Lec 6: Electromagnetism as an Eigenvalue Problem

Photonic Crystals: Fundamentals and Applications

Lecture 06

Scaling Properties of Maxwell's Equations

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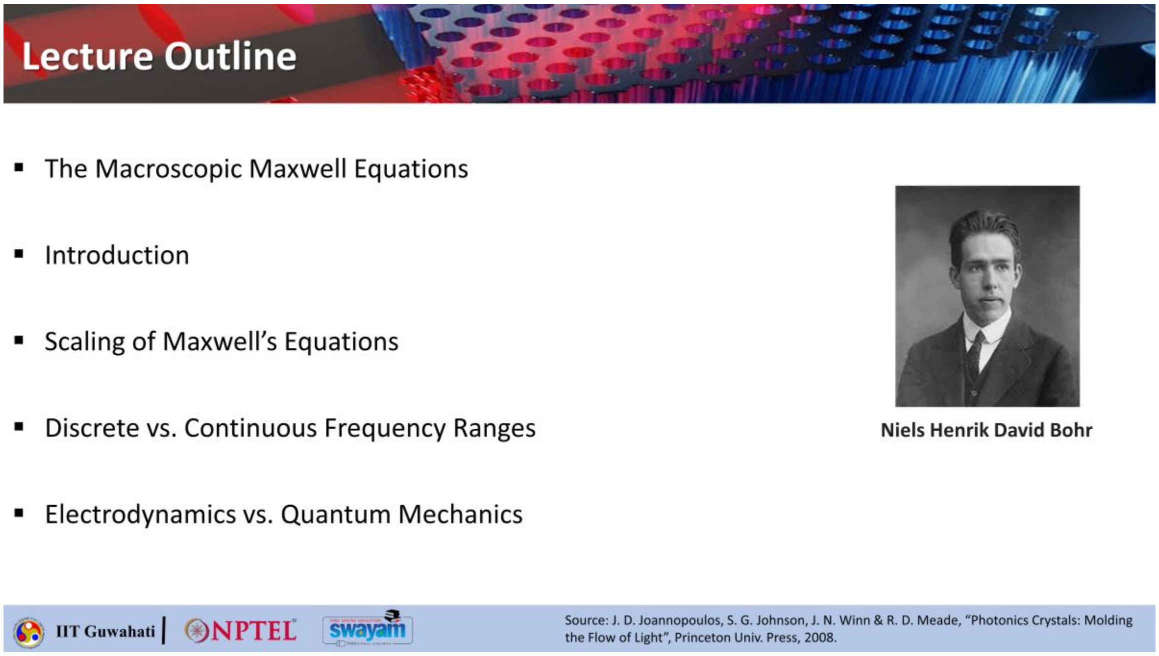
Hello students welcome to lecture 6 of the online course on Photonic Crystals Fundamentals and Applications.



