Intelligent Systems and Control

Prof. Laxmidhar Behera

Department of Electrical Engineering

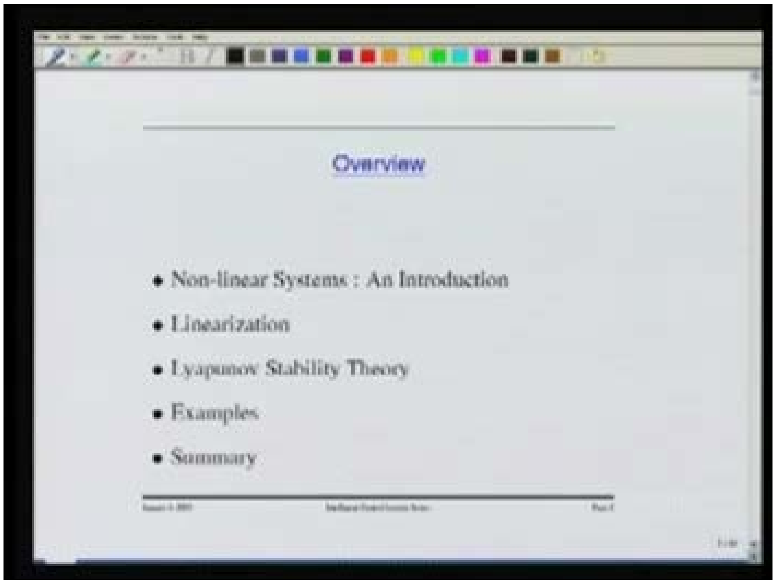
Indian Institute of Technology, Kanpur

Module – 1 Lecture – 4

Nonlinear System Analysis: Part 1

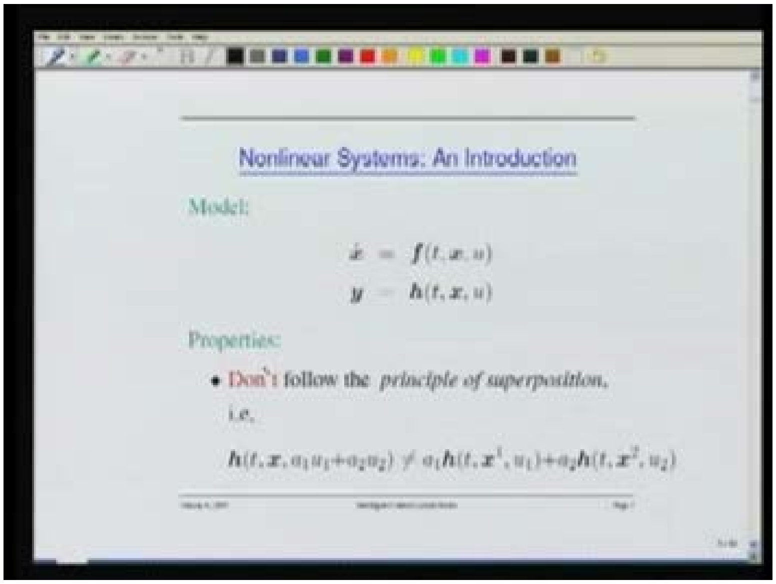
This is lecture 4 of module 1. In this lecture, we will discuss some concepts in nonlinear systems analysis, specifically, the Lyapunov based approach, the reason being that we will use this concept to derive some new training algorithm for neural network based control schemes. Hence, I thought that we must discuss this subject of nonlinear system analysis. You should have some background before you can actually appreciate what we teach in this course.

(Refer Slide Time: 01:16)



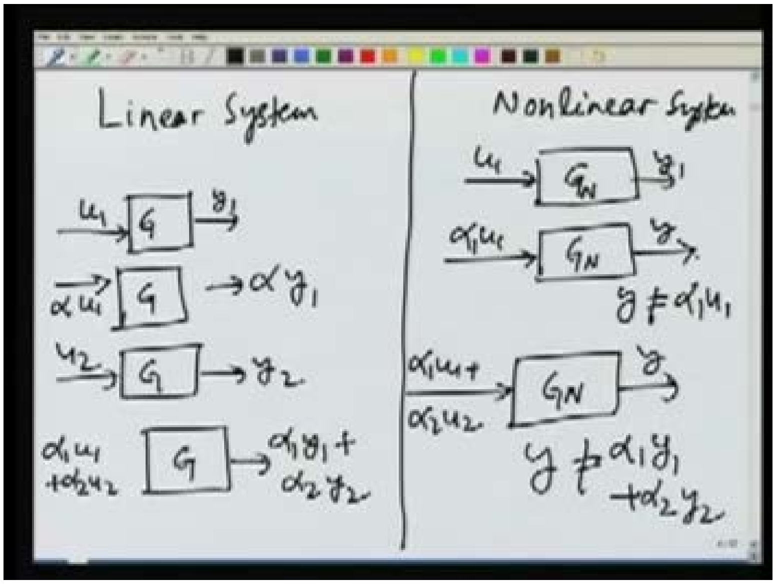
In this course, we will introduce what is nonlinear system, what is linearization, Lyapunov stability theory and some examples.

(Refer Slide Time: 01:27)



What you are seeing is.… This is our general state-space model – nonlinear system model. x dot is equal to f t x u. Hence, it is explicit in time as well as function of x as well as input u. This is a state variable model and output is again a nonlinear function h. You see x is a vector, f is a vector, y is a vector and h is also a vector. This is a multi-input, multi-output system. When I say this is nonlinear, what does it mean? Of course, the first thing that you already know is that a linear system means it follows the principle of superposition and a nonlinear system is expected to not follow the principle of superposition. Let me for your benefit compare. A linear system is different from a nonlinear system in terms of superposition principle.

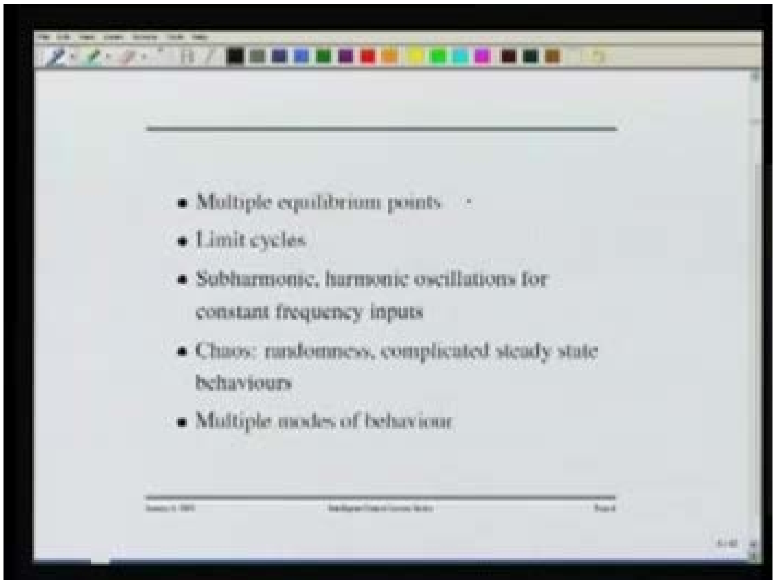
(Refer Slide Time: 02:35)



This is a linear system and this is a nonlinear system. Here, this is my system, which is linear. If I give u_1, I get y_1. In this case, if I give alpha u_1, then also I get alpha y_1; same system, say some system G. Now, if I give another input u_2 to the same system G, I get y_2 and finally, the superposition principle is I give alpha_1 u_1 plus alpha_2 u_2 and I find alpha_1 y_1 plus alpha_2 y_2. A nonlinear system is just the opposite. For example, if I give u_1, I get y_1, same system G nonlinear. When it is a nonlinear system and if I give an input alpha_1 u_1, the output is y and y is not equal to alpha_1 u_1.

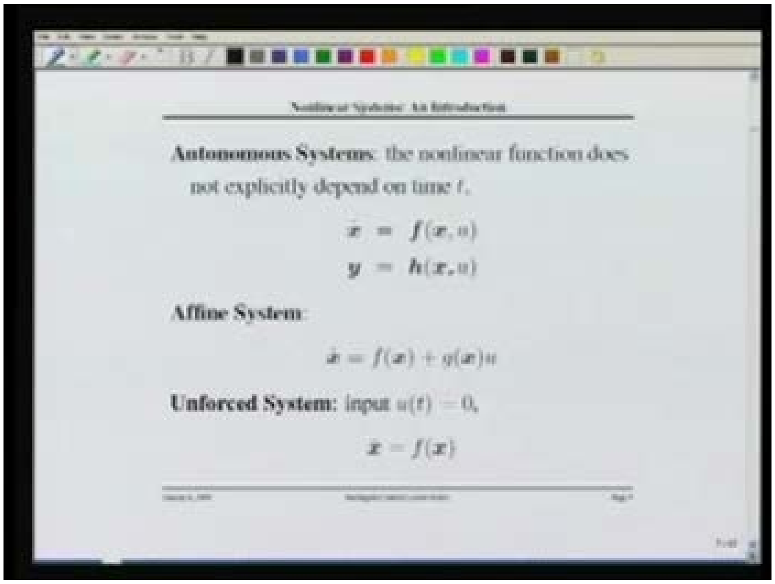
Similarly, if I provide these two inputs alpha_1 u_1 plus alpha_2 u_2 to the same nonlinear system, the output y is not equal to alpha_1 y_1 plus alpha_2 y_2. This is the major difference between a nonlinear system and a linear system as far as differences are concerned, but a nonlinear system is more than just…. It does not follow the superposition principle; it has an even broader perspective and some of these perspectives are summarized here.

(Refer Slide Time: 05:24)



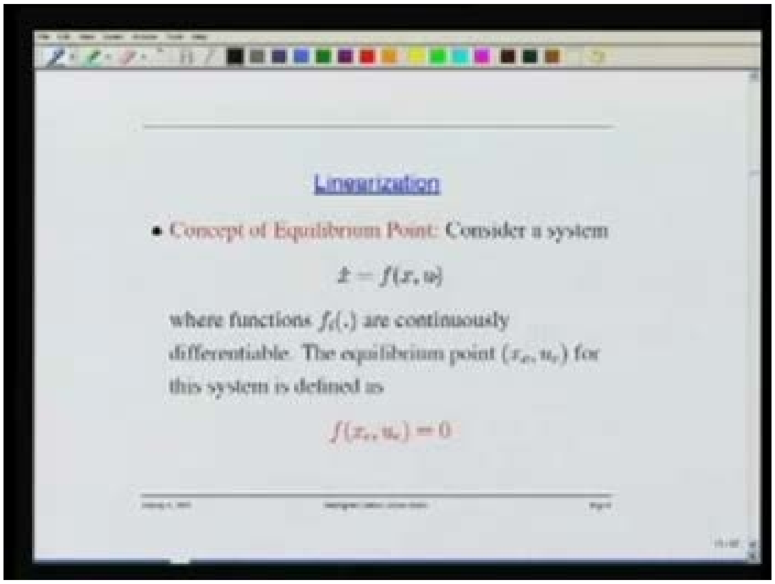
We will just discuss how a nonlinear system has multiple equilibrium points – we will shortly discuss this. It has limit cycles. If you give a specific sinusoid signal of a specific frequency, then in a linear system, the output will have the same frequency whereas a nonlinear system may produce subharmonic oscillations of frequencies. Also, the nonlinear system can behave or can exhibit chaos. Chaos means its steady state behavior is highly unpredictable and random. The nonlinear system also executes multiple modes of behavior and that is called bifurcation. That is one of the examples.

