the dipole that I have been induced and r is the vector from origin to the field point. Now, the potential is actually a function of r minus r prime or the distance between the, if this are the vectors then r minus r prime, let us call this as capital R okay. So this R being the magnitude of the distance between the dipoles in the volume to the field point okay.

So, the potential is basically whatever the charges that are present inside this volume element divided by 4 pi epsilon 0 into R right. We will use capital V for the potential and small v for the volume okay. So, the potential at the field point okay, let the field point is at some point r right. So this at a vector point r. The potential at this point is actually given by 1 by 4 pi epsilon 0, epsilon 0 is a constant so, we are going to take this outside.

And then what is the total charges that are there inside this volume element? This charge density is actually P times d v prime, right. P is the polarization or the net dipoles per unit volume, multiplying that by the volume element will give me the net dipoles sitting inside this volume element d v prime. So, this would be the total charge that I have. Now, P is a vector so I need to actually rewrite this equation in a proper way.

And write this one as dot R hat divided by R square okay. Why is this 1 by R square? If you recall what is the field of a dipole you will remember that if you considered 2 charges 1 minus q and the other one at plus q and this defined q d has the dipole moment right. At any other point where we were trying to find the field, we found that the field is actually P dot R hat divided by R square.

If the field was proportional to P dot R by R square, where P is equal to q d was the dipole moment, right. So, this is, dipole moment field is actually 1 by R square. Therefore, there is a 1 by R square sitting here and R is the unit vector in the direction from this force charge to the field point. However, we will assume that this field point is kept at a very very large distance. Now, that is how we actually note down this field for a dipole earlier right.

So, we assume that the dipole itself or the dielectric material itself is confined to a small region. In other words, this R prime magnitude is very small compared to the magnitude of R okay. This is kind of quiet realistic in many cases that we will be considering. Therefore, we can make this approximation and obtain that the potential is actually going as 1 by R square okay. So, this is the potential.

We should also write down what is the expression for R in order to write down that let us choose Cartesian coordinate system.

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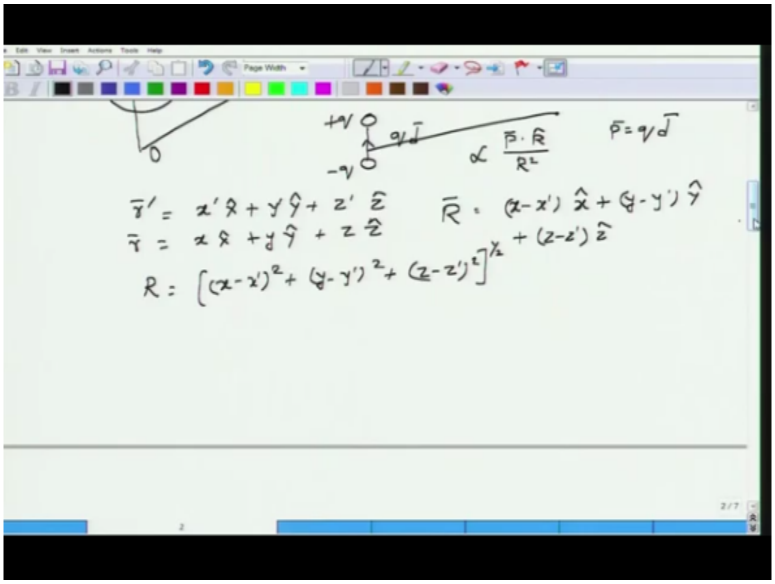

$$\bar{r}' = x'\hat{x} + y\hat{y} + z'\hat{z}$$

$$\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\bar{R} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$$

$$R = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

So, in the Cartesian coordinate system r prime, the source charges is described by the vector x prime x hat plus y prime y hat plus z prime z hat and r the field point vector is defined as x x hat plus y y hat plus z z hat. Note that you could have actually done this entire calculation that we are going to do without really getting into 1 particular coordinate system. It is just that it kind of simplifies our initial you know intuitive understanding of the field that we are about to calculate okay.

