Electromagnetic Theory
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Lecture No - 14
Gauss's Law & its application - II (Contd.)

In the last class, we looked at Gauss's law which helps us in finding the electric field for a given charge distribution, if the charge distribution has some sort of a symmetry then Gauss's law can be used or can be exploited, the symmetry can be exploited to find the electric field. We saw couple of examples of using the Gauss's law. One was the familiar infinite length line charge with the linear charge density of Rho L Coulomb per meter.

And we found that the D field can be obtained by putting up a Gaussian surface which is cylindrical in nature okay and then the field was essentially going as 1/r, so it was inversely varying as r okay. We also saw another example of using Gauss's law, this example was infinite sheet of charges okay and then we took Gaussian surface as a small pillbox okay, this is sometimes called Gaussian pillbox.

And then we found that the field because of this uniform surface charge distribution of charge distribution density Rho S Coulomb per meter square was essentially independent of where you are located, so it was independent of all the location. Of course this two are not realistic in the sense that they are not, you are not going to find infinite line charge or infinite amount of plane charge.

But there are good approximations for a long line charge or a uniformly spread out plane of charges okay. We will continue with couple of examples of Gauss's law and then we will look at what is called as a point form of Gauss's law. So to recap if you have already forgotten what Gauss's law is?

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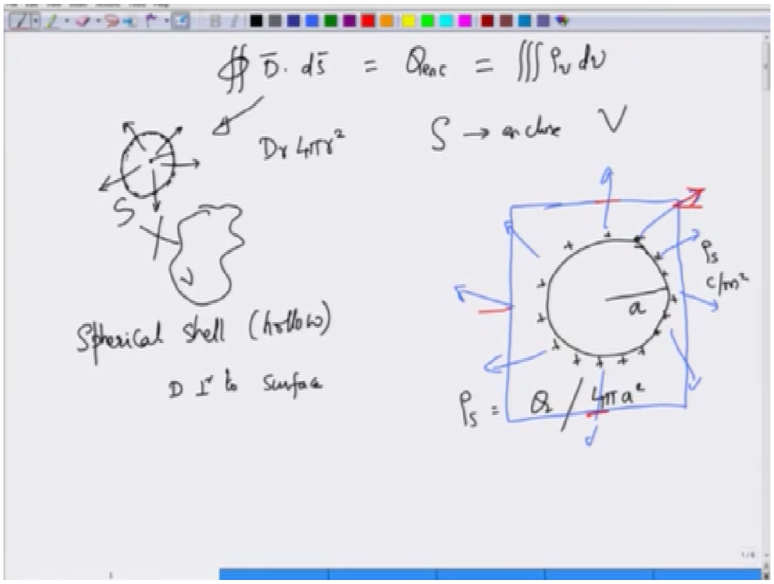

Gauss's law states that the integral, the closed surface integral of a quantity D okay so D dot d S which is integral of this quantity D which is called as a electric flux density and it is measured in Coulomb per meter square, so this quantity if you integrate over a closed surface this should be equal to the total charge that is enclosed. Of course that charge that is enclosed if the charge is uniformly distributed in a region of space will be given by the integral of the charge density over an appropriate integration.

Why do I say appropriate integration? Because if the charge is spread out in the region of space in terms of its volume charge density then you have to integrate the volume okay and if the charge is spread out over a line then you have to integrate over the line if the charge is spread out as a point then as a point charge then there is no integration involved directly you are going to get the charge.

I did not specifically mention the relationship between the surface that you have to use in applying Gauss's law and the volume. For example, if I consider the point charge Q okay so we know that the field lines from earlier Coulomb's law itself we know that the field lines are going to go as 1/r square, we also saw how to use Gauss's law in this example or if we have not seen fairly simple.

You can assume a Gaussian surface okay that would enclose the charge, there are some certainties involved in this one we will come back to the certainties some time later, so you assume a Gaussian surface of a certain radius okay as measured from the origin and then you apply Gauss's law over here okay. Now that will be the left hand side so you can assume as Gaussian surface which is spherical because this is a nice spherical symmetry you can apply.