(Refer Slide Time: 06:26)



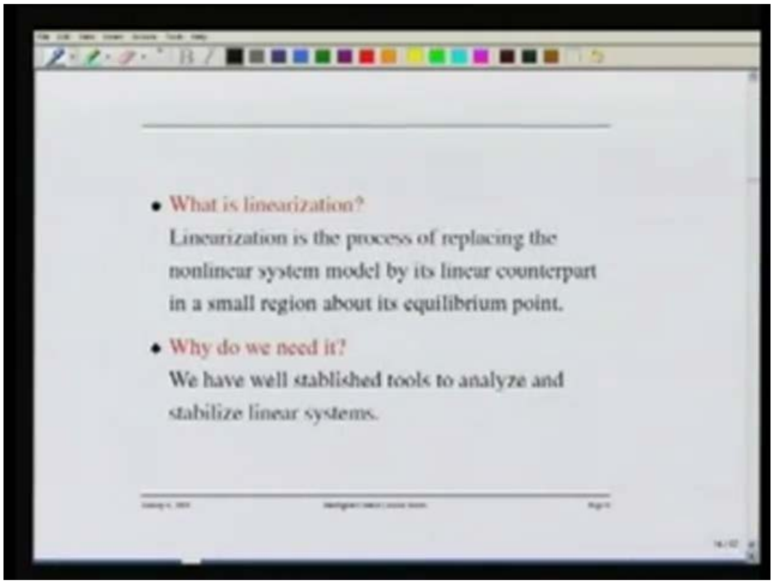
This is the example of an autonomous system, a nonlinear system x dot equal to f x u. Autonomous means it does not explicitly depend on time and y is h x u. This is the generalized state-space model of a nonlinear system that is autonomous. Some systems that have been very widely studied in nonlinear system are known as affine systems. They have a specific structure. In our control system design, many times we will discuss this kind of system while designing a controller, that is, x dot is equal to f x plus g x into u. This is a nonlinear function (Refer Slide Time: 07:15) and this is another nonlinear function, but you can see that this has a specific structure. When you have no external input, then x dot is equal to f x – this is an unforced system.

(Refer Slide Time: 07:31)



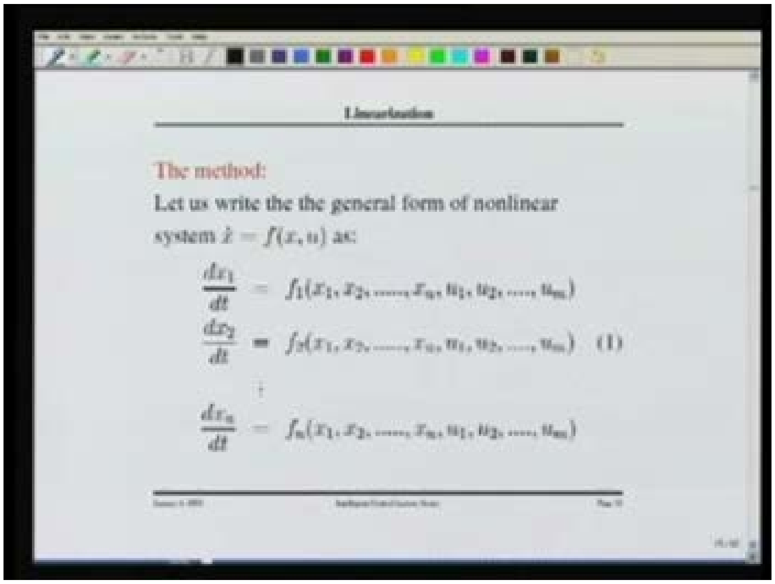
What is the equilibrium point? At equilibrium point, x dot, the dynamics, becomes 0 – the derivative of the states becomes 0; that is the equilibrium point. If x_e and u_e correspond to a specific equilibrium point, you replace and then equate to 0. You will find what the equilibrium point is.

(Refer Slide Time: 08:01)



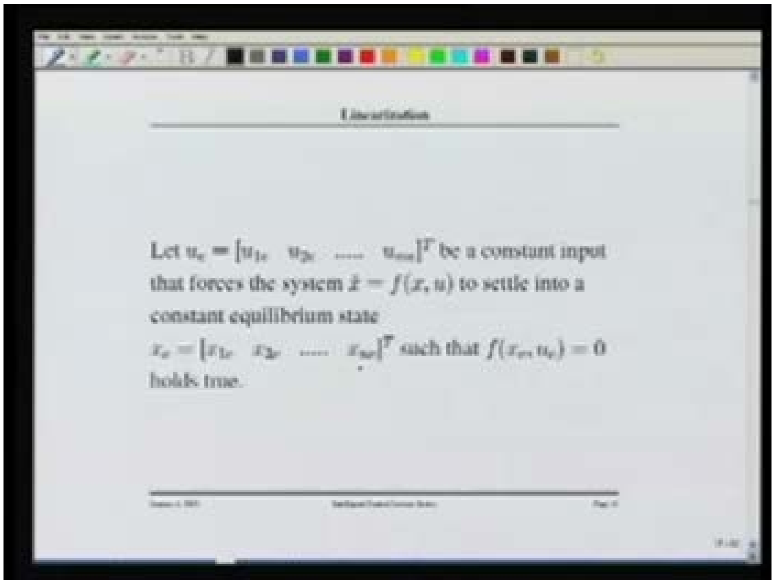
What is linearization? Linearization is the process of replacing the nonlinear system model by its linear counterpart in a small region about its equilibrium point. We linearize because we have well-established tools to analyze and stabilize the linear system.

(Refer Slide Time: 08:20)



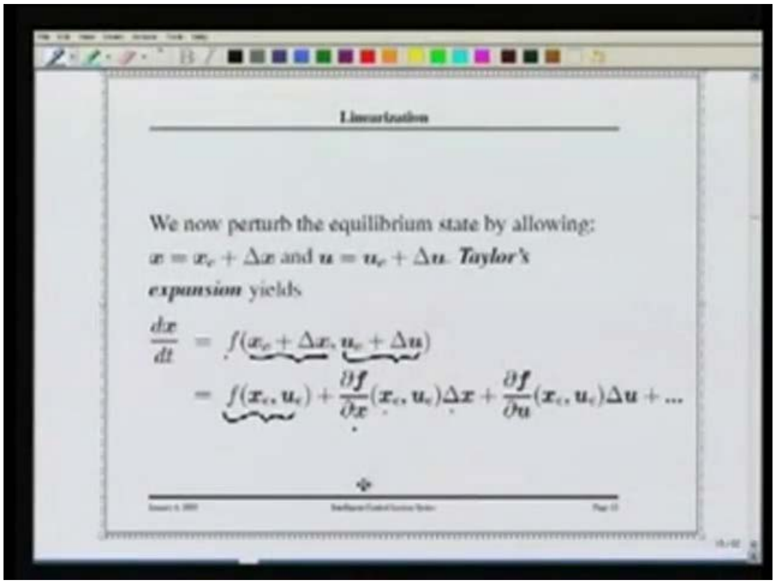
Let us try to understand the basic process of linearization. Let us write the general form of a nonlinear system x dot is equal to f x u as this. We have x_1, x_2 … x_n – there are n states. So dx_1 upon dt is f_1 of x_1, x_2 … x_n and u_1 to u_m. We have m inputs and n outputs. This is the generalized way to write, a very explicit way to write this compact notation x dot equal to f x u. f is a vector. You can see f_1, f_2 … f_n.

(Refer Slide Time: 09:16)



Let x_{1e}, x_{2e} … x_{ne} be a point of equilibrium… the point of equilibrium x_e and u_e, where u_e is a vector – u_{1e}, u_{2e} … u_{me} and x_e is x_{1e}, x_{2e} … x_{ne}. We have x_e and u_e. This is an equilibrium point and this holds true: f x_e u_e is 0.

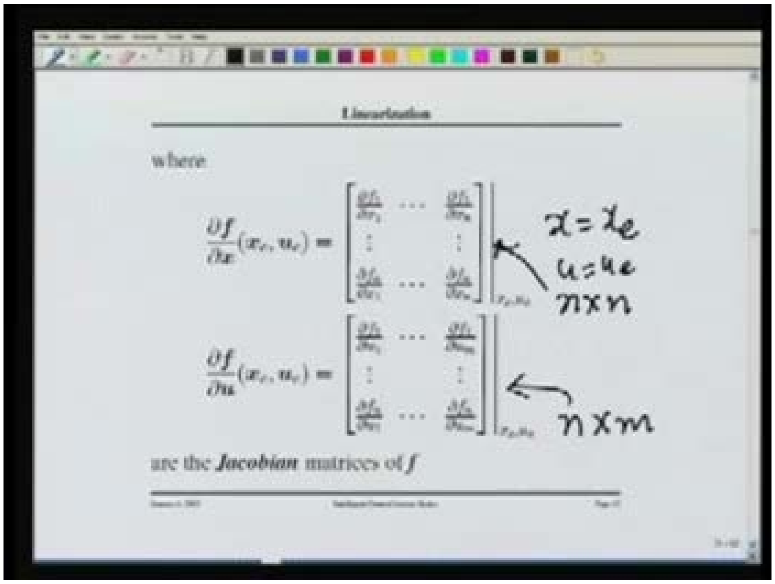
(Refer Slide Time: 10:04)



Then, we perturb this equilibrium state by allowing x equal to x_e plus delta x – little perturbation from x_e and similarly, a little perturbation from u_e by delta u. Then, Taylor's expansion yields a very…. (Refer Slide Time: 10:22). We can write dx upon dt is f and this is your x, this is your u due to perturbation and we expand this using Taylor's expansion. First, this is your equilibrium point, the operating point and then del f upon del x, you know f is a vector, so x is also a vector.