So, if you want, you could completely eliminate a particular coordinate system and work directly with the expressions that we have used here okay. That seems to be slightly not intuitive therefore, we are going to introduce some coordinate system. But, the results that we obtain will be independent of the coordinate system okay. So, we have r is equal to x x hat plus y y hat plus z z hat.

So clearly, the vector R will be x minus x prime x hat plus y minus y prime y hat plus z minus z prime z hat and the magnitude vector R will be x minus x prime square plus y minus y prime square plus z minus z prime square all under square root right. So, this is the expression for R and what you want is R square. In order to obtain R square you can remove the square root R square on both sides and you will obtain R square okay.

Now, we will have to introduce 1 vector identity we will actually give you how to derive that vector identity.

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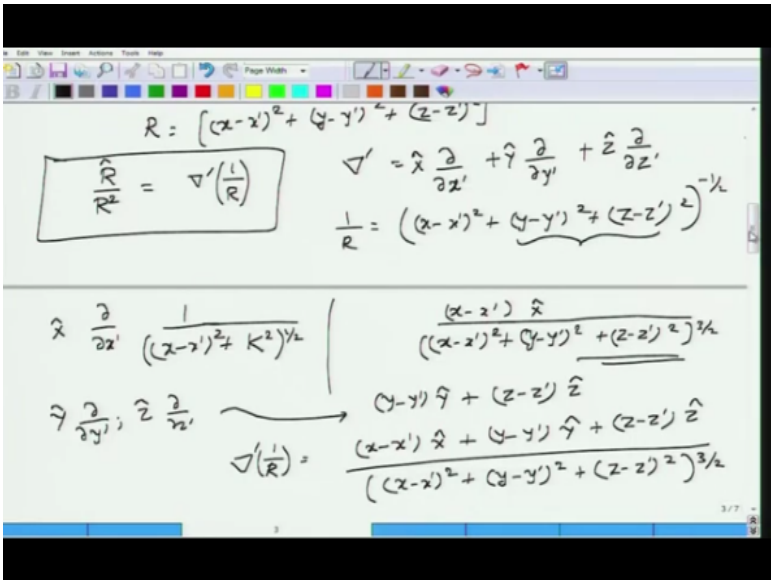

$$R : \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]$$

$$\boxed{\frac{\hat{R}}{R^2} = \nabla'\left(\frac{1}{R}\right)}$$

$$\nabla' = \hat{x}\frac{\partial}{\partial x'} + \hat{y}\frac{\partial}{\partial y'} + \hat{z}\frac{\partial}{\partial z'}$$

$$\frac{1}{R} = \left((x-x')^2 + \underbrace{(y-y')^2 + (z-z')^2}\right)^{-1/2}$$

$$\hat{x}\frac{\partial}{\partial x'}\frac{1}{\left((x-x')^2 + K^2\right)^{1/2}}$$

$$\frac{(x-x')\hat{x}}{\left((x-x')^2 + (y-y')^2 + \underline{\underline{(z-z')^2}}\right)^{3/2}}$$

$$\hat{y}\frac{\partial}{\partial y'};\ \hat{z}\frac{\partial}{\partial z'} \longrightarrow (y-y')\hat{y} + (z-z')\hat{z}$$

$$\nabla'\left(\frac{1}{R}\right) = \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{3/2}}$$

See, I have inside the integral a quantity R by R square okay. This quantity is actually related to this quantity, Gradient of 1 by R now, 1 by R is definitely a scalar function. So, this kind of might look okay because, there is a scalar and on to the left hand side there is vector. So, gradient operates on a scalar gives you a vector that might be alright. However, what is this prime on the gradient operator.

We have not talked about prime on the gradient operator. It just turns out that this prime simply indicates that you have to differentiate with respect to the primed coordinate system that is x prime, y prime, z prime coordinates rather than the unprimed field coordinates which is x, y and z. So, del prime is defined as x hat del by del x prime plus y hat del by del y prime plus z hat del by del z prime okay.

