Electromagnetic Theory
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Lecture - 16
Gauss's law and its application-III

We saw in last class the point form of Gauss's law. If you recall the point form of Gauss's law introduced as to a new operator called divergence.

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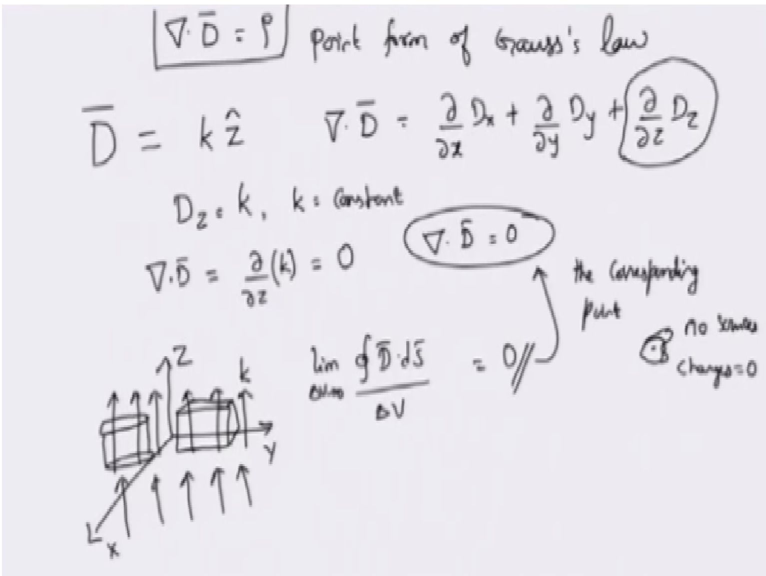

This is a point form of Gauss's law, which was given by del dot D equals rho, rho being the charge that is enclosed or that is charged, that is present at a particular point, D is the electric flux density and this del dot D expresses the vector operation of divergence. So let us look at couple of examples of this divergence or how to calculate divergence and then we will move onto interesting things.

Consider for example that I have vector field D, which is the flux density D given by some constant times z hat indicating that this electric field is constant and it is in the direction of z hat, it is completely independent of x, y and z positions. So if I ask you to evaluate the divergence for this particular vector field how would you proceed, you would recognise that this is in Cartesian coordinates, simple to recognise that one.

And then you can use the formula for del dot D that we developed in the last class, which is given by del by del x of Dx plus del by del y Dy plus del by del z Dz and then substitute for the corresponding components Dx, Dy and Dz. In this case you do not have Dx component, you do not have Dy component, the only component that is present that is non-zero is Dz component and what is the value of Dz, Dz is equal to K where K is the constant.

K is a constant okay. So you now take del dot D, which is essentially evaluating this particular partial derivatives del by del z of Dz and if you do that one, you are going to see that since Dz is constant, its derivatives will also be equal to zero with respect to z and therefore you get del dot D is equal to zero. If for example this particular field, D field would have represented field because of a certain charge distribution.

What is the meaning of this del dot D being equal to zero. For that you try to sketch how the D field itself would like. Now the D field is along the z direction, so let us mark the x, y and z directions on this paper. So you have y, x and z and at all points x, y and z, the vector field D is directed along z axis and has a constant value of K. So it could be return by giving constant K values okay, at all points.

So you have at all points, the vector field D being given by the constant value of K and headed in the direction of z axis. So as you can clearly see if I take any volume over here and then evaluate what is the flux lines that are coming out of the surface, the surface is bounding this particular volume that I am showing here. So there is a D field coming out of the surface. If you try to evaluate what is that integral D dot Ds over the set of surfaces, which correspond to a particular volume and then take or divide the corresponding value by delta V.

This is a close surface and then delta V go to zero, you will see that this particular quantity will be equal to zero simply because the numerator is equal to zero. There are as much flux lines that are entering this surface as those flux lines that are leaving the surface. If you want, you could have try this one in cylindrical coordinates and you would recognise that, that statement is true regardless of your working in Cartesian or cylindrical coordinate systems right.

So you could see that, in this surface which I have taken here, the corresponding value of this integral D dot D will be D equal zero because on the top surface, you have some flux lines coming out whereas from the bottom surface there is as much flux that is going in. So there is balance of the flux going in and flux coming out, which makes the numerator zero and then when you take this ratio of this, when this essentially becomes equal to zero.

