Electromagnetic Theory
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Lecture No - 13
Gauss's Law & its application - I

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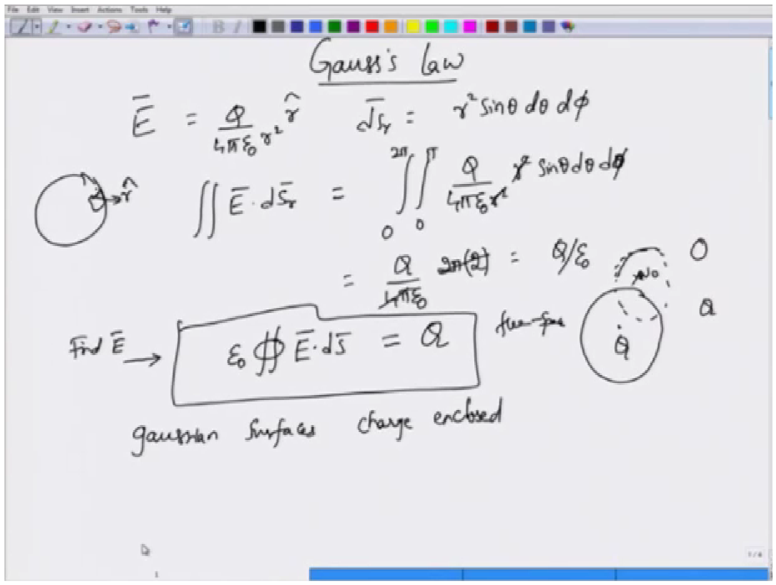

So the subject of this and perhaps next lecture is important because we are going to discuss Gauss's Law and this is the first law of electromagnetics okay. It was first Maxwell's equations, sorry, not law. First Maxwell's equation for electromagnetics. Now Gauss's Law in free space can be formulated in terms of electric field. How? I know the electric field of a point charge; the electric field of a point charge is given by Q by 4 pi epsilon zero r square along the direction r.

If I integrate this electric field over surface, okay of constant radius, if I integrate this one over surface of a constant radius in the spherical coordinates what do I get? We will try it out what we get? So what is the d S r, which is the surface integral of a spherical coordinate system I know what this is right? This is r square sin theta d theta d Phi at a given radius of a r of a given radius r of a unit sphere, this is the surface area element, okay.

So this is the surface area element that we are looking for and this is given by r square sin theta d theta d Phi. Now if I integrate this one what do I get? I have a Q by 4 pi epsilon zero r square multiply by r square sin theta d theta going over d theta and d Phi so theta goes from zero to pi whereas pi goes from zero to 2 pi, correct? So you will see that the r square cancels with numerator and the denominator, Q by 4 pi epsilon is actually a constant.

And the remaining part with integral over d Phi you going to get a 2 pi and integral over sin theta from zero to pi you are going to get another value of two. So you are essentially going to get a 4 pi which gets cancelled on both numerator and denominator, so the result will be Q Pi epsilon zero okay. If you do not think that this is interesting, consider if I bring this two together what happens?

Integral not just any integral this is a closed surface integral right so this is a closed surface integral of the electric field and I just used the surface of a spherical. You can actually imagine any other surface right it does not have to be spherical surface; it can be a cylindrical surface; it can be a point; it can be a cube; it could be any other surface, it could be a general kind of a surface and in free space this is true that if you take such a closed surface and it is important.

It has to be closed surface and then carried out the integral E dot d S on this you are going to see that the value will be equal to total charge. Well it is actually equal to charge by epsilon zero so I can actually push this epsilon zero to the left hand side and I get this expression which is very interesting. It tells me that integral of electric field around a closed surface on the closed surface will be equal to the total charge enclosed right.

If I have a point charge here I take a surface around this and then tried to apply this equation, what I will find is if the charge value is Q the integral answer the left hand side will also give me Q. What if my surface is like this? Right the one that I have shown in the dots, what will be the output here if you try to integrate the electric field? Well the output will be equal to big zero because there is no charge enclosed here no charge enclosed in this surface.

So as long as the surface is enclosing the charge you are going to get charge and such surfaces are called as Gaussian surfaces, a Gaussian surface is one actually encloses and in it is a closed surface which is typically used to calculate the charge enclosed okay. Now this is by itself may not have been very useful unless I do not know what is the electric field but I can somehow guess what is the charge distribution, okay.