Now, you can try and find out whether this equation is true or not by writing 1 by R as x minus x prime whole square plus y minus y prime whole square plus z minus z prime whole square to the power minus half right that will be 1 by R and now applying the gradient operation, so amongst the gradient operation let us first apply only del by del x prime. Let us differentiate this one with respect to next prime and then find out what would be the resultant expression.

And we will be able to find the expressions for del by del y prime and del by del z prime very easily right. So, let me differentiate this 1 with respect to x prime. What do I get if I differentiate this 1 with respect to x prime? So, I have del by del x prime, this quantity y minus y prime square plus z minus z prime square will be a constant right. When I am differentiating with respect to x prime, it would be a constant.

So, if I look at what is del by del x prime of 1 by x minus x prime square plus some constant let us call the constant as k square okay. Why it is constant? It is constant simply because when I am differentiating with respect to x prime, y prime and z prime are constant okay. So, this is a half up there. Now, you can actually differentiate this 1 using the basic rules of differential and you are going to get x minus x prime divided by x minus x prime square plus this constant square.

Constant we will write it down is y minus y prime square plus z minus z prime square in place of half, it becomes 3 by 2 now because there is a differential right and there is x minus x prime in which direction will this be pointing? It would be along x prime. So, if I differentiate and then look at the gradient along the x prime, that is the component along the x prime, I am going to get this x minus x prime x hat divided by this quantity to the power 3 by 2.

Now, there is no big surprise here that if I try this one with y del del y prime and z hat del by del z prime, these quantities would also give me very similar results of y minus y prime y hat and z minus z prime z hat okay and the denominator will still be the same. It will not change right. It would be x minus x prime square plus y minus y prime square plus z minus z prime square to the power 3 by 2.

Now, collecting all these 3 terms you can see that del prime of 1 by R is actually given by x minus x prime x hat plus y minus y prime y hat plus z minus z prime z hat divided by this following the denominator, which is x minus x prime square y minus y prime square plus right and you can easily see that this equation, I mean this expression of the vector that we have on the right hand side is nothing.

But r minus r prime divided by r minus r prime magnitude 3 by 2 which is you know after you factor out r minus r prime you can write this as r hat divided by r square.

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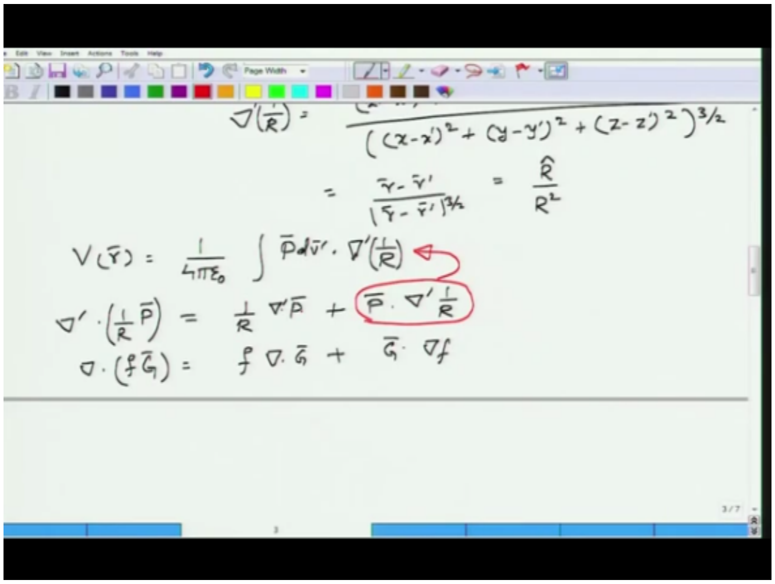

$$\nabla'\left(\frac{1}{R}\right) = \frac{\quad}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{3/2}}$$

$$= \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^{3/2}} = \frac{\hat{R}}{R^2}$$

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int \bar{P}\,d\bar{v}'\cdot\nabla'\left(\frac{1}{R}\right)$$

$$\nabla'\cdot\left(\frac{1}{R}\bar{P}\right) = \frac{1}{R}\nabla'\bar{P} + \boxed{\bar{P}\cdot\nabla'\frac{1}{R}}$$

$$\nabla\cdot(f\bar{G}) = f\,\nabla\cdot\bar{G} + \bar{G}\cdot\nabla f$$

Where r is the vector r minus r prime right. So, we have shown the equation that we wanted to show that is r prime by r square is actually del prime of 1 by R, gradient of 1 by R. Now what good is that equation for us? Well look at this expression here right. Look at this expression here that we have. I have R prime by R square, I can replace that R prime by R square by this gradient prime of 1 by R right. I am going to do that 1 now.