In other words, the fact that del dot D is equal to zero indicates that the corresponding point where you are evaluating the divergence okay or you can imagine a very small volume around this particular point, it contains no sources. Here the source for D field is charges, therefore it contains no charges. So at any point where the divergence vanishes, it simply means that there is no charge enclosed in that volume okay.

So you can take that small volume and see that there will be no charge enclosed. Now that is the meaning of del dot D equal to zero.

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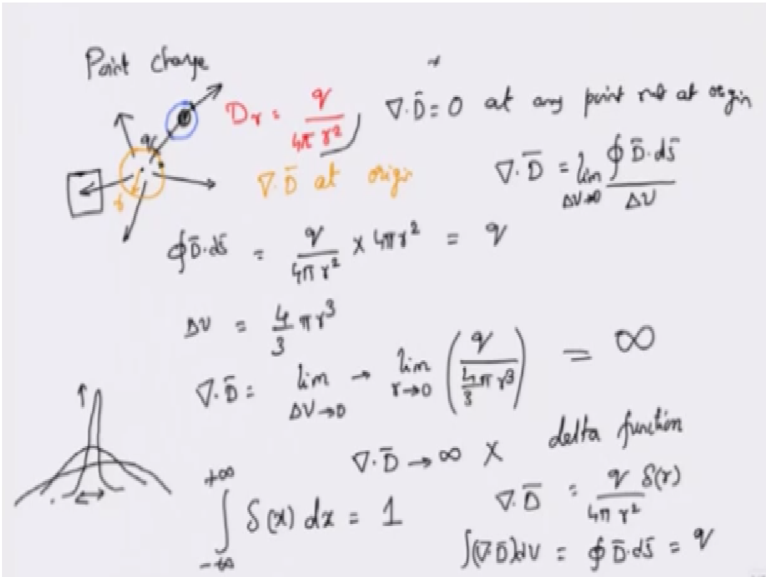

Let us get a little more adventurous and try to evaluate the divergence for a point charge. We know that for the point charge, the D fields would all be in radial direction correct. Let us say the point charge has a value of q, so the D field will all be in the radial direction and this value of the flux density Dr at any particular point r or at any distance r from the origin will be given by q by 4 pi r square right.

It is varying as 1 by r square. Now you imagine that I am going to put a small volume, let us actually try to put a small sphere at any point okay, not at the origin. See, the charge is placed at the origin, but my Gaussian surface that I am placing which enclose as a volume which is essentially a sphere, I am going to place this one at some other point okay. And then you can see this I am not going to derive this one.

But you can see that on this surface the D field will be perpendicular to the surface right. So perpendicular to these two points and at these two points there is as much flux that is going in as much flux that is coming out. Of course, you do not want to take a large sphere that would be like this right, because the D field over here will be different from the D field over there. Yet you would still show that if you start shrinking the volume of this sphere by letting its radius go to zero.

The corresponding point that is at this point del dot D will be equal to zero okay. So del dot D is equal to zero at any point okay, that is not at origin. So this is the statement that we just made that if your volume does not enclose any point charge, then value of divergence will be equal to zero. Now you might ask what if I do not consider this at any point not from the origin.

What happens if I take this sphere around the origin where I know that there is some sphere around the origin where I know that there is some charge enclosed okay. Let us also call this radius of the small sphere a small r. I hope that there will not be too much of confusion here. So I am taking the sphere r, which is now centred at origin and has a radius of r. Eventually, I am going to consider this radius r go to zero, so that I can obtain del dot D at the origin.

So I want to find out del dot D at origin okay by placing a sphere of radius r around that origin okay and then letting the radius r go to zero, so that I can approach del dot D. Now you could do this problem by looking at textbook and then finding out what is the corresponding formula for del dot D. There are formulas for divergence in spherical as well as cylindrical coordinate system that is typically given in textbook, you can do that.

However, let us go back to the definition of divergence and see what would this del dot D turn up to okay. Remember the definition of del dot D at any point, the definition of divergence at any point was the closed surface integral D dot s where the surface s encloses that particular volume divided by the volume element itself and then letting that volume element go to zero okay.