For most analytical problem which possess some sort of symmetry this equation can be used okay to find electric field. We are going to do that couple of examples now but this is the first Maxwell's equation in free space, this equation applies only in free space, okay. What about situation when there is no free space? What if there is a medium in between? Well for the medium in between you have to define a new quantity called electric flux density.

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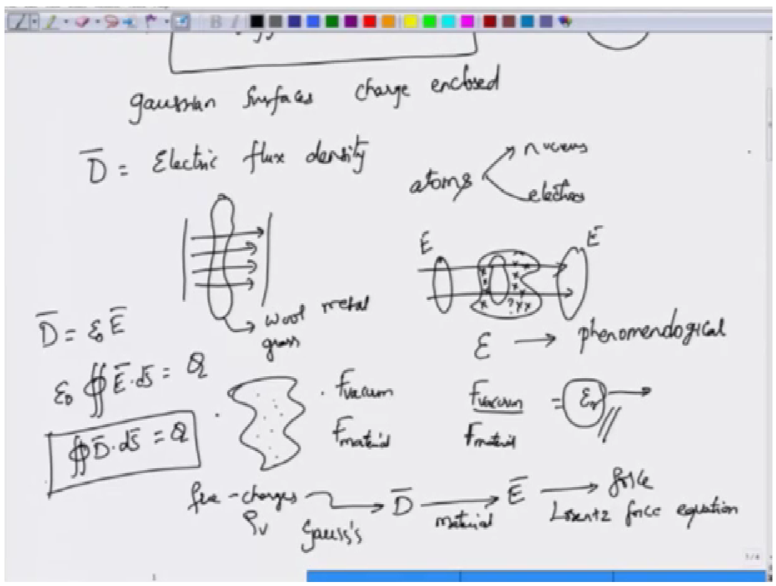

The importance of this electric flux density is this. I consider a parallel plate capacitor for example and then I apply a potential difference which causes in electric field between the two this will be the electric field it would be all located in between okay this would be the electric field okay there is a potential that I have applied to this parallel plate capacitor. What if I substitute the medium which was assumed to be aired here from a medium which is different from here?

What if I put wool or grass or a metal plate inside? What will happen to the field configuration? Well what happens is that the field actually changes because all these matters wool, glass, metal although these are different type of materials, they all are composed of atoms and atoms are composed of nucleus and electrons, okay. So that, there are charges on the microscopic scale, so to any matter, if it -if I apply an electric field, okay through any matter, outside the matter, the electric field would be E outside, it would E but what would be the electric field inside?

It turns out that I have to consider this microscopic structure and find the electric field because of all the microscopic charges that are present inside. Now as we have already seen earlier there will be an enormous number of microscopic charges residing inside and to calculate or to take into account every type of charge that is residing inside and calculate the corresponding interaction will make the problem very tedious.

So what we assume is that the material can be expressed in terms of some measurable quantities and will be defined by certain material properties okay, this are microscopic material properties that we will assume that will describe the material. Much of this is not experimentally obtained although there are very good experiments, much of this is phenomenological, that is to say phenomenological means to say that you observe something observe a phenomena.

And then postulate what could be the type of the material properties okay. If you just take a point charge, calculate the force here in free space you are going to get some force in free space or vacuum okay now if I fill this region in between by some material, the force will change from vacuum to say F with material in place, now I can use the ratio of the vacuum to material okay and then call these ratio as epsilon r, okay.

And this itself will tell me that there has been a material inserted and for as for as the forces that I am measuring on the charges is concern that is all I need to know. I do not need know what is happening in the internal details of this material in between okay I do not need to know what is happening inside of the material, for all practical purposes I just need to know what is this epsilon r and I can use this as a characteristic of the material okay.

However please note that the electric fields are all generated by not by the charges inside but rather by the free charges that you have, okay. What we assume now is that these free charges do not give me the electric field okay in the sense that I am not going to determine the electric field from these charges. These free charges will actually give me the flux density D which I am going to relate it to the electric field right.

If I replace this material from one material to another material my epsilon r changes right. So I need actually a quantity that would be completely independent of the material that I am talking about. A quantity that I can use and then relate later to the electric field through the material properties but as for this one is concerned I do not want to get stuck to a particular material because if I get stuck to a particular material then I have to find out all the interactions that are happening.

