Electromagnetic Theory
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Lecture - 19
Polarization -I

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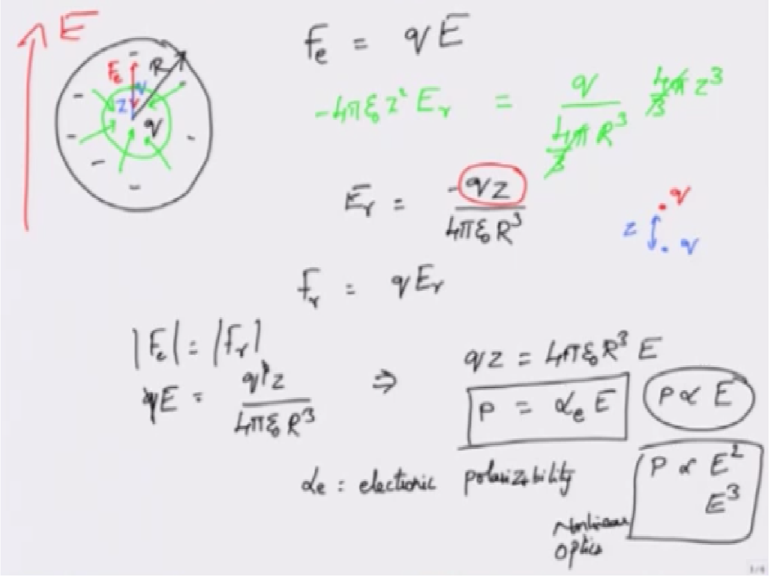

$$F_e = qE$$

$$-4\pi\epsilon_0 z^2 E_r = \frac{q}{\frac{4}{\cancel{3}}\pi R^3}\frac{4}{\cancel{3}}\pi z^3$$

$$E_r = \frac{-\boxed{qz}}{4\pi\epsilon_0 R^3}$$

$$F_r = qE_r$$

$$|F_e| = |F_r|$$

$$qE = \frac{q^2 z}{4\pi\epsilon_0 R^3} \Rightarrow qz = 4\pi\epsilon_0 R^3 E$$

$$\boxed{p = \alpha_e E} \quad \boxed{P \propto E}$$

α_e : electronic polarizability

$$\boxed{P \propto E^2 \quad E^3}$$

Nonlinear optics

Our situation is that I have originally the charge or the nucleus at the centre and electron cloud around it. The electron cloud is uniformly immersed okay and has the radius of R, so it is like a shell of uniform charge density okay and then I have this nucleus q under no applied external field, there is nothing happening here. However, when the external field is applied the charge has moved a small distance z okay forming now a dipole okay.

Now I want to find out what are the forces acting on this charge. There will be one force because of the external electric field. So this is the external electric field that I have applied. So there is one force that is external electric field okay and that is called that as Fe and then there is a restoring force acting to balance out this moment okay because of the electron cloud okay.

And if you equate the two you will get some equilibrium value and that can be used to relate the polarize ability to electric field. Let us also consider the direction of electric field to be along z axis okay. How do I find the force Fe? Well, force Fe is very simple. Force Fe is given by the charge at that particular point times the applied electric field remember this is in fact the definition of electric field right.

So you can think of the charge plus q as or the nucleus as some sort of a test charge for the external electric field and see how much force is gained or how much force is now exerted upon that nucleus and that value will be q times E okay, this is the external force. What about the restoring force are, now here is where we are going to use our knowledge or our expertise in Gauss's law and try to figure out what is the force on that charge?

So momentarily imagine that there is no charge q okay. The radius of this red colour sphere is z, although it is not looking exactly like a sphere but please forgive that one. So now I have this force that I need to calculate at a point z. How do I calculate the force, I can first find out the field, and from the field I can multiply that by a value of q right? Momentarily remove the nucleus and calculate the field at this point and then multiple the field by the charge you are going to get the restoring force.

Now another thing that you will notice is that because the field is given by the electron cloud, the field lines are all directed inwards, the field lines are all directed inwards on this sphere okay. So consider a sphere of radius z okay. On that sphere if you look at the electric fields, those electric fields will all be directed along the radial direction.