And then you get the left hand side which would essentially be some D r into 4 pi r square. If your sphere has a radius of r then the left hand side will give you D r into 4 pi r square, 4 pi r square being the surface area of the sphere of a constant radius r okay. Now what about the volume? Should we actually talk about the volume of the surface that is enclosed, that is, can I take the volume outside of the charge or outside of the Gaussian surface.

So if this S stands for the Gaussian surface that I am using the calculation of the left hand side then can I take the volume over here? No. I have to take the surface and the volume both to be same essentially saying that the surface S must enclose the corresponding volume that we are looking for. In other word choose the surface and also see that the surface which is closed you have to evaluate the volume of the enclosed surface, okay.

You cannot choose a surface here and the volume somewhere else, this two have to be the same okay. So this is some precaution that you have to take when you all solving the Gauss's law, so with this in mind let us go ahead for one example of Gauss's law, something that would come up fairly later as an important example. So here what we are looking for is a spherical shell of charge, okay.

This is a typical example that you are going to use Gauss's law with when there is a spherical symmetry so for this kind of problem Gauss's law is very easy to apply. If you were to try apply this one with Coulomb's law u will struggle a little bit okay. So if I consider a spherical shell and let us also assume that the shell is hollow okay so it is an essentially the shell is made up of conducting material and it is in the form of a nice sphere of radius a okay.

So on this shell if you were to induce some charge by any of the charge induction process, you will see that this charge nicely distributes on it is on the surface of the shell, so the charge nicely gets distributed on the surface of the charge and then you want to find out the field everywhere okay. That is the problem that we are going to solve using the Gauss's law so I have this spherical shell here okay.

The shell has a certain radius, call that the radius as a and the charges that distributed over the surface of this shell okay. So the charge is uniformly distributed over the surface of the shell so with a uniform charge density of Rho S Coulomb per meter square okay. Please remember that this surface area we are considering is actually spherical in nature okay. So if you take the overall surface which would be, I mean the surface area of this sphere will be 4 pi a square.

And if Q is the total amount of the charge then the surface charge density is Q by 4 pi a square, this amount of total charge is spread uniformly over the shell of the shell. Now to apply the Gauss's law let's look at the left hand side okay, we need to of course choose a surface over which the D field will be perpendicular to the surface, I mean it is not always possible but if it is possible you always choose a situation in which D field is perpendicular to the surface.

What is the advantage of choosing D field which is perpendicular to the surface? Now let's say for argument reason this is not correct so let us say the D field is in this way okay to the surface that you have chosen or rather let us go back and write the field here the fields all would be directed radially away okay the field is directed radially away but instead of choosing such a surface a spherical surface what I am going to do is I will choose a rectangular surface okay.

I will choose a rectangular surface. Now what happens? This part of the D field which is of course originating from the shell is not perpendicular to the surface area right so there is some components of D that there would be along the normal to the surface right, similarly over here that D field not perpendicular. Here the D field is perpendicular but such a line is only one and probably on the bottom side there will be one.

So evaluating Gauss's law over such a surface becomes mathematically tedious and that actually defeats the whole point of using Gauss's law, the whole point of using Gauss's law is to minimize the integration the effort that you are going to put in the integration and you can do that provided you choose a surface in which D is perpendicular the surface. Right so that the dot product which is sitting between the Z and ds can be eliminated and it simply becomes scalar integral okay.

I mean it is actually a scalar integral but then the evaluation of the dot product becomes very easy to perform so this kind of an arbitrary surface although Gauss's law actually holds, its not well suited for our purpose because we do not want to increase our effort in integration so with that in mind I am going to choose a Gaussian surface which is going to be spherical.