Now I know that D field for point charge goes as 1 by r square right, the D field decreases as 1 by r 2, therefore at the surface of the sphere which is at a distance r or this at a radius r has measured from the origin will be q by 4 pi r square okay. This needs to be multiplied by the surface area of the sphere because the D field will be everywhere perpendicular to the surface and therefore that integral of D dot ds simply becomes multiplying the value of D with the surface area of the sphere.

So when I do that one and I know that the surface area of the sphere of radius r is 4 pi r square, I see that D dot ds that is this integral of D dot ds over the sphere, which is having a radius r will actually be equal to some constant and this constant value is q okay. Now this is not the end of del dot D, this is just a numerator part of del dot D that we have obtained. Now if you look at what is the volume element or what is the volume of the sphere of radius r that we have, the volume is given by 4 by 3 pi r cube.

I hope that these formulas are familiar to you. If they are not familiar, you can consider this as exercises in evaluating the surface and volume integrals in spherical coordinate system and you can find this out. So coming back to this the volume that is occupied by the sphere of radius r is given by 4 by 3 pi r cube okay. Now del dot D will be over the sphere now, see I am not looking at del dot D at the point as of now.

Because I have considered sphere of radius r and I have not yet let r go to zero okay. Now to obtain the divergence at a point at origin, I am going to let the volume element go to zero, which is equivalent of letting r go to zero okay. If I do that this is equivalent of the same thing. So if I do that one then numerator is constant q, denominator has some 4 by 3 pi r cube and as r tends to zero what happens to this quantity in brackets, this quantity just shoots up to infinity right.

So the conclusion is that del dot D goes to infinity. Now this is mathematically alright, but physically this is not correct okay. The reason why this is happening is because we assumed the charge q has no spatial extension and essentially what we were trying to do was to give that value of r which is in this particular denominator for the flux density equal to zero.

So if you try to set r equal to zero, the field of the point charge at the point charge location itself blows up to infinity right. See in earlier cases, we never went so close to the point charge, but now we are trying to find the field at the location of the point charge, which simply means that this D field essentially goes to infinity just blows up to infinity. So this is clearly unphysical result and to deal with such unphysical results.

Mathematicians have introduced and physicists have extensively utilised this tool called delta functions okay. A delta function is something that shoots up to infinity at a particular point; however, the area of that function under if you integrate that one that would be equal to some finite value. Formally, you define delta function at any point say x as this quantity. So when you integrate from minus infinity to plus infinity and then delta of x, Dx is equal to 1.

In practice, you can approximate this delta function by considering pulses okay, of any shape that you want, but their amplitude keeps on increasing. So you can actually you can approximate this delta function by considering a sequence of pulses whose amplitude keeps on increasing while its width keeps on decreasing. So that the area under this pulse or this curve would always be equal to some finite value.

This is called a delta function and in terms of this delta function, you can show that the divergence of D field at the origin will be given by q by 4 pi r square. If you just leave it like this, obviously, this is going to go to infinity at r equal to zero, which is the origin where the charge is kept. Therefore, you multiply that one by delta of r okay and you will interpret this result as saying that if you integrate this del dot D over the volume that fellow will be equal to zero.

That is if you take that integral of del dot D over the volume, which is equivalent of considering integral of D dot ds over the sphere of radius r that will not be equal to zero and that will actually be equal to the amount of charge q okay. So if you integrate this del dot D over the volume okay or equivalently integrate this D dot ds of the point charge with this divergence given up here.

This will be equal to the total charge enclosed, which happens to be the point charge q. At any other points if you try to evaluate the divergence that divergence value will be equal to zero. Please note that divergence is related to the corresponding volume or the surface that you are choosing okay. It is possible to choose a different type of surface okay. However, the evaluation of the D field in any orbitally chosen surfaces is going to be difficult and not usually recommended.

So let us just review back what we said. We started with a point form of Gauss's law okay and we considered two examples of calculating the gradient. In the first example, D was equal to some constant directed along z axis at every point in space. So if you try doing the divergence at any point, this would be equal to zero because the volume or the surface would not enclose the volume, actually would not enclose any charge.

There is as much flow of flux lines into the volume as there is an outflow of the flux that was the first example. In the second example that we considered, we had a point charge placed at the origin whose field decreases as 1 by r square. So if you try to consider sphere, which is at any other location than the origin. Then the corresponding del dot D value evaluated at that point will be equal to zero.