You can use Gauss's law and apply that to obtain 4 pi epsilon zero z square, z is the radius of the sphere okay, 4 pi z square is the surface area. Times epsilon zero into Er, I am indicating this r because I want to emphasise that this is a radial electric field and there is a minus sign because electric field is closing inwards rather than spreading out this is inwards. So this is the left hand side of Gauss's law.

This must be equal to the total charge enclosed by this volume. How much charge is enclosed, well I know what is the charge density, charge density is q by 4 by 3 pi r cube. However, the fractional charge enclosed by the sphere of radius z will be given by 4 by 3 pi z cube okay. You can equate the two right, so 4 by 3 pi on both sides cancelled and find out what is the expression for the electric field.

Electric field due to the cloud okay. Er is the electric field due to the cloud, E is field that is applied externally okay. So Er is given by minus q divided by 4 pi epsilon, there is an r cube here, z square you can bring it down to the denominator, z cube is there in the numerator, so that z square will cancel and you get minus q z by 4 epsilon zero r cube. Now, just now we have talked about dipole and this is looking like a dipole thing right.

Because if you look at this qz you can see that this should correspond to a situation where I have a positive charge q and a negative charge minus q and the separation between the two is z okay. Forget the minus sign, the minus sign is only indicating the direction of the electric field okay. So this qz itself can now be related to the dipole moment, of what dipole moment, this is the dipole moment of the atom okay.

Previously, we considered a hypothetical situation in which I had a positive charge and a negative charge separated by a distance. Now I have actually created a dipole inside a matter, matter consisting of atoms by making the nucleus move a distance z towards the electric field. So this qz, that I am writing over here and emphasising so much is actually the dipole because of the application of the external field and this dipole is that of the atom itself.

So we will come back to that in a minute. So you would be now tempted to write qz is equal to p and you will be correct. We will come back to that in a minute because our idea was to still calculate the forces right. So what is the restoring force acting. Restoring force that is acting on the nucleus at this point because of the electron cloud will be equal to q times Er and magnitude of the restoring force must be equal to magnitude of the pulling force.

So I have Fe magnitude that must be equal to Fr magnitude okay. So if you find this substitute for Fe and Fr, you can now obtain a relationship between dipole moment p and electric field, we will do that. So you have qE on to the left hand side okay, which is equal to q square z by 4 pi epsilon zero r cube. So q from left hand side goes and cancels on to the q on the right hand side giving me turning around the equations giving me qz is equal to 4 pi epsilon zero r cube into electric field.

I know qz is the dipole moment p okay of the atom that we are considering. This should be equal to some quantity called as alpha E times E okay and this quantity alpha E is called electronic polarizability, and this is electronic polarizability because this is corresponding to a particular atom of a given matter. So this is electronic polarizability okay and this quantity is something that you will find in material constant books or handbook of materials okay.

What is important to notice that, the dipole moment p was actually directly proportional to the electric field. This direct proportionality is a consequence of linearity that is electric fields are assumed to be so small that they are not going to disturb appreciably the nature of the matter. Therefore, the dipole density or dipole moment will be proportional to the electric field okay.

This breaks down if you go to strong intense field, that is if you apply a very large amount of electric field. Then this relationship breaks down and you will start seeing the dipole moment terms being proportional to E square, E cube and so on. Giving you a rich area called non-linear optics because mostly this is found in optical system. So you get non-linear optics okay. We are of course not going to discuss that one. For us p is equal to some constant times E will suit everything.