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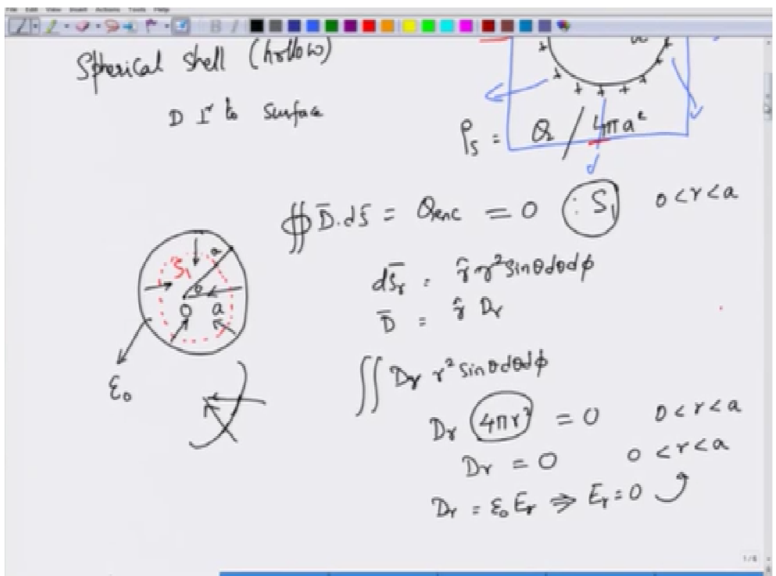

So let me redraw that Gaussian surface over here and apply the left hand side of the Gauss's law, radius a, now I have to choose the Gaussian surface which also will be in in the form of a sphere. Let us say I choose two surfaces, one surface I am going to choose inside of the shell okay so the center of this sphere will be the origin and I am going to choose a surface S1, let us call this surface S1 which is inside the shell okay, now what happens to the left hand side of Gauss's law?

Left hand side of the Gauss's law tells you that over the closed surface if there was any D field over here this must be equal to the total charge enclosed by the surface. Now clearly the right hand side will be equal to zero for this case. Why? Because this surface S1 you have to assume that this surface S1 is closed, of course it is closed and then you have to see that there is actually no charge enclosed by this okay so there is no charge enclosed by the surface S1.

And therefore this will be equal to zero on the surface S1 that we have considered. Of course the surface S1 we did not specify what the radius of is, the radius of the sphere S1 could be anywhere from zero all the way up to a, right so as long as r is less than a which defines the surface S1 the corresponding integral over here will be equal to zero right now the integral is zero is does it actually mean the d is zero? That is an interesting question right?

So if you are looking at this integral that integral is zero but what happens to the field D? Let us do this on the surface, assume that there is a D field if there is D field then it would be perpendicular right to the surface, it would be perpendicular, so if this is a sphere all the line which are along the radial direction will be perpendicular to this surface of the sphere. So what is the incremental surface area that we have on the surface of sphere S1?

That incremental area will be directed along the radial direction and that is given by r hat, let's call the radius of this one as r itself so I have r square sin Theta d Theta d Phi okay. Please do not get confused between this r and this r, this r actually stands for the radius of the sphere S1 okay and D has only the r component or the radial component so that becomes r hat D r, so if I consider the dot provided between the two.

I am going to get D r and then r square sin Theta d Theta d Phi integrated over the sphere radius alright. Now I also know that my D field does not depend on Theta and Phi because of the symmetry condition the D field does not depend on Theta and Phi okay. So D r will essentially be constant over the surface and actually can be moved out of the integral and the remaining portion of the integral will simply give me the surface area 4 pi r square.

So I actually have a constant D r that is D r is constant over the surface, that is getting multiplied by the surface area of the sphere S1 which is 4 pi r square but the right hand of this side is equal to zero because r is less than a, right? Since this will not be equal to zero right, the only conclusion that we are going to have from this expression is that D r must be equal to zero inside a, so inside a hollow shell D r will be equal to zero.

The radial component of D which is the only component of D that would be equal to zero but we also know that if the material is made up of some homogeneous material or if it is just free space or air then the material will be described by the constant Epsilon zero the permittivity of the medium which relates D r and E r right, the radial component of the electric field. So if this is if D r is equal to zero.