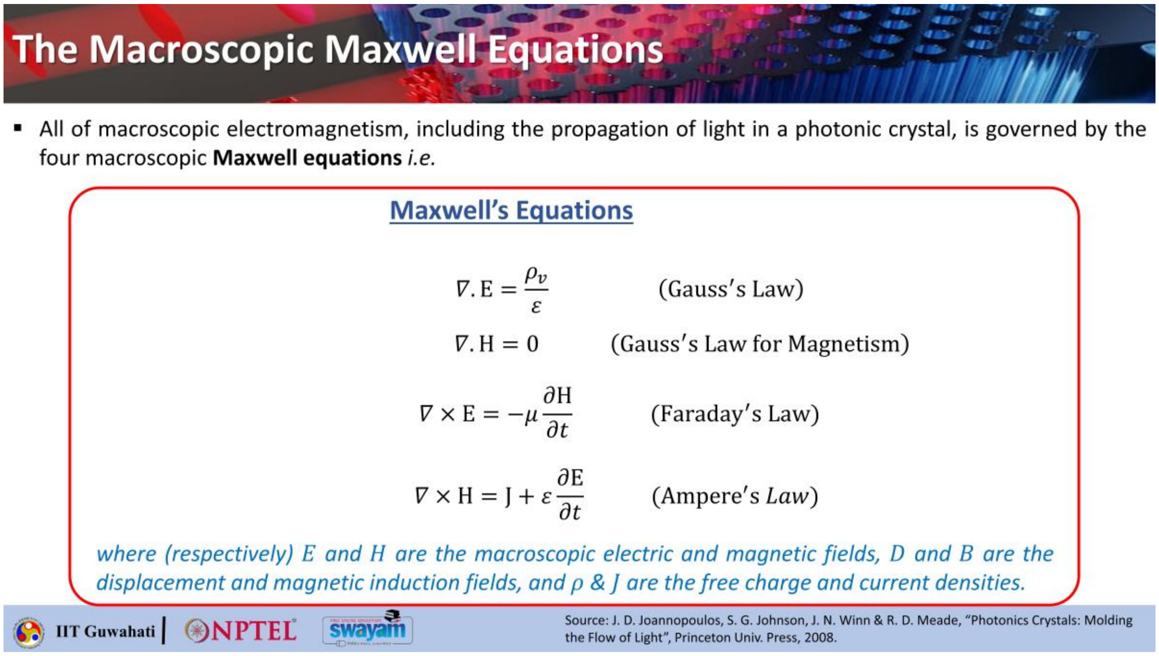
Lecture Outline

- The Macroscopic Maxwell Equations
- Introduction
- Scaling of Maxwell's Equations
- Discrete vs. Continuous Frequency Ranges
- Electrodynamics vs. Quantum Mechanics

Niels Henrik David Bohr

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

Today's lecture we will cover the scaling properties of Maxwell's equations. Here is the lecture outline we will first look into the macroscopic Maxwell's equation this is basically a recap. And then we will introduce this scaling property of Maxwell's equations. We will discuss about discrete versus continuous frequency ranges and also we will discuss electrodynamics versus the quantum mechanics.



The Macroscopic Maxwell Equations

- All of macroscopic electromagnetism, including the propagation of light in a photonic crystal, is governed by the four macroscopic **Maxwell equations** *i.e.*

Maxwell's Equations

$$\nabla . \mathrm{E} = \frac{\rho_v}{\varepsilon} \qquad \text{(Gauss's Law)}$$

$$\nabla . \mathrm{H} = 0 \qquad \text{(Gauss's Law for Magnetism)}$$

$$\nabla \times \mathrm{E} = -\mu \frac{\partial \mathrm{H}}{\partial t} \qquad \text{(Faraday's Law)}$$

$$\nabla \times \mathrm{H} = \mathrm{J} + \varepsilon \frac{\partial \mathrm{E}}{\partial t} \qquad \text{(Ampere's } Law\text{)}$$

where (respectively) E and H are the macroscopic electric and magnetic fields, D and B are the displacement and magnetic induction fields, and ρ & J are the free charge and current densities.

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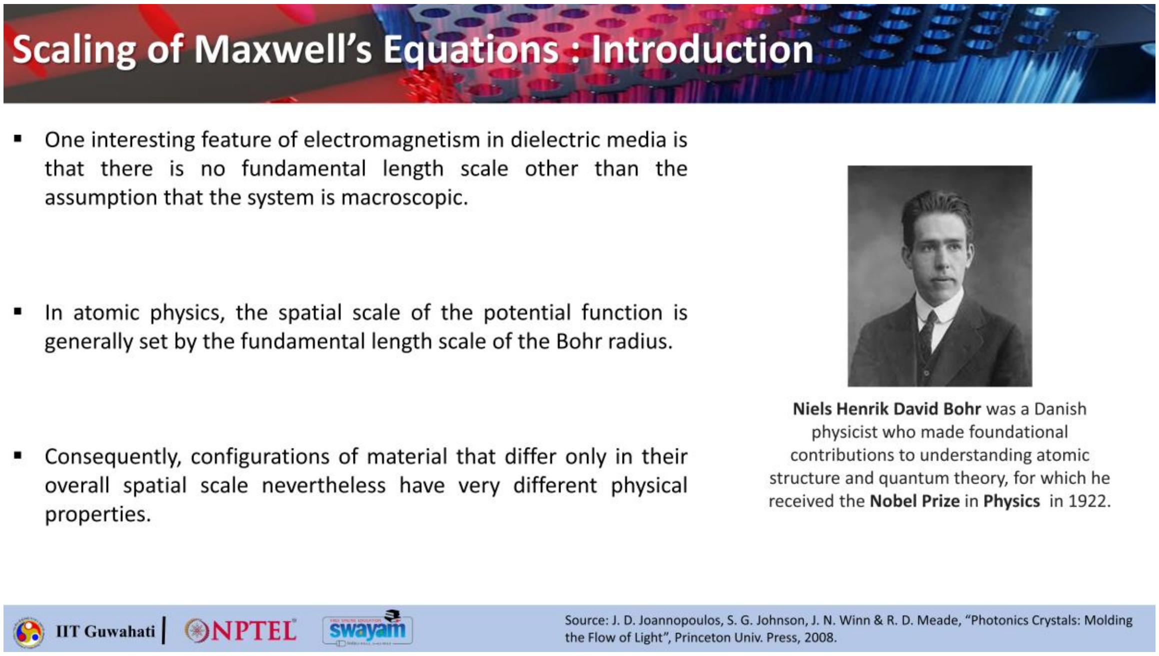
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, macroscopic Maxwell's equations.