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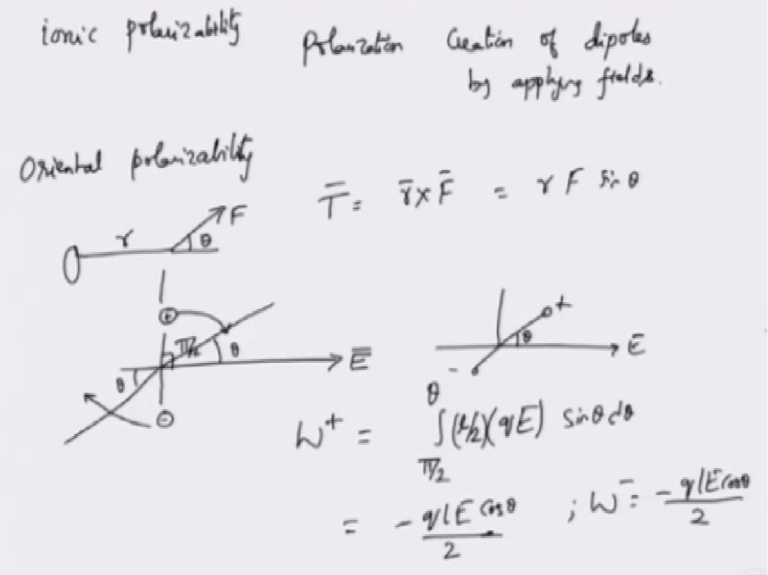

In addition to electron polarization, there is something called as ionic polarizability oaky. Ionic polarizability tells you how easy it is to polarize an ion, what do you mean by polarize an ion that is create a dipole okay. Polarization is a process of creation of dipoles by applying appropriate electric fields okay, creations of dipoles by applying fields. Now there is another type of polarization or polarizability that comes up very often and this is called as oriental polarizability okay.

Oriental by its word tells you that this has got something to do with directions and you are right okay. To discuss oriental polarization, let us look at first the concept of a torque okay.

What is torque, imagine a point O okay which is an r and then you imagine a rope that you have tied to this origin O and then you are applying a force F okay let us see in this particular direction. What would happen imagine moving a gate okay.

So when you are moving a gate, the gate, the hinges are supported at one end and they are connected to the wall and what you do is, you just move the gate like this right. So you are applying a force and then there is a radial length from the hinges to the point of application of the force. What would happen, the gate would rotate right. So the amount of force that you need to apply would also vary with the contact.

You can actually try this very simple exercise. You know you go the end of the gate and then push it, it is very easy to move however, if you go to the, nearby, near the hinges of the gate and try to move, it is that much more difficult than to move it from the very outermost distance. In fact, that is because torque is given by r cross F okay. It is given by r which is the length from the origin, F sin theta, where theta is the angle between r and F.

Now you have a dipole right, so this was just to introduce you to the torque concept of a dipole. So I have a dipole here okay and let us say to this dipole, I apply an electric field along this direction. What will be the moment of the dipole well there will not be any moment of the dipole because when I apply the electric field there is no dipole moment happening right.

When I apply the force here, the dipole actually moves, its sin theta right, so when theta is equal to pi by 2, I actually can move it right. So when I apply the electric field there will be a moment of the dipole that is the plus charge will move a certain distance okay. It will move an angle theta okay. The angle theta is measured with respect to the electric field okay, so it moves an angle theta.

Similarly, the minus charge will move away from the electric field again by the same angle theta okay. So the net effect of this moment is that my dipole has shifted its orientation from original axis towards some angle theta okay with respect to the electric field okay. In shifting this something else has happened.

Since the plus charge has come closer to the electric field or in as rotated in the direction of the electric field, the potential energy of the plus charge has actually decreased. Why is it that it has decreased and not increased because the moment is towards the electric field for the positive charge and moment is away from the electric field towards the positive charge which are it is natural tendencies.

So the potential decrease can be calculated now. If you assume that the electric field is constant, the potential decrease because of the charge moved from an angle of pi by 2 to an angle of theta okay of the positive charge is given by integral from pi by 2 to theta, which you can think of as the work done in order to move the charge over this and what component of the electric field that I want, the component of the electric field that I want is the Sin theta component right.

So this is E, there has to be a sin theta component that will give rise to the torque okay. Only when F is perpendicular to r, the component of F that is perpendicular to r will rotate it. If you try to pull it horizontally, the gate would not actually open right. That is essentially telling you that for torque or for rotation, the force F must be perpendicular to the vector r.