So at any points not at the charge itself the divergence will be equal to zero. However, the value of the divergence at the point goes up to infinity, we accommodate that infinity by defining a function called delta function okay. Let us consider one additional example of divergence and we will wrap this divergence up to introduce you to another vector quantity.

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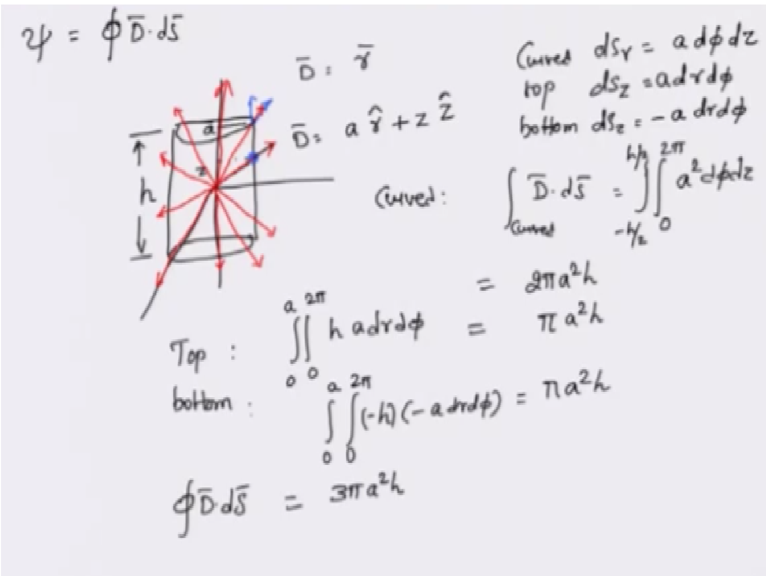

So let us try to calculate or let us try to evaluate this D dot ds over the surface that I am going to give you now. So I want you to evaluate the total flux that is coming out integral of D dot ds over a cylindrical surface okay of radius a and the height h okay. So I consider a cylindrical surface of radius a and the height h where the D field okay, the flux density is given by r. What is r, r is the position vector.

So if you take this as the origin, r is the position vector. So at the surface of the cylinder, which is at this point the corresponding value of D will be equal to a, this will be a times r hat, r hat is the unit vector in the radial direction. So this is the vector that I am going to consider. So a times r hat plus z hat where z is this height z okay. And on this surface you need to find out what is D field?

Of course, D field would also come out of the top and bottom surfaces and you would need to find that also. So if you just sketch how the D field would look like, this is how the D field would look okay. At each point, it is going to be a vector okay. Although I am showing you vectors of different magnitudes, they are not actually different magnitudes. They would all be of the same magnitude up here okay.

So this is how the D field would look if you try to find this one on the surface okay. At each point on the cylindrical surface, you have to breakup that vector r into two vectors, one vector along the radical direction and the other vector along the z direction okay. One vector along radial direction and one vector along z direction. So you have break this vectors all up oaky. At the top surface, this particular line will always be along the z axis.

This line will be along minus z axis okay. So to evaluate this integral of D dot ds over the cylindrical surface, close cylindrical surface that I am considering. I need to first express the electric field at the surface. So now this cylindrical surface has three surfaces in itself right. There is a curved surface, there is a top surface and then there is a bottom surface. On the curved surface, the surface element Ds is directed along the radial direction.

So Dsr is equal to ad5dz because r is equal to a, the radius. On the top surface, the surfaces are directed along the z axis that will be given by drd5a. At the bottom surface, the surface element would be directed along minus z direction and that is given by minus adrd5 with appropriate integration limits that you need to consider okay. So on the curved surface, we have the value of D field being given by ar plus zz hat.

So the integral of D dot ds on the curved surface becomes integral of only the r component will be there, because on the curved surface the surface element is directed along radial direction; therefore, only be the r component that would be non-zero here on the curved surface and what is the r component of the D field, it is a right. You will have a square d5dz and the integral is over 5 is from 0 to 2 pi.

And integral of z is whatever the height of this D field h okay. So you could consider for example this to be from minus h by 2 to plus h by 2, so that a total height of a cylinder is h and if you look at what is this, this could be 2 pi a square h okay. So this is one partial result that you can keep. The other results are the top and bottom results. For the top, the D field is directed along z axis, right.