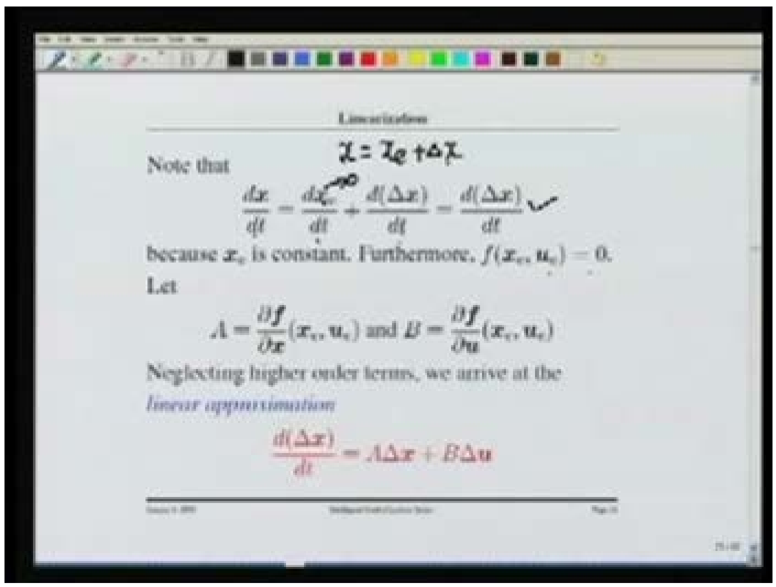
If you differentiate a vector with respect to a vector, you get a matrix. So, del f upon del x into delta x and this matrix is computed at the value x equal to x_e and x equal to u_e. These are first-order terms and this is the operating point or equilibrium point. This is del f upon del u and you compute this – again, another Jacobian matrix; you have to compute at x_e and u_e and multiply by delta u. This is the first-order expansion plus second order, third order and higher orders. But we only go up to the first-order expansion of Taylor series because that is where we can apply our linear system theory. If you go to second order, again the system becomes nonlinear. It is not always true that we can always approximate a nonlinear system around an equilibrium point using first-order Taylor series expansion – there is some limit on that. Let us compute and see what the Jacobian matrix is.

(Refer Slide Time: 12:31)



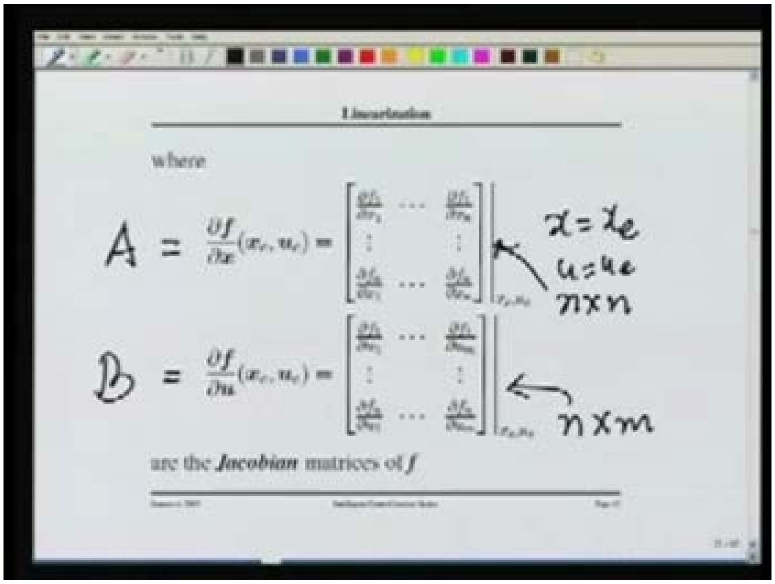
del f upon del x. As I told you, this is the matrix. The easiest way to remember is that the first vector in f_1 is differentiated with respect to each element of the x vector and that is in the first row. Similarly, in the last row, the last vector of f, is the n th vector of f, is differentiated with respect to all the elements of x and that is the last row. You can easily remember this: del f_1 by del x_1 to del f_1 upon del x_n; similarly, del f_n upon del x_1 to del f_n upon del x_n. These values have to be computed by replacing x as x_e. So x has to be x_e and u has to be u_e. This will be a constant matrix x_e. Once you replace x_e and u_e, this becomes a constant matrix (Refer Slide Time: 13:35). Once you replace x_e and u_e, this becomes a constant matrix. Similarly, you can compute also del f upon del u – similar principle. Obviously, you can see that this is n into n matrix and this is n rows and m columns, so n into m matrix. These are all Jacobian matrices.

(Refer Slide Time: 14:13)



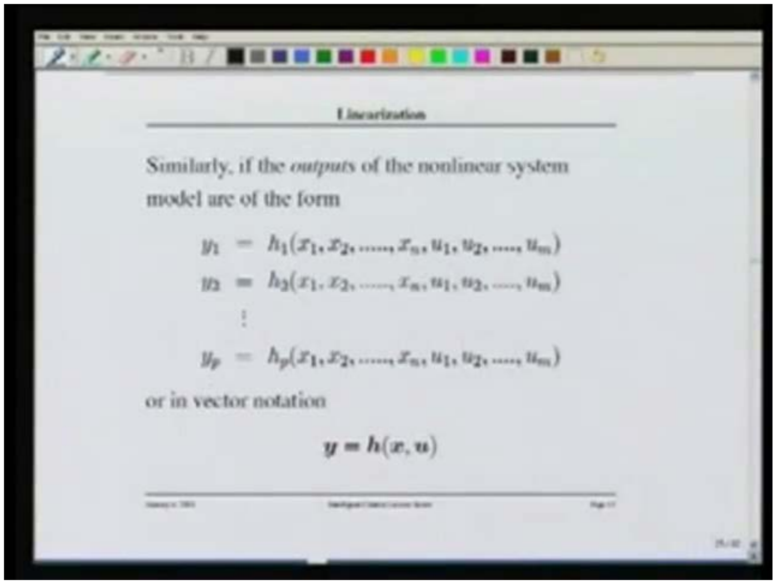
Let us note here that dx upon dt is dx_e upon dt plus d delta x upon dt, because x is x_e plus delta x. When I differentiate x with respect to dt, I differentiate x_e with respect to dt and delta x with respect to dt, but you can see that x_e is an equilibrium point, and at that point, x dot is 0, the derivative is 0, so this is 0 (Refer Slide Time: 14:49). You only get this term: d delta x upon dt. Furthermore, f x_e u_e is also 0 because that is how you computed x_e and u_e – by making it 0. At the equilibrium point, x dot (the derivative) is 0.

(Refer Slide Time: 15:22)



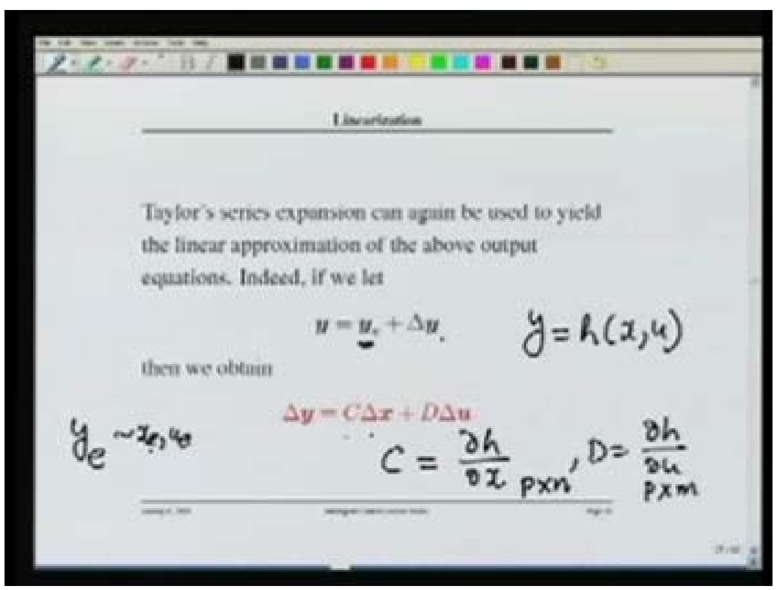
We have already computed in the previous slide. This matrix is actually matrix A and this matrix is matrix B. As we defined, A is del f upon del x and B is del f upon del u. We have neglected the higher order terms and we arrive at the linear approximation – d delta x upon dt is A delta x plus B delta u. This gives sometimes we also say small signal models, that is, the behavior of a nonlinear system around an equilibrium point, where the dynamics is assumed to be linear. Hence, the dynamic is in the linear state-space model delta x dot equal to A delta x plus B delta u.

(Refer Slide Time: 16:23)



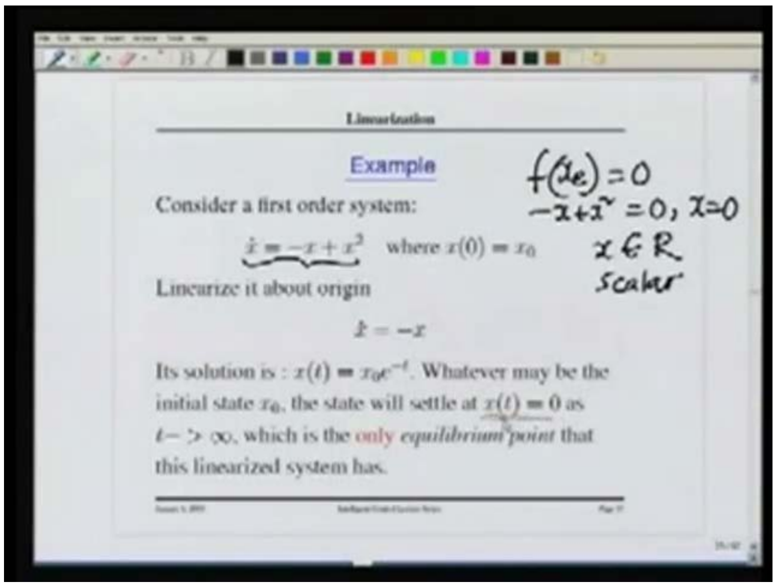
Similarly, we had P outputs. For P outputs, the functions are h_1, h_2 … h_P. In vector notation, you can write y equal to h x u.

(Refer Slide Time: 16:36)



We can again use Taylor's series expansion: y equal to y_e plus delta y. That is being perturbed – equilibrium point y_e with delta y. Then, we have the actual system is y equal to h x u. Using Taylor's series, we can always write delta y is C delta x plus D delta u, where C and D are again Jacobian matrices (computed). Can you guess now what is C and what is D? As we have already said, C has to obviously be del h upon del x. This has to be a P into n matrix. Similarly, D is del h upon del u and this has to be a P into m matrix. That is how we also linearize the output y around the equilibrium point y_e. y_e corresponds to x_e and u_e. At value x_e, u_e, the system response is y_e. So, y_e is the system response when the system state is x_e or system is at the equilibrium point. Let us take an example.