So with decrease of the potential energy w plus of the positive charge, this will be integral of pi by 2 to theta okay, F into r into sin theta. What is r, r is L by 2, F is the force which is q into E on the positive charge times sin theta d theta okay. If you integrate this, you are going to see that this will be equal to minus q l E cos theta by 2 okay. Similarly, you can show that w minus would also be equal to minus q l E cos theta by 2.

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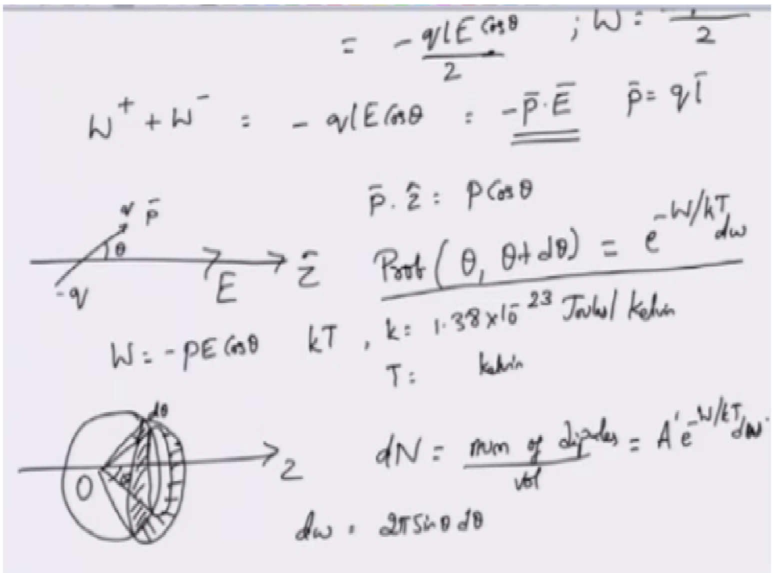

$= -\frac{qlE\cos\theta}{2}$; $W = \frac{\quad}{2}$

$W^{+} + W^{-} = -qlE\cos\theta = \underline{\underline{-\bar{p}\cdot\bar{E}}}$ $\bar{p} = q\bar{l}$

$\bar{p}\cdot\hat{z} = p\cos\theta$

$\text{Prob}(\theta, \theta + d\theta) = e^{-W/kT} d\omega$

$W = -pE\cos\theta$ kT , $k = 1.38\times10^{-23}$ Joule/ kelvin

$T =$ kelvin

$dN = \frac{\text{num of dipoles}}{\text{vol}} = A' e^{-W/kT} d\omega$

$d\omega = 2\pi\sin\theta\, d\theta$

In fact, because of this moment of the rotation of the charges the total potential of the dipole has actually decreased. The total dipole potential decrease or the potential energy decrease is given by w plus, plus w minus which is equal to minus q l E cos theta. Again q into L which is the distance between the two charges can be thought is actually the moment and cos theta indicates that there must be some dot product happening right.

So the dot product can be obtained by writing q l E cos theta as minus p dot E where p is given by q into L vector, of course where the vector is going from minus charge to the plus charge and E is the electric field okay and the amount of potential decrease that happens depends on the angle between the p and electric fields okay. What happens when p is opposite to electric field, then the potential energy would be positive indicating that the potential energy has actually risen because of application of the electric field okay.

Well that different thing that we were going to not discuss now. This p dot E finds extensive use in quantum electrodynamics where such coupling is considered to be the coupling because of the external light and matter. So if you think of the electric field that is applied as is because of light then assuming this matter is p, then this p dot E is called as p dot E coupling and you will find this one extensively in quantum electrodynamics.

We are not going to discuss that. If I consider electric field along z axis, you know application of the electric field along z axis and my dipole is at an angle theta okay. This is minus q and this is plus. This is the dipole moment p at an angle theta, the component of p along z axis is given by p cos theta correct. Now if you consider matter which is made up of many many such atoms okay.