And then I need to keep evaluating that interactions okay. So because of that we assume that free charges generate D. Which is the independent of the material properties and the D will generate in turn electric field E which the connection between the two phenomenological and that describes the material properties okay. The electric field E is the one which gives you the force on the other charge.

So how does free charge generate D or what is the relationship between the charges in general free charges Rho V to the electric flux density D? This is given by Gauss's law okay and the relation between D and E is material dependent and this is given mostly by phenomenological arguments or by some measurements plus phenomenological arguments, how does the electric field apply force on the electric charge is given by Lorentz force equation, okay.

So this is the new structure, earlier we have seen that free charges are there free charges generate electric field, electric field will in turn exerts a force on the charge. Now because of the material properties we have free charges generating D; D generating E; E generating force. In free space you can define D as epsilon zero into E or epsilon not into E. So that the equation which we had written epsilon zero integral surface integral of electric field over the surface was equal to the total charge contained can be replaced.

And written as the surface integral over this flux density D that should be equal to the total charge contained. This is so important and this is called as Gauss's law, so we are going to discuss application of Gauss's law now.

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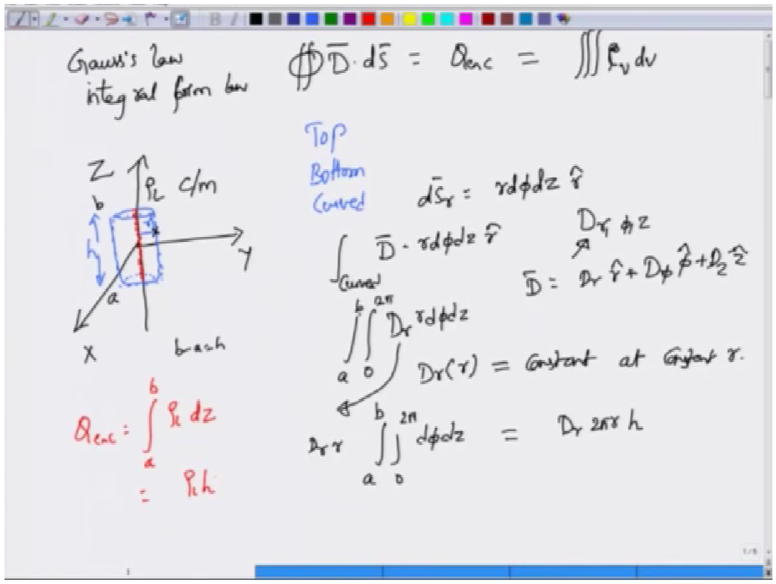

So Gauss's law tells you that if you take a closed surface and then integrate the electric flux density you are going to get the total charge enclosed. Now I know that charge enclosed can be in general written as the integral of the volume charge density okay this equation is Gauss's law or Maxwell's first equation, I mean first, he did not actually numbered them but it is something that we can conveniently take that.

Because this is the first equation that we are encountering which will eventually become Maxwell's equation okay. So this equation is Gauss's law or Maxwell's equation; Maxwell's first equation and this is an example of an integral equation or an integral form of the law okay. Why integral form? Because there are integrations on the both left hand and the right hand side of this law okay. How useful is this Gauss's law for me?

Consider the infinite line charge; we have seen infinite line charge so many times it might actually feel that we are actually revisiting just a friend okay. So I do not have to explain too much about the line charge for you except telling you that the line charge density is Rho L

Coulomb per meter. See we struggled a lot to find the electric field because of this infinite line charge right we first start with the Cartesian coordinate system.

We had to do three integrations and we found out that the problem could be simplified somewhat by using a different coordinate system which was much better suited for it, which was cylindrical coordinate system. Then again we could use symmetry to deduce that the only electric field component that we are going to get will be radial component and that the electric field also will be radially directed okay so we saw that in the last class.

Now we will still use the symmetry arguments but let us apply Gauss's law to find out what will be the expression for the electric field okay. To find Gauss's law I need to first find the Gaussian surface. Let me assume a Gaussian surface which is cylindrical in nature okay this is the Gaussian surface that I am applying, this is a cylindrical Gaussian surface okay of some height h. Does not matter what is the height of this one, this is some height h, okay.