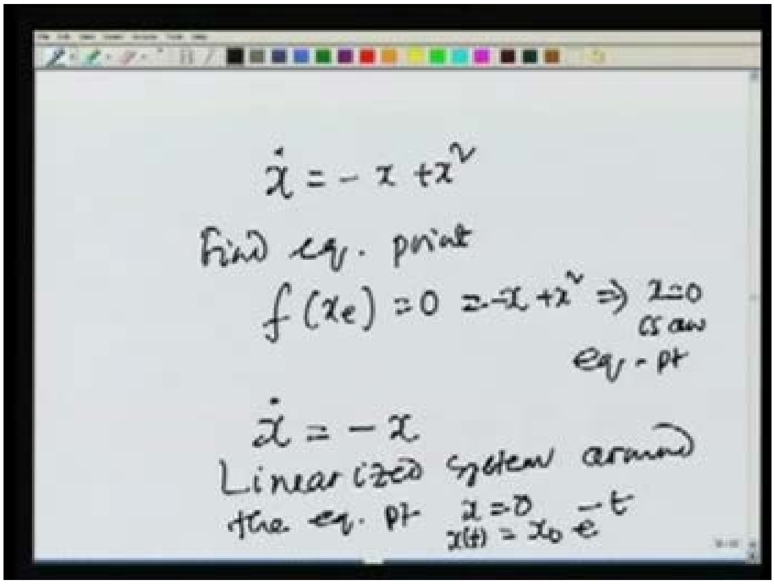
(Refer Slide Time: 18:46)



You can see here that this is a simple scalar differential equation. x is a scalar – one-dimensional, simply one-dimensional; it is a scalar quantity. Here is a scalar differential equation but nonlinear because of this term x square. We can verify that this equation will not follow the principle of superposition. You linearize it about the origin and you can easily see that if I write f x_e as 0, then you will find when you make minus x plus x square equal to 0, this will lead you to x is 0. So origin (Refer Slide Time: 20:02) equilibrium point; x = 0 is an equilibrium point. Linearize it about the origin and you get x dot equal to minus x. The solution is x t equal to x_0 e to the power of minus t. It is very simple because this is a simple linear differential equation: x dot is equal to minus x. I hope that you are very well aware on how to get the solution of this. You can verify this now.

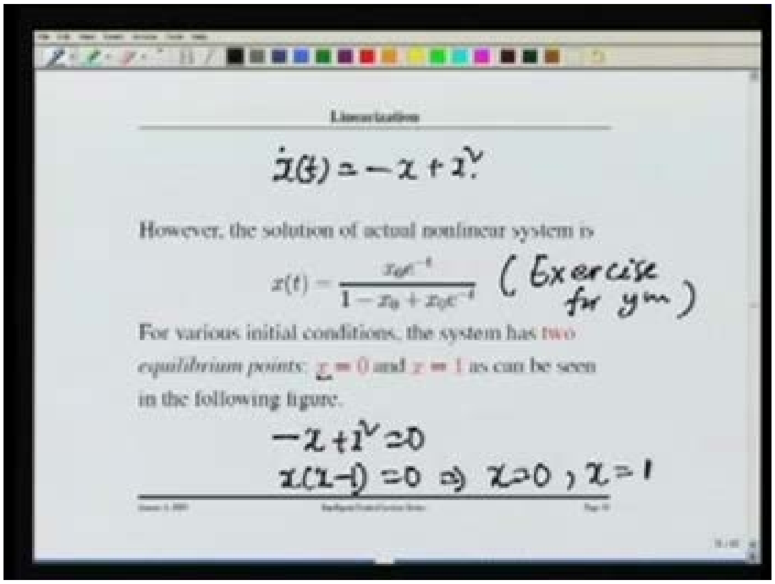
You put x_0 e to the power of minus t here (Refer Slide Time: 20:38), differentiate it and at x of 0, when t equal to 0, put this value x_0 and you see this equation is fine. Whatever may be the initial state x_0, the state will settle at x t = 0, because whatever the initial condition, when you perturb the system from its initial position, the equilibrium point x_0 will always come to the origin. The state will settle at x t (Refer Slide Time: 21:12). This is your equilibrium point, which is the only equilibrium point that this linearized system has. Let me again clarify what we actually did.

(Refer Slide Time: 21:27)



We had x dot equal to minus x plus x square. First, we found out the equilibrium point, that is, f x_e is 0, which is minus x plus x square. So x = 0 is an equilibrium point. At the equilibrium point, the linearized equation is… if you follow the linearization scheme, it is minus x. This is your linearized system around the equilibrium point x = 0. The solution we found out here is x t is x_0 e to the power of minus t – that is the solution to the equilibrium point.

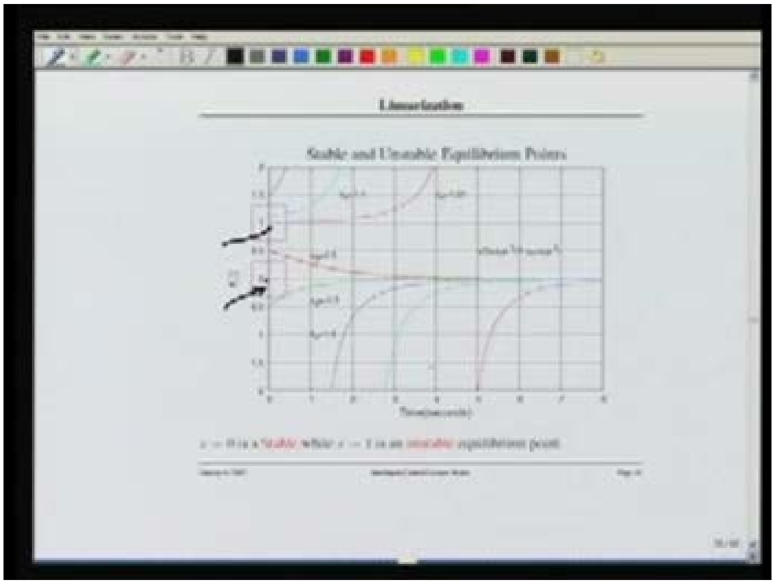
(Refer Slide Time: 22:48)



Now, if I actually take the exact solution of x t dot, which is minus x plus x square, if I find the actual solution of this nonlinear system, then x t equal to x_0 e to the power of minus t upon 1 minus x_0 plus x_0 e to the power of minus t. You can try it. It is an exercise for you. This is a simple nonlinear system (Refer Slide Time: 23:17), for which we find a closed loop solution and this is the closed loop solution for this nonlinear solution. In general, we cannot express the solution of a nonlinear system in a closed form but because this is a simple one, we can actually find the closed form solution of this nonlinear system.

Earlier, we showed that the origin was one of the equilibrium points, but there is another equilibrium point, that is, when I solve minus x plus x square is 0, you have x of x minus 1 is 0. This implies x = 0 and x = 1. There are two equilibrium points. We talked about linearizing the system around only one equilibrium point x = 0, origin, but there is also another equilibrium point x = 1. Let us see the system behavior around these equilibrium points.

(Refer Slide Time: 24:36)

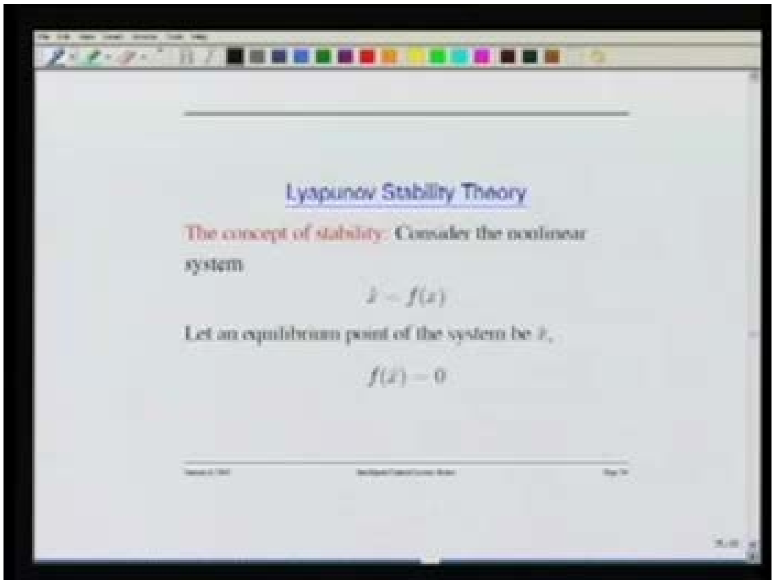


This is a plot. You can see this is one equilibrium point (Refer Slide Time: 24:39). What we are showing is this is the time axis and this is your x t – we are showing the response. The value of x t in the beginning t = 0…. We can start either near the first equilibrium point, that is, 0 – this particular point and another is 1 – this is another equilibrium point. If I give the initial condition as —0.5, you see they all settle. —0.5 and +0.5, wherever I give in this zone, they are all going towards 0.

Finally, the system at t tends to infinity, goes back to the equilibrium point 0, that is, if the system is disturbed slightly from its equilibrium point 0, it is near the origin, it goes back to the origin, whereas if you look here, this is a little interesting. One: if you go back here, it diverges from 1. Very near it diverges, but surprisingly what happens is that after it goes to a very high value, it comes back and then again goes back to 0. Similarly, we can see this one (Refer Slide Time: 26:08). Just after 1, it goes and again from here to here, it jumps and then again goes to 0.

Similarly, another point here, this particular line and from here, it jumps here and again goes to 0. What you are seeing is that as I said in the beginning, the equilibrium point 1 is actually an unstable equilibrium point. From here, the system diverges from this point, whereas if you start somewhere near the origin, then you could disturb the system near the equilibrium point, which is the origin. Then, it goes back to the origin. The conclusion is x = 0 is a stable equilibrium point, whereas x = 1 is an unstable equilibrium point. At the end of the lecture, we will discuss a little more about these equilibrium points – we will solve through examples but let us discuss some notion of stability in a nonlinear system. What we talked about now is that a nonlinear system can be linearized around an equilibrium point. Some equilibrium points are stable equilibrium points and some equilibrium points are unstable equilibrium points. Now, what is the concept of stability on a nonlinear system?