You now cannot say that all the atoms which effectively form dipoles would rotate only by an angle theta right. Different atoms, you know of the matter would rotate by a different angle theta. So what you need to actually ask is; what is the probability that the dipoles would rotate with an angle between theta and theta plus d theta okay. Such a probability distribution will not be uniform right.

It not everything will be rotating by a same angle, the angle of rotation would actually be a probability distribution and this will be equal to e power minus w by kT d omega known as Boltzmann distribution okay, w is of course minus p E cos theta okay and k into T is a thermal energy where k is one point three eight into ten to the power minus twenty-three joules per Kelvin okay.

And T of course is measured in temperature in Kelvin giving you k T as thermal energy okay. So this is the probability that the dipole will be, you can find a dipole in the matter with an angle of rotation from theta to theta plus d theta okay. If this is theta, then there is another theta plus d theta, very small displacement. The probability of finding dipoles would actually be equal to E power minus w by k T into d omega, where d omega is the solid angle okay.

So imagine that this is the z axis. So if this is the angle theta, right? measured from (()) (17:33) theta and this will be theta plus d theta right? So this angle is d theta, so this angle from the z axis will be theta plus d theta, this is d theta, so this solid angle is the angle that is because of this strip okay. So this fellow will actually be a circle and this one will also be a circle right. So the area of this strip is this solid angle d omega.

That is the element of solid angle subtended from the origin O okay. Now if I call d N as the number for dipoles okay, as the number of dipoles per unit volume okay with the angular distribution of theta and theta plus d theta. This can be written as some constant A times e to the power minus w by k T into d omega okay. Where A is a constant that we will evaluate now.

So what is this d omega, well I showed you what d omega is, you can actually employ spherical coordinates and do some little calculations which I leave it as an exercise to give you d omega as 2 pi sin theta d theta okay. And let us call this as A prime rather than A okay.

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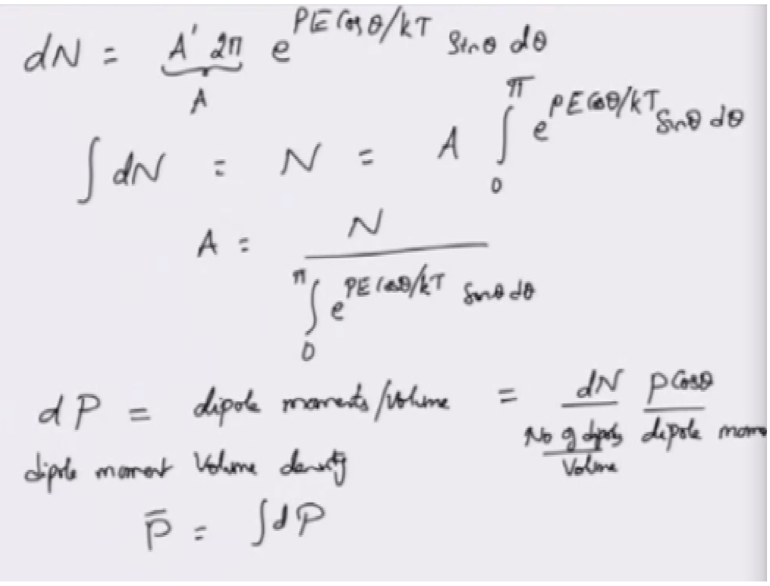

$$dN = \underbrace{A' 2\pi}_{A} e^{PE\cos\theta/kT} \sin\theta \, d\theta$$

$$\int dN = N = A \int_0^{\pi} e^{PE\cos\theta/kT} \sin\theta \, d\theta$$

$$A = \frac{N}{\int_0^{\pi} e^{PE\cos\theta/kT} \sin\theta \, d\theta}$$

$$dP = \text{dipole moments/Volume} = \underset{\frac{\text{No of dipoles}}{\text{Volume}}}{\underline{dN}} \; \underset{\text{dipole mom}}{\underline{P\cos\theta}}$$

dipole moment Volume density

$$\bar{P} = \int dP$$

So substituting that, you get dN as A prime 2 pi e to the power PE cos theta, why PE cos theta because w is minus PE cos theta right. So divided by kT times d omega, d omega is 2 pi here, sin theta d theta. I can combine this 2 pi into A prime as some constant A okay and how do I obtain A, I know that I am considering matter, there will be a certain total number of molecules or total number of dipoles that would be present right.