After I do that I get the potential V as 1 by 4 pi epsilon 0 the constant outside, and inside the polarization or the net dipole moment P d v prime dot gradient of 1 by R well that is it. The R square has been observed in this gradient correct. Now, we might want to use or we will be using another vector identity and that vector identity is gradient prime of 1 by R into P okay not the gradient, the divergence of this 1 okay. So, I am hoping that this is visible to you guys.

So, this is the gradient right, sorry this is the del operator with respect to the primed coordinates. So, del prime dot 1 by R P, P is a vector, 1 by R is a scalar right. So, if I have a scalar f and a vector G okay. So, if I have a scalar function f and a vector field G then, del dot of f G is actually f del dot G plus G dot gradient of f. Does it make sense? Yes, it makes sense because, this is a divergence operation on a vector field.

Just because I am multiplying the vector field G by a scalar function f does not mean that I have changed this in to a scalar field. So, this is actually a vector field. So, divergence of this vector field will be equal to f times del dot G. This is all right because del dot G is the scalar. So, scalar times scalar will still be scalar and gradient of f is a vector, G dot gradient of f will become a scalar right. So, I have this identity and I can apply this identity to 1 by R into P okay.

So, if I apply this identity what do I get? I get 1 by R del dot P now, this del is with respect to the primed coordinates. So therefore, this is del prime dot P and this is nothing but P dot gradient prime of 1 by R. Now, if you look at this second term, this is precisely the term that is sitting inside this volume integral. D v prime is just a scalar that can be taken out without any change in this expression right.

So, d v prime can be just put outside and then you have to just move it outside of this term okay, outside of the dot product I mean. So, you have P dot gradient prime of 1 by R and that gradient can be written as, I mean that expression can be written as del prime dot 1 by R P minus 1 by R del prime dot P.

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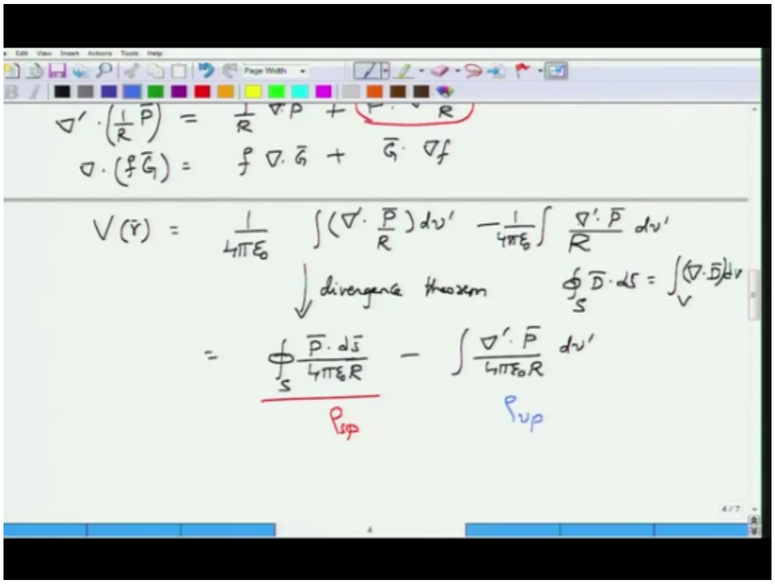

$$\nabla' \cdot \left(\frac{1}{R}\bar{P}\right) = \frac{1}{R}\nabla \cdot P + \boxed{P \cdot \quad R}$$

$$\nabla \cdot (f\bar{G}) = f\,\nabla \cdot \bar{G} + \bar{G} \cdot \nabla f$$

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int\left(\nabla' \cdot \frac{\bar{P}}{R}\right)dv' - \frac{1}{4\pi\varepsilon_0}\int \frac{\nabla' \cdot \bar{P}}{R}\,dv'$$