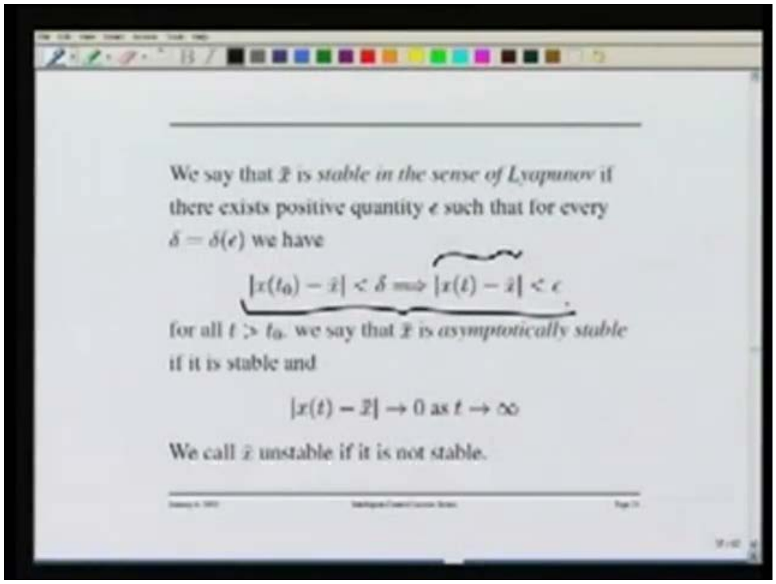
(Refer Slide Time: 27:47)



In linear system, we had only one equilibrium point. We never talked about locally stable, globally stable – you never heard these kind of notions in a linear system, because it has only one equilibrium point. But when it becomes a nonlinear system, we always talk about whether it is locally stable, globally stable – all these, because an equilibrium point can be locally stable, it can also be globally stable. This is not a course on nonlinear systems and we will not go into details, but we will cover whatever minimal is necessary in this course. x dot is equal to f of x. This is your nonlinear system and the equilibrium point is x bar. This is your equilibrium point.

(Refer Slide Time: 28:44)

(Refer Slide Time: 29:04)

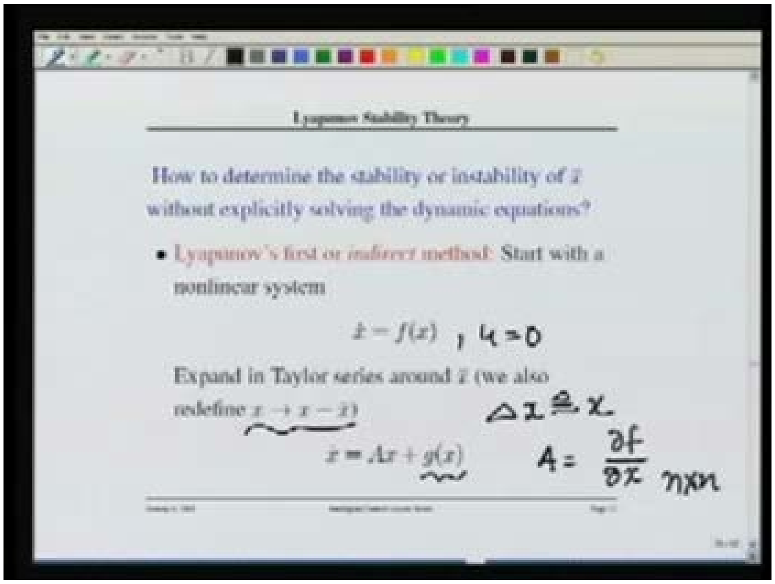


The equilibrium point x bar is stable in the sense of Lyapunov…. This is the definition we are giving – what we are seeing on the board is the definition. We say that x bar is the stable equilibrium point in the sense of Lyapunov if there exists a positive quantity epsilon such that for every delta (this is a function of epsilon), we have this particular…. (Refer Slide Time: 29:24) and this is satisfied, that is, this is my initial state. My system state has been disturbed from the equilibrium point x bar and so, there is some disturbance. After you disturb the system, now I am relaxing the system, that is, system dynamically evolves and after dynamically evolving, where does it go? What happens to the system states?

If this disturbance that I have given is less than delta, I cannot infinitely disturb the system and then conclude that system still remains stable. I assume some value delta, that is, I put an upper bound by which I give a disturbance to the system. If this is true, if this is my disturbance, this is the way I disturb the system, then, x t minus x bar, that is the future state of my system is not…. If x t minus x bar is less than epsilon, then the system…. What is the meaning of that? When I say this statement (Refer Slide Time: 30:55) that if this is my disturbance, finally, always the magnitude or absolute magnitude of my future state minus the equilibrium point is always less than epsilon, what does that mean?

If I am giving a simple disturbance to the system, the future states do not go far away from the equilibrium point at any point of time – they remain very close to their equilibrium point. That is the concept of stability that we defined. Then, we define a concept of asymptotically stable or asymptotic stability, where this particular difference that we said in the beginning is less than epsilon (Refer Slide Time: 31:42), we are now saying that if it goes to 0, that is, my future state converges to the equilibrium point as t tends to infinity, then, this is called asymptotically stable and we call x bar unstable if it is not stable. If it is neither this nor this (obviously, this is neither of these), then the system is unstable. These are all definitions we are talking about. We defined stability, that is, it is just stable and then we defined the notion of asymptotic stability. Now, how do we determine this notion – whether the system is stable or asymptotically stable?

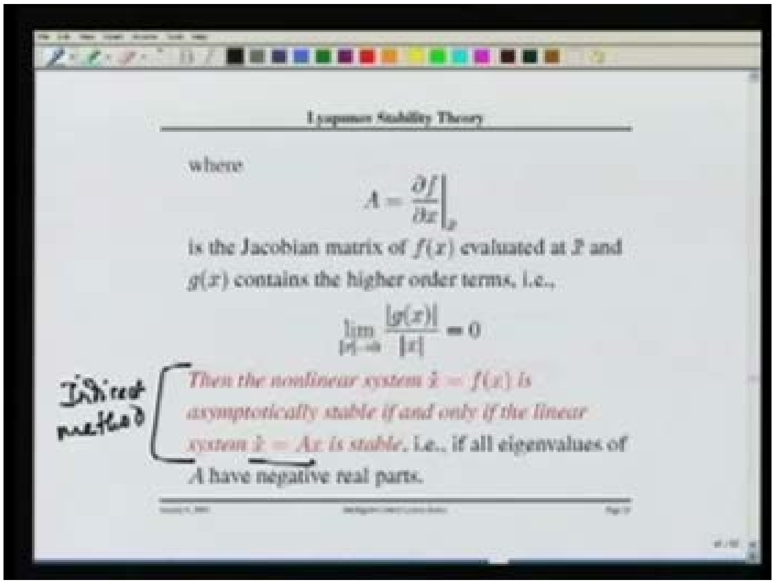
(Refer Slide Time: 32:38)



We will discuss two methods in this class today: one is the indirect method and another is the direct method. The indirect method is that we linearize the system, that is, x dot is equal to f x – that is my nonlinear system. Imagine that we are all…. In the beginning, we are only considering the unforced system u = 0. So, x dot is equal to f x means my u is 0 and I have no external force on the system. Now, what I do is that I linearize the system and the delta x that we wrote earlier, I am again redefining that delta x again as x just to make our notations very simple.

Here, delta x is defined as x. That is here (Refer Slide Time: 33:36). Now, x dot is equal to A x plus g x. This g x represents all higher order terms – second order, third order and so on. What is A? A has to be del f upon del x and this is an n by n matrix. This is A.

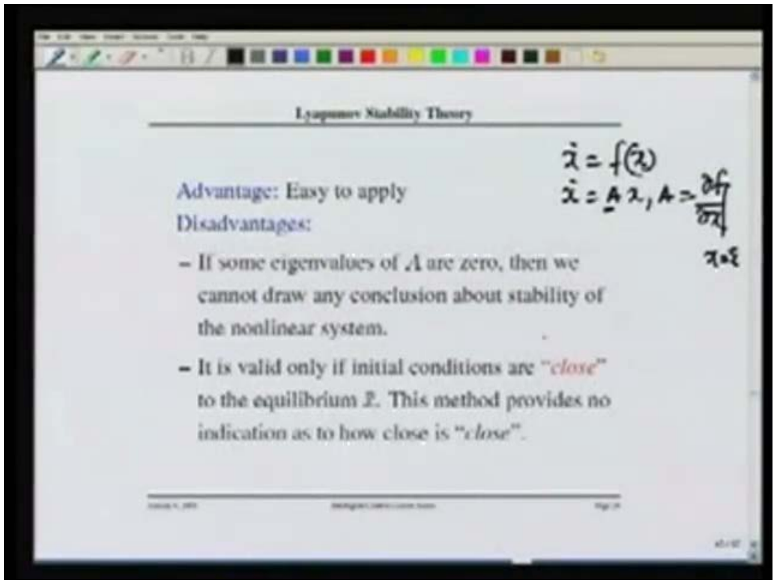
(Refer Slide Time: 34:12)



Now, g x is my higher order term and I assume that this higher order term satisfies this condition. You can see that the absolute magnitude of g x upon the absolute magnitude of x as x tends to 0 is 0. It means that this approximation linearization is valid. Using first-order Taylor's series expansion, this linearization scheme is valid when this higher order term at g x becomes smaller, it becomes very small in comparison to x – then only, it can become 0 and that is the meaning. The nonlinear system x dot = f x is asymptotically stable if and only if the linear system x dot equal to A x is stable. This is our theorem or definition of the indirect method.

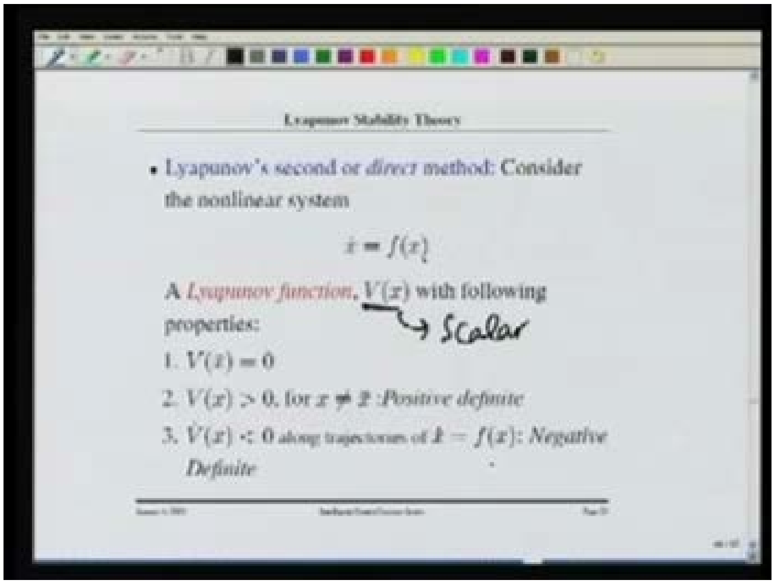
How to determine whether the system is stable or not? This is the definition of how to determine whether a system is locally stable – stable around an equilibrium point. If a system is stable around an equilibrium point, then the linearized system around that equilibrium point, which is x dot equal to A x and if we analyze the stability of the system, all the Eigen values of A must be in the left half of this plane, that is, the real part of the Eigen values must be negative. If the real part of the Eigen values of A are negative, then the system is asymptotically stable around the equilibrium point and that means all Eigen values of A have negative real parts. What is the advantage of this method?