I mean the D field has z components along positive z direction and the integral of D dot ds on the top surface is given by h into a d r d5 right, where 5 goes from 0 to 2 pi and r goes from zero to a. So if you evaluate this integral, you are going to see that this integral will be pi a square h. Similarly, for the bottom integral, you have integral from zero to a, 0 to 2 pi okay, h will be h.

However, the value of D component on the bottom surface is directed along minus z direction, therefore this will not be h, this would be minus h, but the surface element is also directed along minus z; therefore, I have minus adrd5, there are two minus signs, which can be eliminated because it becomes 1 and if you evaluate this integral, this integral is the same as the integral of the top surface.

And you are going to get a flux of pi a square h leaving the bottom surface okay. So the total contribution to the integral D dot ds from all the three surfaces can be added up and for this cylindrical surface, that we have considered the integral of D dot ds will be equal to 3 pi a square h okay. Now let us find this value by a different method okay.

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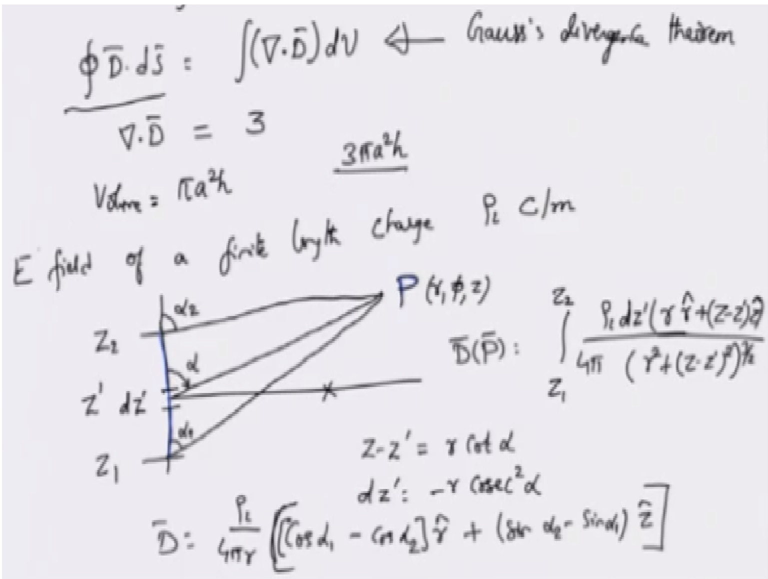

We have already discussed that integral over the closed surface D dot ds can also be written in terms of its divergence that is del dot d over the volume enclosed by that particular surface right. So let us try to evaluate what is del dot d and in this case, you need to evaluate del dot d for cylindrical coordinates okay and if you try to evaluate del dot d for the cylindrical coordinates, you are going to get a constant value of 3 okay.

Now this if you multiple by the volume of the cylindrical element that we have considered, the volume will be pi a square h, therefore the right hand side of this fellow, this is the volume okay, the right hand side of this quantity is actually given by 3 pi a square h. This is rather simple version of finding out this integral of D dot ds instead of trying to use the surface integrals.

This integral which is a very simple because the divergence would be constant and it can be taken out of the volume integral. So that the integral can be simplified gives you the same result as that you would have obtained from the left hand side. In fact, this particular relation we discussed had a name called Gauss's divergence theorem. Gauss's divergence theorem is important.

Because this will allow you to replace the closed surface integral of D okay by a volume integral over the divergence of d inside the entire volume okay. So please keep this theorem in mind, because you are going to discuss next the topic of dialect in the next class then you will be requiring to recall this divergences theorem okay. You can solve some additional problems and show that this divergence theorem always holds, we will not do that one.

Let me go to one additional problem okay in the electric fields and we will then stop discussing the electrostatic fields okay. We will stop electrostatic fields and then discuss some of the other things that we want to discuss okay. Here I want to consider calculating the electric field of a finite length charge okay. I want to consider calculating the electric field of a finite length charge, which has uniform charge density of rho l column per metre okay.

So the problem here is the charge is lying along the z axis assume that it is lying around the z axis and between the points or the planes z1 and z2 okay. This is the length of the charge, so let me mark that one separately. Over this the charge has a uniform charge density, the line charge of the uniform charge density of rho l column per metre and I want to find the field at point p okay away from this charge.