And Epsilon zero is not zero which simply implies that E r will also be equal to zero, so as long as you are inside the shell, the electric field components will be zero okay, alright so this was for the electric field component, inside the shell.

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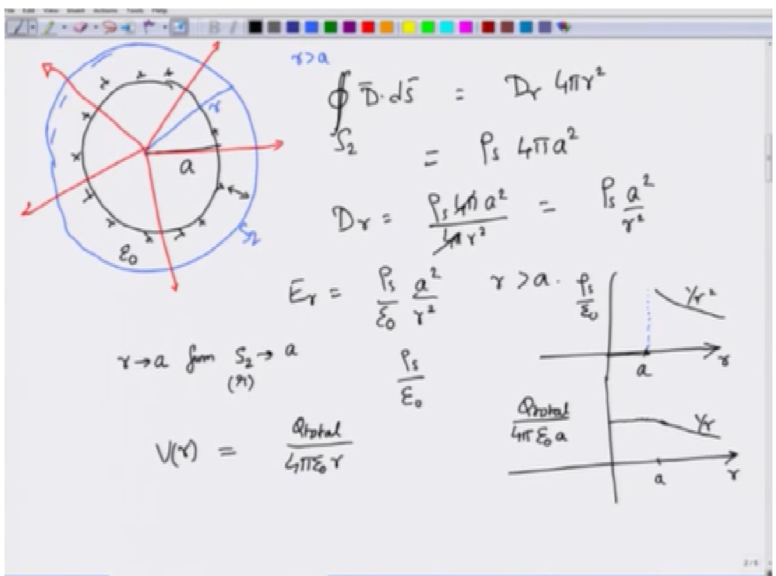

Now what happens if I choose the Gaussian surface outside of the shell? Right so outside of the shell if I choose what will happen? So let's say I choose the Gaussian surface, let's call that as S2 okay this has a radius of r, again in this case r is greater than a, so if I choose this surface, call this as S2 as the Gaussian surface which is outside of the shell and there are of course charges on the shell. Let us write that one otherwise this become like two circles written.

And we will lose what we are actually trying to solve okay. So this is our situation we have chosen a surface S2 which is outside of the shell okay so what would happen to the right side and left side of the Gauss's law? The left side again will have a situation where that D field will be radially directed as well as there would be constant on the surface S2 okay.

There would be constant on the surface S2 allowing us to rewrite this D dot d S integral over the closed surface now with respect to surface S2 as D r 4 pi r square 4 pi r square being the surface area of S2 okay. Please remember that r is greater than a, so this is the left hand side of Gauss's law so what happens to the right hand side of Gauss’s law now? You must now see that this should be equal to the total charge enclosed by the surface S2.

Now there is no charge in the intermediate region over here, right the amount of charge that is enclosed must come by integrating the charge density over the surface area of the sphere which is the original shell charge that we are considering and that original shell of charge has a radius of a. Therefore, the right hand side becomes Rho S into 4 pi a square. Is that correct? So the right hand side will be equal to Rho S 4 pi a square, okay.

That is the total charge that we have okay so Now we will simply equate the left hand side and right hand side of Gauss's law thereby getting D r as Rho S 4 pi a square divided by 4 pi r square so 4 pi in the numerator and 4 pi on the denominator will go away and now you all left with Rho S a square by r square, okay. This is the radial D field, the electric field E r again assuming that the medium outside here is filled with Epsilon zero.

So the radial electric field E r will be equal to Rho S by Epsilon zero a square by r square provided r is greater than a, now at r is equal to a, if you see from the surface area S2, see what happens as you come close to a from shrinking S2 which was originally with a radius r to sphere of radius a. So what happens as you start shrinking S2, that is if you come outside of the sphere so if you start coming towards to the shell spherical shell of the charge that contained.

So what would be the value of the electric field predicted? The value of electric field predicted will be obtained by writing r is equal to a, so when you substitute r is equal to a, this fellow will become Rho S by Epsilon zero right, so from the outside of the shell as you start moving towards inside you are going to predict electric field of Rho S by Epsilon zero. However, if you remember, what we have done in couple of minutes earlier?