(Refer Slide Time: 36:37)



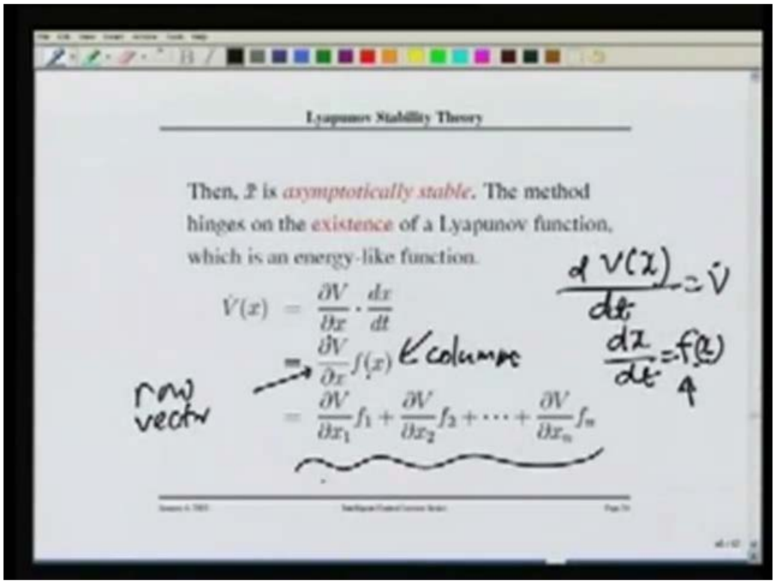
It is very simple, because if x dot equal to f x is given, find out what is x dot equal to A x, that is, A is del f upon del x at equilibrium point x is equal to x bar or x_e or whatever is the equilibrium point. Then, check if A has all the Eigen values whose real parts are all negative, but the problem here is that if any of these Eigen values are 0, the real parts are 0 or if some of the Eigen values are imaginary values, then we cannot conclude that the system is locally stable. So, if some Eigen values of A are 0, then we cannot draw any conclusion about the stability of the nonlinear system and also if the Eigen values are simply pure imaginary quantities – the 0 real part. It is valid only if initial conditions are close to the equilibrium point x bar, because of the condition that the higher order term g of x upon x limit x tends to 0 should be 0. These are two drawbacks. Although this is very simple, now we must look for a very nice method of determining stability, which is known very popularly as Lyapunov stability theory or Lyapunov's second or direct method.

(Refer Slide Time: 38:29)



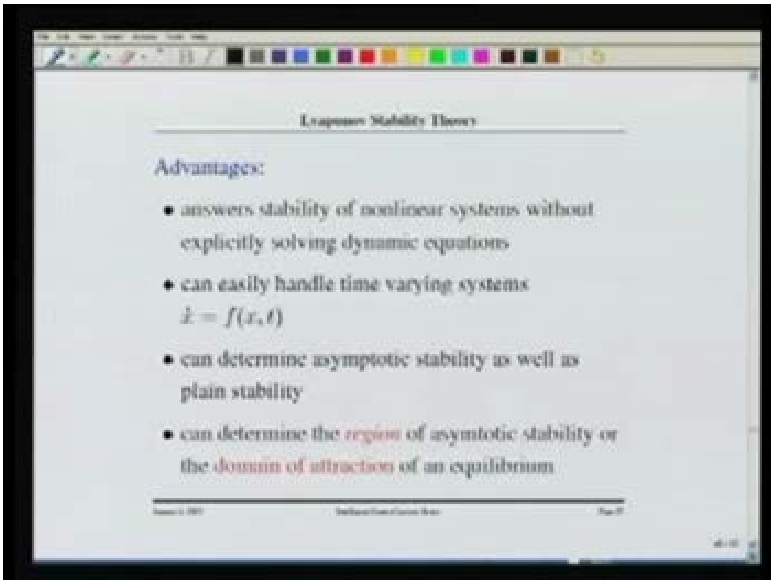
For this, x dot is equal to f x. We define a Lyapunov function V x with the following properties. This Lyapunov function at the origin or at the equilibrium point is 0; this Lyapunov function is a scalar function, this scalar function is always greater than 0 and x is not in the equilibrium point – it is always a positive definite; and the rate derivative of this Lyapunov function is negative definite along the trajectory x dot = f of x. Then, we say the system is stable.

(Refer Slide Time: 39:18)



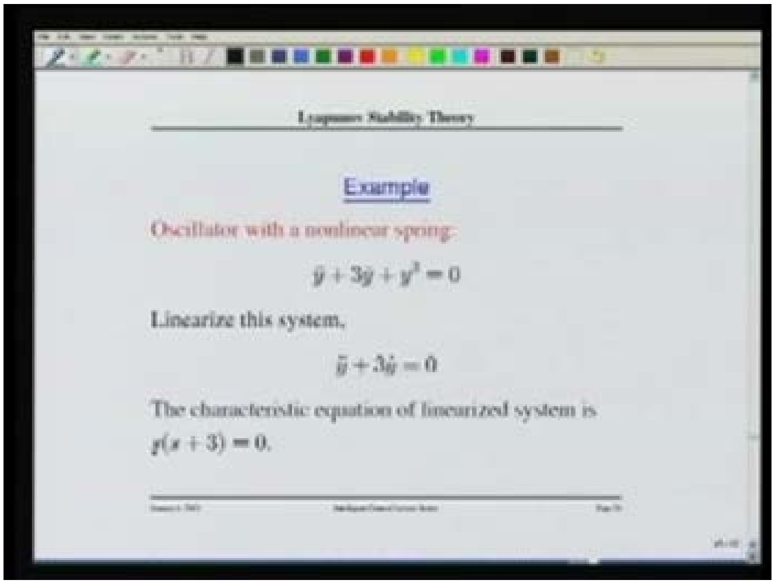
Then, x is asymptotically stable. The method hinges on the existence of a Lyapunov function, which is energy-like function. We defined a function V x. If I want to differentiate V x with respect to dt, so d V x by dt is the rate derivative (Refer Slide Time: 39:35). Then, del V by del x into dx by dt. This dx by dt is already my f x, this is my system. So I replace this f x here. dow V by dow x into f x. What is f x now? V is a scalar function and x is a vector and my f x is also a vector. You can see that dow V by dow x is a row vector and f x is a column vector. This is my column vector and this is a row vector. If I expand, we can easily see that V dot x is dow V by dow x_1 into f_1 plus dow V by dow x_2 into f_2 and so on plus dow V by dow x_n into f_n. What is the advantage?

(Refer Slide Time: 40:56)



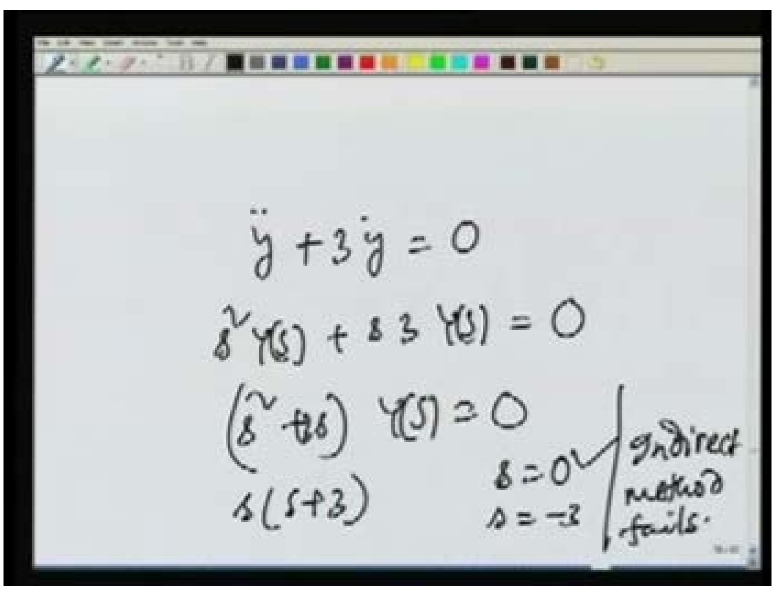
This answers the stability of nonlinear system without explicitly solving the dynamic equations. It can easily handle time varying systems. It can determine asymptotic stability as well as plain stability. It can determine the region of asymptotic stability or the domain of attraction of an equilibrium. We will see all these things through examples.

(Refer Slide Time: 41:13)



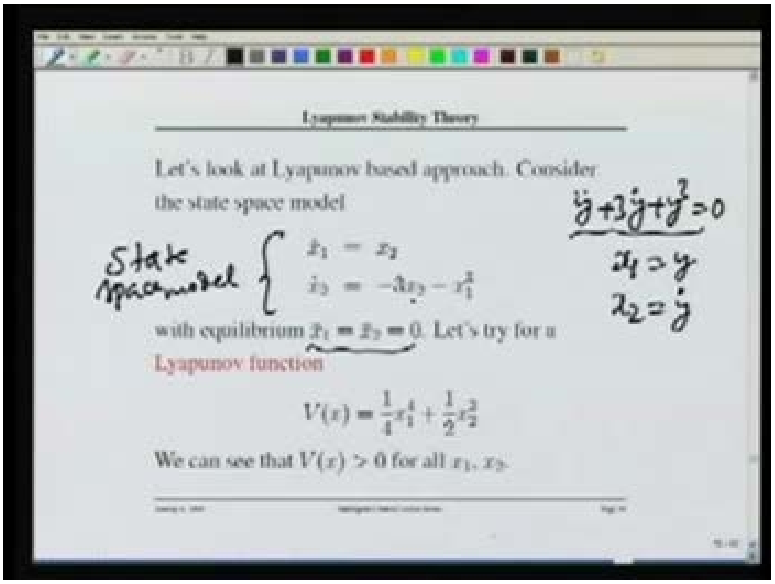
Here is an example. This is a second-order system with cubic nonlinearity – y cube is there. If you linearize this system, you get y double dot plus 3 y dot equal to 0. The characteristic equation of the linearized system is s into s plus 3. You see that you cannot say that by using Lyapunov's indirect method, the system is stable, because one of the poles s is 0. So, the indirect method fails here. So, there is a 0.