I want to find the field at this point p away from this charge, which is defined by r5 and z. How do I do that, first question would be since we have been discussing Gauss's law, can I actually use Gauss law okay. Well unfortunately, Gauss's law cannot be used in this case because there is no symmetry along z axis. If you recall for an infinite line charge, there was symmetry about the azimuthal angle 5 as well as there was symmetry about z.

You could go up and down along the direction of the charge, but you would not see any difference in the line charge density. However, for a finite length charge there is obviously no symmetry along z because you can imagine moving from a lower portion z1 and then you keep moving up to the point z2 right and beyond that if you try to move, there will not be any charge that is visible to you, there is no charge extending beyond z2.

Similarly, if you try to move below z1, there will not be any charge. So the symmetry in z axis is broken, which is another way of saying that the corresponding electric field will be a function of z. However, if you go around the charge inside at any point you will see that the charge would be independent of 5 okay. For a infinite line charge, the charge would be independent of 5.

But in this case the symmetry around the line charge element will be visible only if you are going around the circle on this axis okay, because at this axis if you keep going around the fields corresponding from the top portion as well as the bottom portion would cancel each other out and there will be a symmetry along 5. However, in this case, if you go at any other point right, there will not be such symmetry.

For example, you can imagine this point p located somewhere up here and in this case you can keep moving around and then will not be a symmetrify that you can easily verify, I mean that you cannot easily see that there will be symmetry up there okay. Finally, symmetry along r because the field will be dependent on r okay, so let us try to evaluate this one. If you try Gauss's law, we cannot really use Gauss's law as I have said, but instead we can use Coulomb's law okay.

So you Coulomb's law in order to calculate the electric field okay. So let us use that one, I am going to use Coulomb's law to calculate d and then I am going to relate d and e okay. Strictly speaking, Coulomb's law gives you the electric field e okay; however, in the medium that we are considering d and e are related just by a constant epsilon zero. Therefore, I can use Coulomb's law to calculate this electric field okay.

If I consider any point okay, a small length dz prime and then find the electric field at this point, I can label the angle here as some alpha okay. Alpha will be the line; I mean alpha will be the angle that this line from the charge element dz prime makes on to the point p okay. There will be this line alpha, there will be this angle alpha measured from the z axis. So you can now write down what is the D field at point p okay.

The D field at point p is given by integral from z1 to z2 because you will have to go up and down the charge covering from z1 to z2. At any point, you have the charge rho l dz prime okay divided by 4 pi epsilon 0 r square or other magnitude of r cube times the unit vector times, the vector at point p right. So if you recall Coulomb's law, this would be rho l dz prime r divided by or other r hat divided by 4 pi epsilon 0 r square where r is the distance from the charge location to the field location.

This is Coulomb's law, so if you apply the Coulomb's law to this one, you are going to evaluate the vector distance from dz prime to point p okay. This is the point at any z that I am considering whereas the location of dz prime is at some z prime okay. So this would be z minus z prime along z axis divided by the magnitude of this vector. So we are going to get r square plus z minus z prime square to the power 3 by 2 okay.

There is no epsilon here because I was looking for Coulomb's law. So this D is actually epsilon e, so you can actually relate the two as we have said. So how do I integrate this particular integral, well in turns out that I cannot really integrate this to give you a nice analytical expression that I have gave you in the case of infinite line charge or an infinite plane of charge or a sheet of charge.

So this is the brute force method that you need to employ, you can solve this integral or you can evaluate this integral by a certain substitution method, so you can try z minus z prime is equal to r cotangent of alpha r cot alpha, so which implies that dz prime will be equal to minus r cosecant square alpha. You remember that cot is one by tan and cosecant is cos by sin right, so I hope that you remember these two formulas from trigonometry.

So you can substitute for this, what you have to see is that, at the ends where you go, there will be two extreme angles alpha 2 and alpha 1. Alpha 1 will be the angle from the bottom portion of the line charge okay, at the below of the line charge to the field point p and alpha 2 will be the angle that the line from z2 to p makes with respect to the z axis. So the limits of integral from z1 to z2 will be converted into the limits from alpha 1 to alpha 2.