The total number of dipoles would be obtained by its simply integrating this dN right. So if you do that you are going to get the total number of dipoles in the matter that I am considering. So you carry out the integration, left hand side was integral over dN, right hand side will be integral of theta from 0 to pi. So integrate that one you are going to get the value of A.

So A is a constant that can come out and integral from 0 to pi e to the power PE cos theta divided by kT sin theta d theta okay. So this is the value of A, I mean this is the equation that gives you value of A, A is given by N by integral 0 to pi e power PE cos theta by kT into sin theta d theta. You do not really have to evaluate this one. This is some sort of a total number. This A is just simply giving you the normalisation factor okay.

Now let us introduce another quantity okay. Here is where a little bit of confusion can result. I will explain as much as possible over here what we are doing. Let us call this dP as the number of dipole moments per unit volume okay. dN was a number of dipoles per unit volume, dP is a number of dipole moments per unit volume okay and how are these two related, obviously you can relate this as the number of dipoles per unit volume dN times that dipole itself right.

What is the dipole moment, dipole moment itself is P cos theta, why P cos theta because this is the component of the dipole along the z axis for an angular distribution of theta okay. This is the dipole moment okay and this left hand side, this dN is actually a number of dipoles per unit volume giving you this fellow, which is number of dipole moments per unit volume. So this is some sort of a dipole moment volume density correct.

So this is some sort of a dipole moment volume density and if you want to obtain what is the total number of dipoles, you simply need to integrate, this p has to be small right, to integrate this p over the corresponding volume right. So if you do that one, you are going to get the dipole moment okay. So what is dP, dP is dN into P cos theta, I just found out what is dN, So I will substitute for dN in this equation okay.

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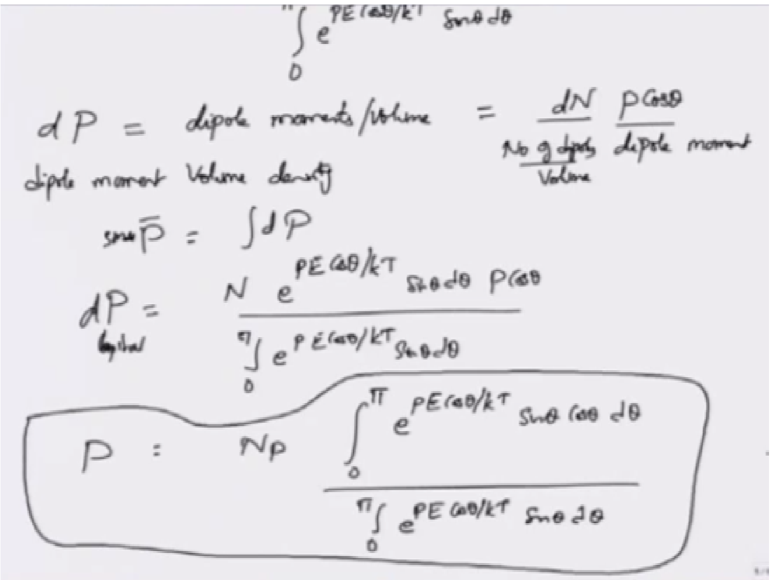

$$\int_0^{\pi} e^{pE\cos\theta/kT}\sin\theta\,d\theta$$

$$dP = \text{dipole moments/Volume} = \underset{\text{No of dipoles/Volume}}{dN}\;\underset{\text{dipole moment}}{p\cos\theta}$$

dipole moment Volume density

$$\text{small } \bar{P} = \int dP$$

$$\underset{\text{total}}{dP} = \frac{N e^{pE\cos\theta/kT}\sin\theta\,d\theta\;p\cos\theta}{\int_0^{\pi} e^{pE\cos\theta/kT}\sin\theta\,d\theta}$$

$$\boxed{P = Np\,\frac{\int_0^{\pi} e^{pE\cos\theta/kT}\sin\theta\cos\theta\,d\theta}{\int_0^{\pi} e^{pE\cos\theta/kT}\sin\theta\,d\theta}}$$