(Refer Slide Time: 41:59)



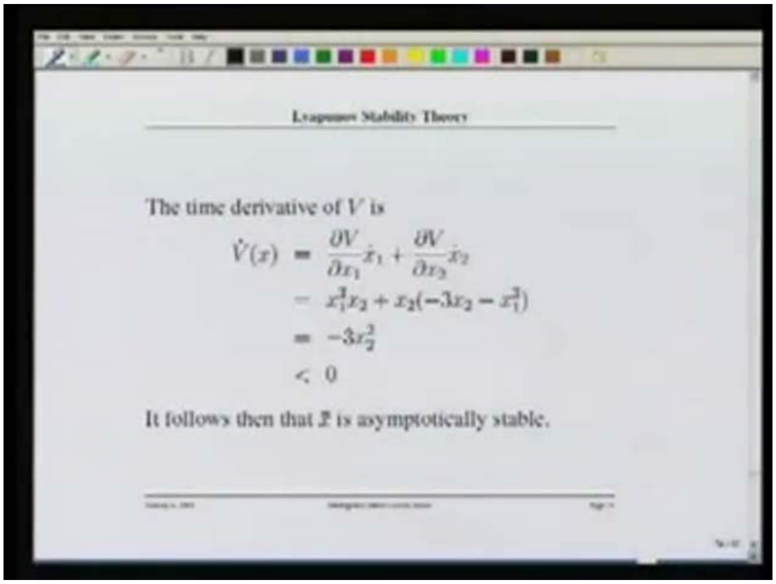
We found out y double dot plus 3 y dot is 0. If I convert this into Laplace domain, s square y s plus s 3 y s is 0. If I look at the characteristic polynomial, s square plus 3 s into y s is 0. The poles are at s = 0 and s = —3 and because one of the poles is at 0, the indirect method fails in this case. We saw that the indirect method failed and so now, we will apply Lyapunov's direct method.

(Refer Slide Time: 43:16)



We had the system, which is y double dot plus 3 y dot plus y cube is 0. I select the states as x_1 is y and x_2 is y dot. I select these two states and then, this equation (Refer Slide Time: 43:38) can be written as a state-space format. x_1 dot is y dot, which is x_2 and x_2 dot, which is y double dot, is —3…. This is y double dot plus 3 y dot. So minus 3 x_2 minus x_1 cube. This is my state-space model. For this, the equilibrium point is obviously the origin. The origin is the equilibrium point. Now, the Lyapunov function I select for the system V x is 1 upon 4 into x_1 to the power of 4 plus 1 upon 2 into x_2 square. This is an energy-like function.

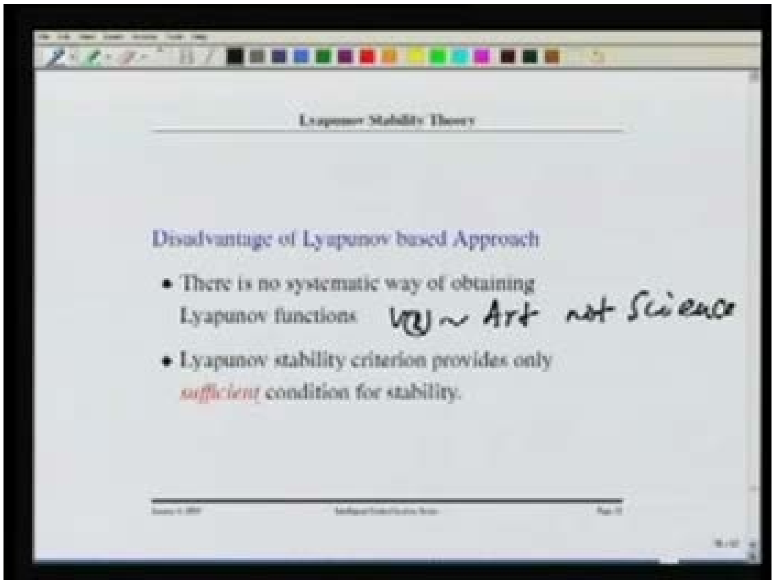
(Refer Slide Time: 44:28)



I take the derivative of this. V dot x is…. I put the formula dow V by dow x_1 into x_1 dot plus dow V by dow x_2 into x_2 dot and you can check that x_1 dot is x_2 and x_2 dot is minus 3 x_2 minus x_1 cube. This is your f_1 (Refer Slide Time: 44:51) and this is your f_2. This is f_1 and this is f_2. dow V by dow f_1 plus dow V by dow x_2 f_2. So, f_1 is x_2 and f_2 is minus 3 x_2 minus x_1 cube. If you simplify it, V dot x is minus 3 x_2 square, which is always less than 0 because x_2 square… other than origin; when x_2 is not in the origin or when x_2 is not 0, it is always negative. It follows that the equilibrium point is asymptotically stable, which we could not conclude using the indirect method.

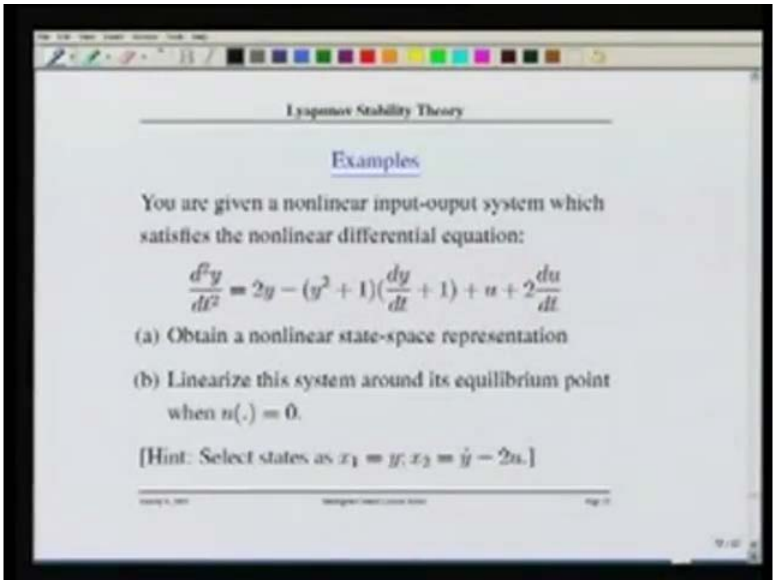
What is the disadvantage of this Lyapunov based approach? Although we showed an example where we could not solve or we could not determine the stability using the indirect method, we could determine that stability using the direct method.

(Refer Slide Time: 46:00)



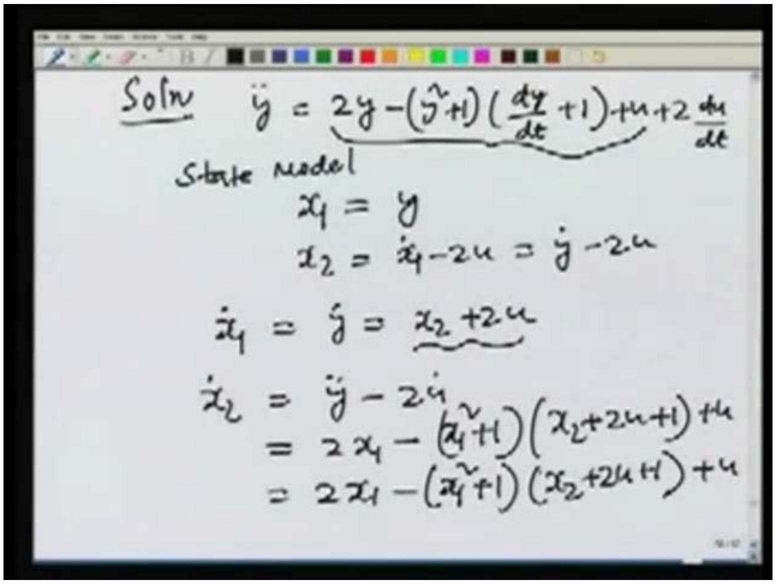
The problem is that there is no systematic way of obtaining Lyapunov functions. Finding a Lyapunov function is more of an art than science. This is an art, not science – how to find out a Lyapunov function. The Lyapunov stability criterion provides only sufficient condition for stability. To end the lecture, there are two examples that we will solve today.

(Refer Slide Time: 46:39)



The first is to make your concept very clear of how to linearize a nonlinear system. We will always encounter these in our intelligent control course. More importantly, the Lyapunov stability theory will be used to determine or to derive some new algorithm – the training algorithm as well as current control algorithm. So, you see here (Refer Slide Time: 47:09), please note the system dynamics. Obtain a nonlinear state-space representation of this and linearize this system around its equilibrium point. This is your question. Let us solve it.

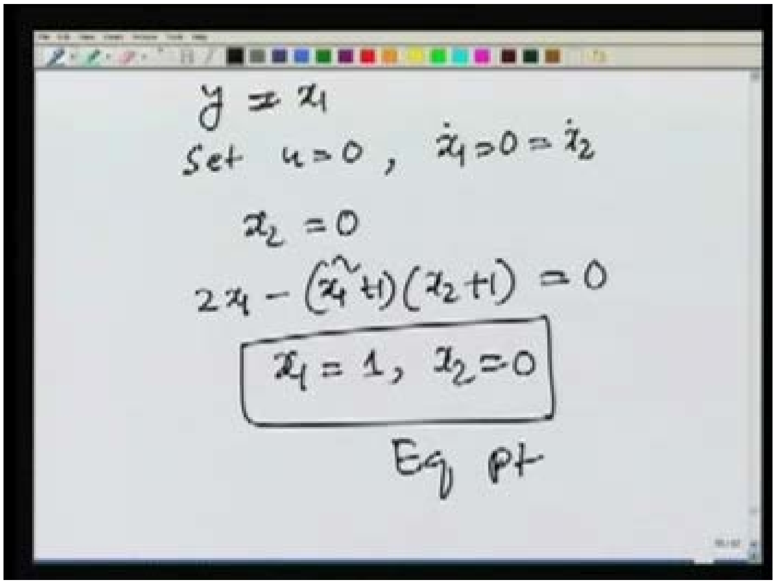
(Refer Slide Time: 47:29)



This is the solution. We have y double dot equal to 2 y minus y square plus one dy upon dt plus 1 plus u plus du by dt. This is the nonlinear dynamics. Now, we have to find the state-space model. In the state model, we first define the states. Obviously, let us take x_1 equal to y. From our experience looking at the form of the structure of this equation, we define x_2 as x_1 dot minus 2 u, which is y dot minus 2 u – this is our definition. With this definition, you can write down x_1 dot is y dot and y dot is x_2 plus 2 u.