You can evaluate this integral and find that D field is given by rho l by 4 pi r okay. So there is a 1 by r at least that is somewhat a good thing for us because in the infinite line charge also we had a field going as 1 by r. So there is a 1 by r dependence, which is nice multiplied by some cause alpha 1 minus alpha two okay. You will get this when if you evaluate the integral by making these substitutions you can evaluate the integral.

And after changing the limits you will get this one plus sin alpha 2 minus sin alpha 1 along z axis okay. So there is a dependency on both r as well as z in the electric field D as we have return in this particular thing. Alpha 1 and alpha 2 themselves depend on the point p okay. If you are going to go around the point p these values of alpha 1 and alpha 2 will also change.
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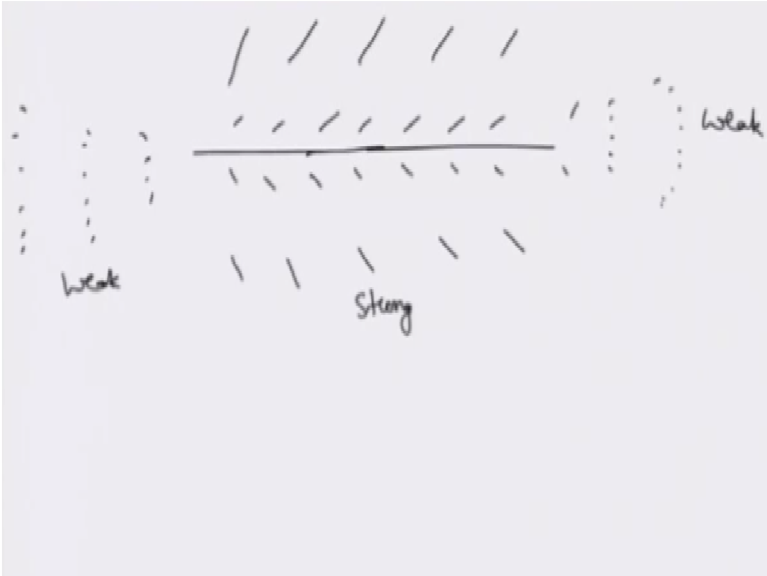

If you want to sketch the field pattern for this assume that the charge is lying along here okay. If you sketch the field patterns, you will actually see that the field starts to become very thin as you go to the top or bottom okay. So the field actually becomes very thin as you go here okay. However, the field is stronger at the centre of the charge and it progressively goes to lower values as you move away from the charge okay.

So you can see that the field is actually very weak over here okay and the field is actually quite going as 1 by r in the middle of the charge okay. Again at the bottom also the field will be very week as you go away from the charge okay. So the field is actually very weak at these points. The field is strong only here. Such line charges are not uncommon okay. If you are familiar with precipitation, you will see that, later we will discuss precipitator.

In precipitators, you will see that there is a finite length of wire, which is getting charged okay. Actually it is getting electrified by passing a current, but in the simplified version we can consider this to be a uniform line charge of charge density that is given by electrification process how much charge density we are imposing, but the field around this lined charge would look like this okay.

Alright so we are sort of closing up on electrostatics now, there is lot to discuss in electrostatics we will come back to that, but before that we will have to introduce you to one more vector operation and then we will review electrostatics, that is we will close electrostatics and then we look at applications of electrostatics okay. What is that I want to introduce you to, we have seen two vector operations.

One vector operations is called the gradient operation, which allows you to express the electric field as gradient of a potential function, the potential was a scalar field and therefore this was very interesting to us because measurements on the potential are were easy to make compared to the electric field measurements. So you could make measurements or you could specify the potentials in a region of space.

And from there evaluate the electric field, potentials are scalar's whereas electric field is a vector and you could do that one by two operations, one an integral in order to evaluate the potential and from the potential you need to take the gradient, which is differential operation or derivatives that you have to take in order to get the electric field okay. The other vector operation that we introduced due to worse divergence.

And divergence tells you how much flux is emanating from a particular point. So if the flux is outgoing at a particular point, then it is called, then the point actually has a source, otherwise if the flux lines are closing in on a particular point, then at that point we have a think. So you could think of a positive charge as a source because the field lines would all emanate from that and you could think of the negative charge as a thing.

Because all the field lines would converge on to that particular point okay. So we have seen two vector operations and there is actually a link between gradient and the next vector operation that we are going to discuss.