And this at radius of r okay this is a radius of r it, this r could be anything you just imagining different cylinders of different radius r, so you fix a radius of the cylinder in this case to equal to r and of course r itself can be variable and the height of the cylinder is h. Now how many faces this cylinder have? Three faces; top, bottom and the curved surface right. Now symmetry already tells us that no matter where on this surface point on the cylindrical surface I am located.

Symmetry already tells me that there cannot be z component there cannot be a Phi component correct, so the only components I can have will be along the radial component and that is good for us because on the curved surface the surface of the cylinder is given by r d Phi d z right going along the radial direction. So I can substitute that into this expression. So I have the closed surface integral broken up into three parts; top, bottom and curved.

So I can consider the curved surface integral which will be D dot r d Phi d z and r dot. So clearly if D has components of r Phi and z correct the only component that will be non-zero in this expression will the D r component, so writing that as D r so writing D as D r r hat plus d Phi Phi hat plus d z z hat, okay. I can see that the dot provided r make r dot Phi go away r dot z go away so I the inside of this one for the curved surface will be just D r r d Phi d z.

What are the values of d Phi and d z? Well for this curved surface Phi should go from zero to 2 pi and Z would go from any point, which will in total will give me a height of a h okay. So let's say this is some a to b such that b minus a is equal to h then this integral will be a to b okay. This is the expression right to the left side of the Gauss's law; this is the left side of the Gauss's law. Now I can proceed to do the integral over here.

If I know that the D r will not be a function of Phi and z and we know that D r is not going to be a function of Phi and z because the symmetry if you move around the line charge you are going to see the same line charge everywhere so d r cannot be a function of Phi. Similarly, you move up and down the on the infinite line charge you are going to see the same line charge which means that D r will again be independent of z so D r is a function of r only, okay.

And this is the integral over Phi and z. However, this is crucial d r of r is actually a constant. Well I just said that the d r is a function of a r but how could it be constant? Well remember this is a constant at constant r right. So I am actually considering a cylinder at constant r therefore on that constant value D r will be a constant so which means that I can move this D r out of the integral.

So that I am left with r d Phi d z, r will also be a constant. So r also can be moved out, okay and I am left with integral from zero to 2 pi; integral a to b, so if we evaluate this one you are going to get D r 2 pi r into h. This is the left side of the Gauss's law. What about the right side? To get the right side you need to know what is the total charge that is enclosed here by the surface, the charges enclosed only on the line, so what is the total charge enclosed?

Total charge enclosed will be the integral of the line charge right so you have a Rho L and then d z over the same height; a to b and this will be equal to Rho L into h. Now equate the left hand side and the right hand side of the Gauss's law.

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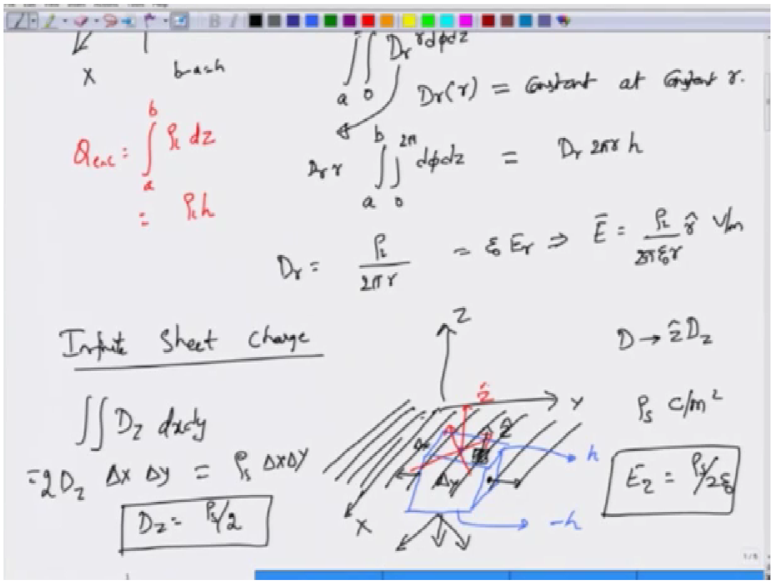

So that you obtain D r as Rho L by 2 pi r. D r will be also equal to epsilon zero into E r right, in free space assuming that this is in free space this will be equal to epsilon zero E r which implies that electric field will be radial and it is given by Rho L by 2 pi epsilon zero r volt per meter along the radial direction okay. Now let me get you to a different charge distribution, we have seen line charges enough that it might bore us, so we will look at infinite sheet charge, okay.