So when I substitute for dN in this equation, I get dP, the dipole moment per volume, the only difference I am able to show here is that this is capital P and I am hoping that you will not get this one by a small p that I have written here. You do not get confused. This is a small p and this is the capital P. Unfortunately, this is the normal usage that is there in the literature, it's confusing I understand, but there is nothing else that I can do.

If I use a different notation you cannot go back to the text books and read it or literature and read this one. So please bare with a little bit of confusion over here. This is small okay and this is capital P. Small p stands for dipole moment. Capital P stands for polarization or dipole moment volume density. So dP is equal to dN into p cos theta, dN we have already written which is e power PE cos theta divided by kT okay, sin theta d theta.

And there was a constant A here, so constant A has been evaluated. Constant A is N 0 to pi right, e to the power PE cos theta divided by kT, sin theta d theta multiply this two P cos theta and to obtain the total polarization P you need to integrate this one over theta right. So you are going to get Np integral from 0 to pi e to the power PE cos theta by kT sin theta, cos theta, d theta divided by integral 0 to pi, e to the power PE cos theta by kT sin theta d theta okay.

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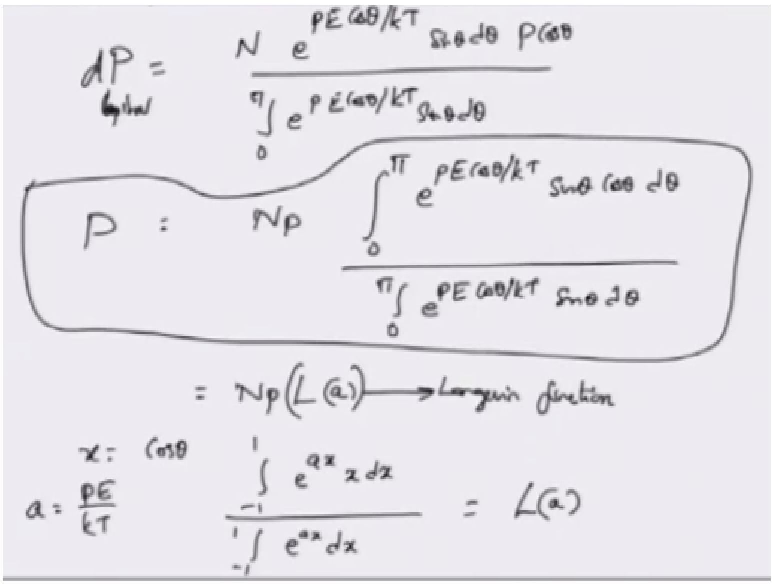

$$\underset{\text{total}}{dP} = \frac{N\, e^{PE\cos\theta/kT} \sin\theta\, d\theta\; P\cos\theta}{\int_0^{\pi} e^{PE\cos\theta/kT} \sin\theta\, d\theta}$$

$$\boxed{P = Np\, \frac{\int_0^{\pi} e^{PE\cos\theta/kT} \sin\theta \cos\theta\, d\theta}{\int_0^{\pi} e^{PE\cos\theta/kT} \sin\theta\, d\theta}}$$

$$= Np\left(L(a)\right) \longrightarrow \text{Langevin function}$$

$$x = \cos\theta$$

$$a = \frac{PE}{kT} \qquad \frac{\int_{-1}^{1} e^{ax}\, x\, dx}{\int_{-1}^{1} e^{ax}\, dx} = L(a)$$

Now you do not have to evaluate all these integrals because this is well documented in literature under a name called Langevin function okay. In fact, this can be written as N p L of a okay and this L of a is called Langevin function okay, this L of a is called Langevin function. And what is this function, you can actually obtain that by going back to this polarization integral and substituting x equals cos theta okay.