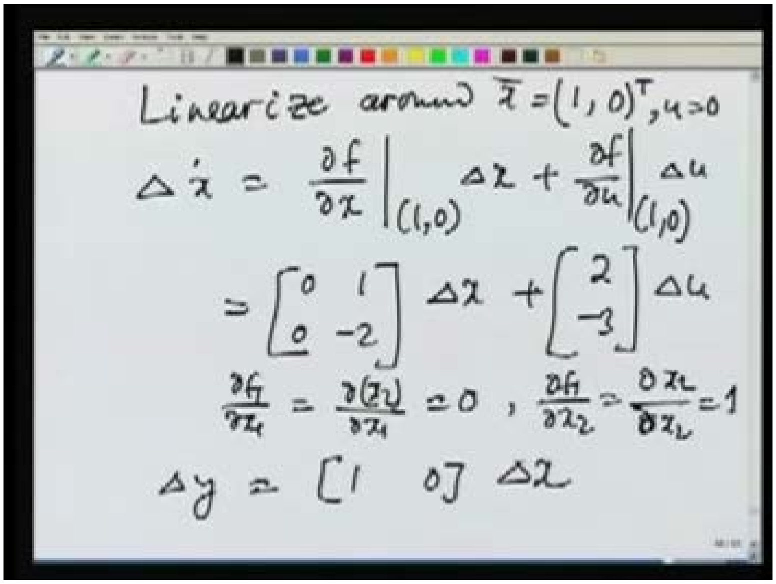
State-space means we always write the derivative of the state in terms of other states and input. Here, x_1 dot is a function of x_2 and u – in terms of states and u. Similarly, for x_2 dot, you can do little algebraic manipulation – x_2 dot is obviously y double dot minus 2 u dot. You can see that y double dot minus 2 u dot is this particular quantity (Refer Slide Time: 49:37). We can write that… 2 y, so 2 x_1 minus y is x_1 square plus 1 and dy upon dt, that is, x_1 dot is x_2 plus 2 u plus 1 plus u. On simplification, this becomes 2 x_1 minus x_1 square plus 1 into x_2 plus 2 u plus 1 plus u. Actually, we did not simplify – it is the same.

(Refer Slide Time: 50:37)



Finally, y is x_1 – that is my output. How do I linearize it? Set u = 0 and x_1 dot = 0 and so also x_2 dot. x_1 dot and x_2 dot are both made 0. If you make u = 0, the first equation is x_2 = 0 and the second equation is 2 x_1 minus x_1 square plus 1 into x_2 plus 1 is 0. The solution is x_1 = 1 and x_2 = 0 – this is the equilibrium point. At this equilibrium point, we have to linearize the system.

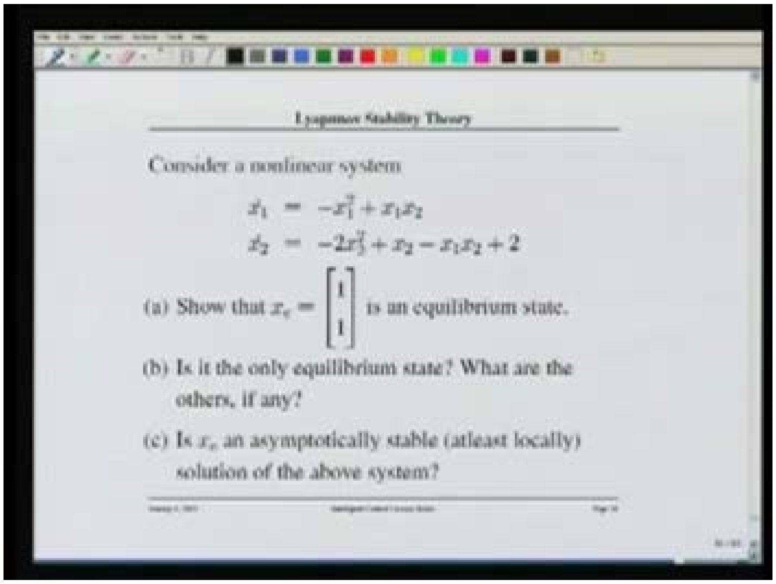
(Refer Slide Time: 51:48)



Linearize around the equilibrium point, which is 1 and 0. Apply the formula that we derived earlier using Taylor's series expansion, that is, delta x dot is del f upon del x at 1, 0 delta x plus del f by del u at 1, 0 into delta u. (Refer Slide Time: 52:46). We are not writing the input here because we have assumed u to be 0 and we are linearizing around the equilibrium point 1, 0 and u is 0. That is the question that is asked to you. If you compute del f by del x, it is… please verify, this will be your A matrix. The B matrix is…. (Refer Slide Time: 53:36).

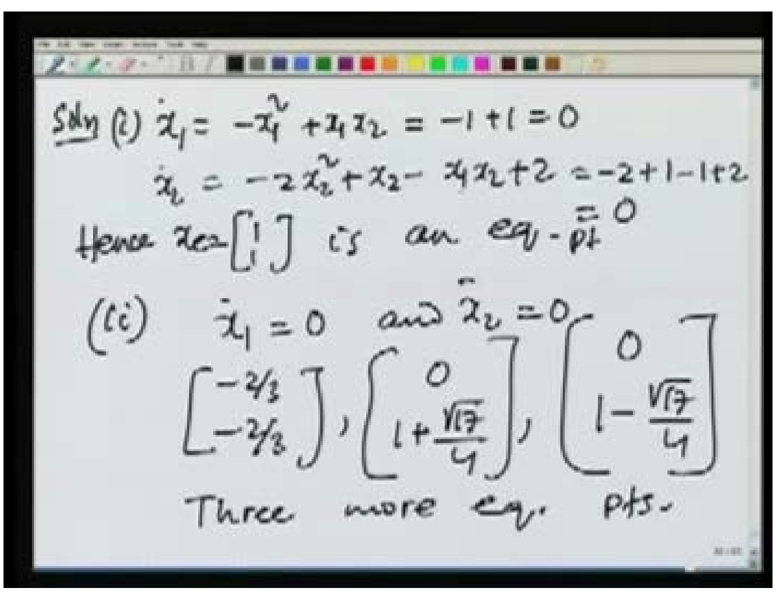
How do I find it? 0 is del f_1 by del x_1 and f_1 is actually del x_2 by del x_1. By definition, this is 0. Similarly, the second one del f_1 by del x_2, which is this value (Refer Slide Time: 54:07), is del x_2 by del x_2 and this is 1. You are differentiating x_2 with respect to x_2. Like that, please verify this particular solution that we got. Finally, delta y is obviously (1, 0) delta x – this is the answer. Now, we will take the second example.

(Refer Slide Time: 54:44)



The second example is that we again have a nonlinear system and you are asked to find out whether (1, 1) is an equilibrium state – first question; the second is that is this the only equilibrium state or there are many; the third is whether the first one that is given – x_e – is asymptotically stable. Let us try to solve this equation again.

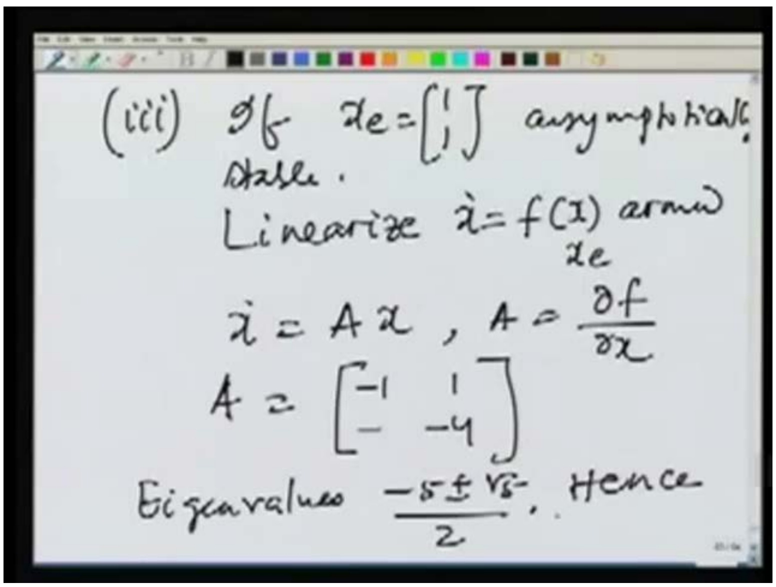
(Refer Slide Time: 55:13)



The solution is that you are given x_1 dot is minus x_1 square plus x_1 x_2 and x_2 dot is minus 2 x_2 square plus x_2 minus x_1 x_2 plus 2. How do you know that x_e = (1, 1) is an equilibrium point? Just put these values here (Refer Slide Time: 55:54) and see whether they are 0. If you put (1, 1) here, this is minus 1 plus 1, which is 0. In this case, when you replace x_2 as 1 and x_1 as 1, again you have minus 2 plus 1 minus 1 plus 2 and again, it is 0; hence, x_e is an equilibrium point. This is the first part of the answer.

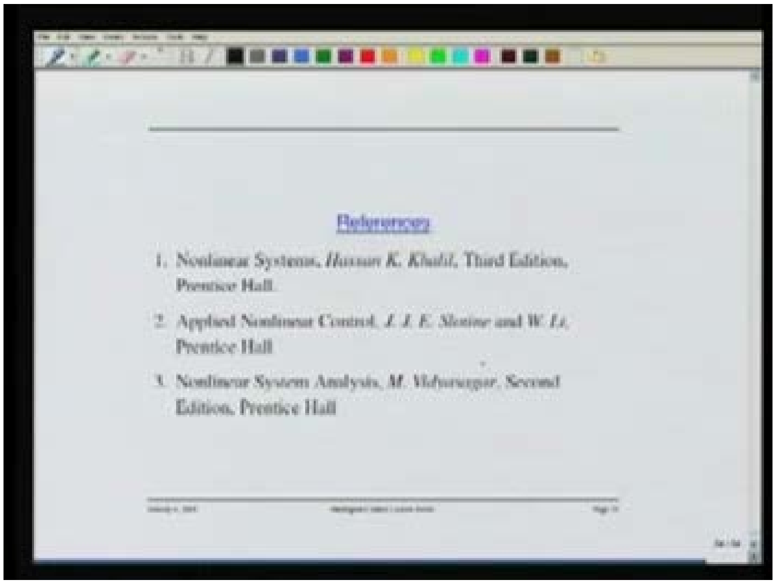
The second part is to find whether there are other equilibrium points. A simple way is there to solve it – put x_1 dot equal to 0 and x_2 dot is equal to 0 and solve. You will find these equilibrium points are…. We have more equilibrium points: (minus 2 by 3, minus 2 by 3) is one more, then (0, 1 plus root 17 by 4), and (0, 1 minus root 17 by 4). There are three more equilibrium points. You can verify this by just putting x_1 dot equal to 0 and x_2 dot equal to 0 – you will find this solution.

(Refer Slide Time: 57:39)



The next question, the third part, was whether x_e is asymptotically stable. How do we do it? We linearize the system around x_e. If you linearize around x_e, linearize f x around x_e, then you get x dot equal to A x, where A is del f upon del x. Please verify that A is (—1, 1, —1, 4) and the Eigen values are —5 plus or minus root 5 divided by 2. You can check that both the Eigen values have negative real part —5 and hence, it is asymptotically stable. We will continue further discussion on nonlinear system in the next class.

(Refer Slide Time: 59:31)



Here is a little help for you to follow some reference books. You can follow one of the books by Khalil – Nonlinear Systems, Prentice Hall. Another one is Slotine and Li's Applied Nonlinear Control – a very nice book. I like this book on Applied Nonlinear Control. M. Vidyasagar has also written a nice book – Nonlinear System Analysis, Prentice Hall. Thank you again.