There is a hard way of doing this problem and there is an easier way of doing this problem. What we mean here is that there is an infinite, there is an infinite sheet of charge that we have placed okay so this is x this is y, so I have placed this charge at x is equal to, sorry, that z is equal to zero plane and I want to find out the electric field at any point here. How do I find the electric field?

Well choose again Gaussian surface okay so although I have drawn lines here they should be actually a surface okay this should be a surface for me, lets proceed by writing the block okay so I take cube okay and then cover the cube such that the cube is halfway between, so this is the surface the cube the top surface of the cube is located at some height along z and the bottom surface is located z is equal to minus h, okay.

What will be the electric field? Consider a point here okay consider a point on the surface, now for every point that is residing here I can actually find from the surface charge, I can find two points such that or two lines such that the electric field will be directed in this way but the resultant of the two electric field will be directed along z axis okay. So you can imagine that this is a plane charge that I am showing okay this is a plane charge that we have.

For this plane charge I am going to consider a Gaussian surface, that the Gaussian surface will be halfway located like this okay for this halfway located surface so this is equal to h and this is equal to minus h, now on this point right now imagine that on this point I am going to find the electric field. There is a line here or there is line inside this surface which would run like this. There is another line that would run like this. There is or along this line is line charges all uniform right?

I can actually think of a plane of charge as an infinite number of line charges right a plane of charge can be written as infinite number of line charges, so this line charge which is running here and you have to imagine the box here would give me an electric field in this direction right because at this point I want to find the electric field it could give me the electric field in this direction.

Similarly, there will be an electric field because of another line charge at the same distance and the same magnitude so this would also be directed in this way. Now I have two charges; one directed this way the other, I mean two forces one directed this way the other one directed this way so the resultant electric field or the resultant D field would have to be perpendicular and it would be along the Z direction okay.

You can do this for any point on the pillbox that we have considered, so at any point on the pillbox that we have considered, you can show that there are two line charges which gives you electric fields of equal magnitude such that the resultant electric field is always along Z axis okay. Then I know that the D field is going to be along Z axis only right so it will be say z D z okay, so what will happen to the left hand side of the Gauss's law?

It would be the integral of D z over the surface area. Now what is the surface area here, the surface area is directed along the Z axis and I know what is the surface area that is nothing that but d x d y over a surface delta x delta y right that is to say that this surface that I am considering will have delta y and delta x so the area of the surface total will be equal to D z delta x delta y okay.

This is the left hand side of the Gauss's law. Why was this happening? Because at any point on the surface I could consider there were two equal line charges which would give me the electric field which was away from the Z and it would actually be constant okay, similarly there will be electric field in the or the D field at the bottom, they would also be obtained because of the two equal magnitude lines and the resultant will be along the minus Z direction.

So therefore the total D field that is here will be two times this value, okay. Why two times? Because the total flux enclosed are the total flux that is coming out after integrating will be on this top plate dz dz along Z but on this bottom plane surface area is along minus Z, right? the surface area is along minus Z that the D field along also minus Z okay. So there will be essentially add up. Nothing will happen to the side because the electric field is not or the D field is not present there at all.

The D field is all directed along Z and the surface area on to the right and the left as well the back and the front will not contribute anything to this integral. So I have the left hand side 2 D z Delta x Delta y, the right hand side will be the total charge enclosed. What would be the total charge? If I assume a uniform line charge, sorry, uniform surface charge density of Rho S Coulomb per meter square which is entirely on the Z equal to zero plane.

The total charge enclosed by this surface area will be Rho S into Delta x Delta y which will give D z of Rho S by 2 electric field E z will be equal to Rho S by 2 epsilon zero righ?. So this is interesting because the electric field directed entirely along the Z axis and that this electric field is completely constant it is not depending on the x; not depending on y; not depending on z, an electric field is the same strength as it is at hundred meters as it could be at one million kilometers from the source, right?

Of course, this is completely an idealization but a good example would be this tube light okay so if I had only one tube light here that the lighting of the tube light provides would essentially have the same strength near the tube light or on this table or on the floor okay so it does not change anything if I go from here I am seeing the exact illumination and if I take the top I am seeing the exact illumination.