So that your d x will become minus sin theta d theta and appropriate integral changes can be done and then also defining PE by kT as a okay, that is the ratio of dipole moment times the electric field, which is some sort of a force because of the external electric field divided by the thermal energy okay. So you can actually substitute this to write down this integral as minus 1 to 1 e to the power ax x dx divided by integrals from minus 1 to 1 e power ax dx okay. This is called the Langevin function.

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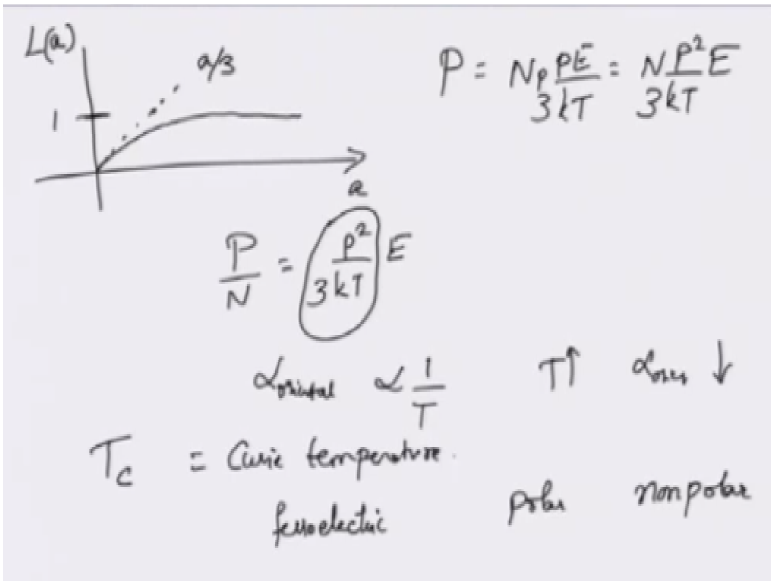

Let us quickly sketch this Langevin function okay. So if we sketch the Langevin function as a function of a, you will see that for small value of a, this will actually be equal to a by 3; however, as a turn starts to increase, this will curve like this and it will asymptotically approach 1 okay. That is for large values of a, the value of L of a can be made equal to 1 or it will be equal to 1 okay.

For small values of a, this is equal to a by 3. So for small values P, the polarization density will be equal to a by 3 and a I know is PE by kT, there was N into P earlier. So this becomes P square by kT N electric field. Now I want to find out what is the polarization per molecular per dipole that would be P by N okay. This will be equal to P square by kT into electric field okay. Sorry there is a three here because this is a by 3 right.

So this is a by 3. So this becomes P square by 3 kT into electric field okay. This quantity is called the oriental polarization okay. This is alpha oriental that we were looking at. This alpha orientation or the orientational polarizability is actually inversely proportional to T okay. As T increases the alpha orientation actually decreases and remember what alpha orientation was, alpha orientation was the polarizability in a particular direction okay.

The polarizability, the oriental polarizability how easy is it to induce the polarization and as temperature increases this polarization goes down indicating that it is not so easy to align the molecule okay. So if the molecules are all at haphazard angles okay by application of an electric field you can sort of bring them to align to the electric field, but such an alignment process will become very difficult as the temperature rises okay because at high temperature, the molecules have a lot of agitational energy.

So they are jumping up and down, jumping up and down and that energy is sufficient to overcome the external electric field and create a sort of random order. All this happens at a temperature greater than what is called as a Curie temperature. So at a critical temperature after Tc, it is simply difficult to align all the molecules okay. So this Curie temperature is very important.

And there are certain dielectrics in which when you apply an electric field okay and take out the electric field, the dipoles will not actually go away. Those are called as ferroelectric materials. There are also materials, which have some permanent dipoles already present. Those are called as polar dipoles okay. Non polar dielectrics are those that do not have an inbuilt electric field okay.

So if you apply an electric field, there will be dipole induced, but dipoles will go back when there is no electric field. Polars have permanent dipoles. Ferroelectrics have this nature of getting aligned with an electric field okay.