Inside the electric field was actually equal to zero right so if you now start expanding the surface S1 so you start expanding the surface S1, which means that you are you are r which was the radius of surface S1 is now becomes closer to a, that electric field will become that electric field will still be equal to zero. So there is some amount of discontinuity between what the values you get when r is less than a and you approach a from the left hand side.

And what you will get from r greater than a then you approach the shell on right hand side. So this discontinuity in fact gets is this discontinuity is not completely incorrect because that is precisely the amount of discontinuity that is required to maintain the charge distribution okay. We are going to see this type of discontinuities when we discuss boundary conditions later.

But for now remember that this discontinuity is the reason why I am going to why there is a charge distributed on the shell okay. So if you want to sketch you can sketch the electric field, so let us sketch the electric field of the function of the radial distance r, so you will see that until r is less than a, there will not be any field. At r is equal to a, the field will be equal to Rho S by Epsilon zero okay and then it starts to go away as 1/r square, okay.

So if you are very far away from the spherical shell the charge would essentially will be 1/r square and the entire charge would look like as though it is coming from the center of the spherical shell. So it looks like the entire charge that was actually distributed on the surface is actually concentrated on the point at the center of the shell okay. Now what about the potential V of r?

We know that the potential and electric field are related with each other and we can actually calculate what is the potential, the potential for an electric field which is going by r square will go as 1/r right, so this was this is similar to the point charge potential that we have seen so that we can calculate the potential which you can obtain by integrating the electric field from one point to another point and you know dropping taking the point of origin as infinity.

And considering all potentials with respect to infinity as the absolute potential at a point, this v of r will be equal to the total charge enclosed divided by 4 pi Epsilon r okay so the potential is going as 1/r. Now here is an interesting thing, what will be the potential if you start sketching the potential what will be the sketch of potential that is electric field was discontinuous here right the electric field was discontinues over here.

What happens to the potential first consider r much greater than a, so in this case the potential is going as 1/r so there is a 1/r so this behavior is 1/r okay. Now as you start to moving towards the shell right so the potential keeps on increasing and at r is equal to a, you are going to get a total potential of Q total which is the total charge enclosed which is into Rho S 4 pi a square if you want to write that down, divided by 4 pi Epsilon a.

This must be the potential on the spherical shell of the charge right so this will be the value Q total by 4 Epsilon a at r is equal to a. Now what will be the potential to the left? Will it be zero or will it be some finite value? It is interesting to see that one, that the potential will not actually go to zero okay the potential will be constant and it would be continuous at this point whereas the electric field was discontinuous this potential is actually continuous.

Now why would potential be continuous and why is the potential not going to zero inside the shell? Well if you recall what is the definition of the potential, potential was the energy that was required to move a charge from one point to another point, right? so if you start far away from infinity and then you move saying that radial path you keep moving, the potential will keep on increasing, right?

The potential energy keeps on increasing and as you reach the shell you are at the maximum potential right so you are at the maximum potential you have done whatever the work that is required to move the test charge from infinity to the shell. Now let’s try to move hypothetically of course, we will try to move the test charge inside the shell. Now, inside the shell do you have to do any additional work in order to move the charge? Answer is no. Why?

Because there is no electric field inside the shell, the shell has no electric field. Therefore, you do not have to do any additional work in order to move the charge once you have brought it to the edge or at the surface. The surface of the shell then inside you do not have to spend any more energy in order to move the charge which means that the potential must remain constant, also the potential cannot change discontinuous manner, okay.

So the potential will be continuous and the potential will actually be equal to Q total by 4 pi Epsilon a, inside right up to the center of the shell, okay. So this is very critical. Please note that electric field can be zero. And electric field is zero inside the shell but the potential is not zero okay alright. So with this we will sort off close Gauss's law now. There is a lot of other symmetry charge distribution that you can keep working on, okay. So we are not going to consider them.