$\downarrow$ divergence theorem

$$\oint_S \bar{D} \cdot d\bar{s} = \int_V (\nabla \cdot \bar{D})\,dv$$

$$= \underbrace{\oint_S \frac{\bar{P} \cdot d\bar{s}}{4\pi\varepsilon_0 R}}_{\rho_{sp}} - \underbrace{\int \frac{\nabla' \cdot \bar{P}}{4\pi\varepsilon_0 R}\,dv'}_{\rho_{vp}}$$

So, if I make that substitution, I can write down for the potential V of r, I can write this down as 1 by 4 pi epsilon 0, there is an integral and this becomes gradient prime dot P by R right. 1 by R into P is P by R over the volume minus 1 by 4 pi epsilon 0, the second term is del prime dot P divided by R right because, this is 1 by R times del prime dot P so, that is up here integrated over the volume.

The volume integral is actually over the dielectric that you should keep in mind okay. Now, from here we can sort of apply couple of other theorems, I know how to apply Gauss divergence theorem so that I can convert this del dot integral into a surface integral right. So, I can apply divergence theorem here, so I will apply divergence theorem. So, if you are confused what a divergence theorem is, integral of D dot d s is actually equal to integral of del dot D d v right.

This is the Gauss divergence theorem. Surface integral can be converted into volume integral. Here what I have is a volume integral therefore, I can convert that into a surface integral. So this becomes integral of P dot d s divided by 4 pi epsilon 0 R right and this integral must be closed over the surface that the dielectric material is bounded in. So, the dielectric material is bounded by a surface s which is what we are considering.

And therefore this volume integral can be replaced by the surface integral. This let us not change anything. So, I still have del prime dot P divided by 4 pi epsilon 0 R d v prime correct. Now, this kind of starts looking very suspicious like a surface charge density and volume density correct. This first one which is P dot d s by 4 pi epsilon 0 could very well be obtained by an equivalent surface charge distribution.

We will call that as rho s p which is this surface charge because of the polarization and 1 by R indicates that is the potential function right. So, this kind of looks like a potential that could be developed or that could be induced because of a surface charge density, an equivalent surface charge density and that equivalent surface charge density is rho s p, okay. Similarly, if I look at this del dot p right, this could be a volume charge density.

If you go back to that Gauss law integral of del dot d, I mean Gauss law in point found del dot d is equal to rho. This kind of indicates a volume charge density and that volume charge density would be say rho v p. Of course, it is possible to incorporate the surface charge into volume charge, right. So, you can always define a volume as a vanishingly small height with a finite area and include surface charge into volume charge distribution itself in which case we will have only one volume charge equivalent volume charge, okay.

We can do that and that is what is normally done, okay. So, I have a volume charge and the surface charge and please note that I have put in a subscript p here, okay. In addition to the usual s standing for surface and v standing for volume, I have put in additional p here to indicate that this charges are not the charges which are freely available to you. But, they are the charges because of the polarization, that is the material has been polarized.

And these equivalent surface and volume charge densities are available, but they will act as though they are free charges and produce an electric field. Now, you might be rightly asking are these free charges available to us? Yes, and no, yes in the sense that you can actually device measurement on an atomic scale, okay and these charges will be available to you, but, we are considering phenomenon which is not such microscopic skill. We are considering phenomenon over macroscopic situations.

You know i take a length for example which would be a dielectric media and I do not want to put an ionic or an atomic measurements inside and try to find out this or try to access this surface and volume charges, okay. I will be happy as long as I am given an equivalent description, okay. So, I am not looking to actually extract this surface or volume charges, if I choose to, I can but that is not the goal of our study.

So, for us these are not available as free charges rather they are bound inside the material, okay. So, these are sometimes called as bound charges just to emphasize the fact that they are bound to the material. They are not available as free charges at least on the macroscopic scale.