So, we have seen that all of the macroscopic electromagnetism including the propagation of light in a photonic crystal is governed by the four macroscopic Maxwell's equation. that is this metaphor equations which you have been seeing over the last couple of lectures. So, quick repetition ∇.E=ρ_v/ε that is Gauss law and ∇.

H=0 that is Gauss law for magnetism. You have Faraday's law which is written as ∇×E=$-\mu$ ∂H/∂t and then you have Maxwell's Ampere law which is written as ∇×H=J+∂E/∂t. So, you can also write this as ∂D/∂t. So, here E and H are basically the macroscopic electric and magnetic fields, whereas capital D okay and B are the displacement and the magnetic induction fields, rho and J are the free charge and the current densities okay.



Scaling of Maxwell's Equations : Introduction

- One interesting feature of electromagnetism in dielectric media is that there is no fundamental length scale other than the assumption that the system is macroscopic.
- In atomic physics, the spatial scale of the potential function is generally set by the fundamental length scale of the Bohr radius.
- Consequently, configurations of material that differ only in their overall spatial scale nevertheless have very different physical properties.

Niels Henrik David Bohr was a Danish physicist who made foundational contributions to understanding atomic structure and quantum theory, for which he received the **Nobel Prize** in **Physics** in 1922.

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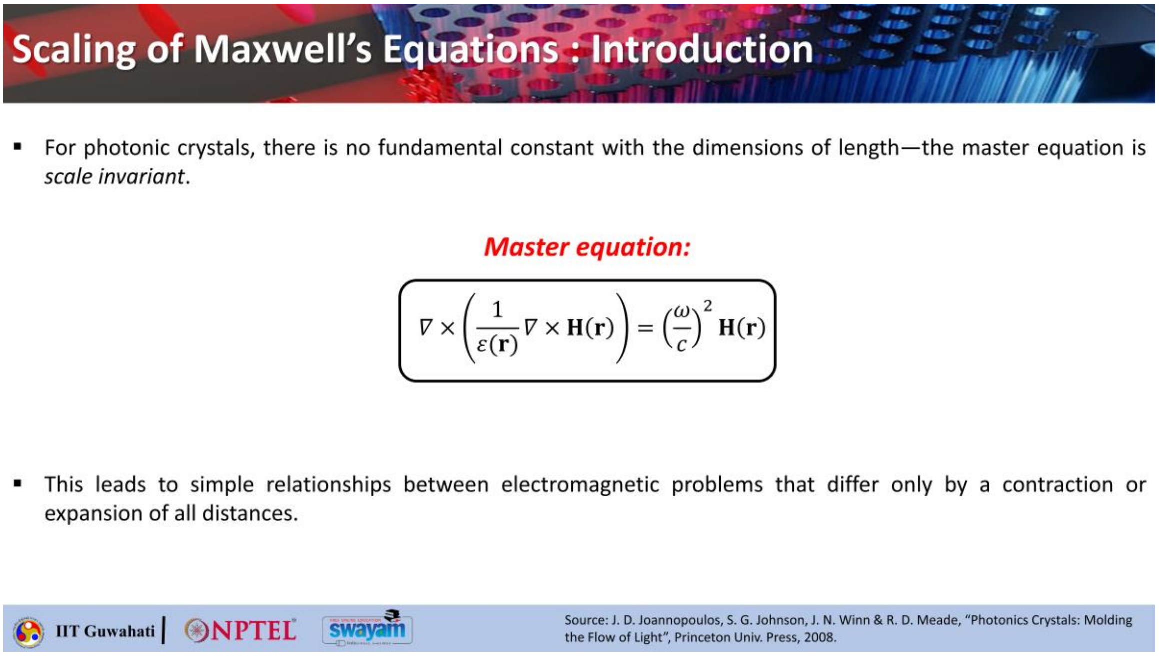
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So now we'll move into the topic of scaling of Maxwell's equation.

So one interesting feature of electromagnetism in dielectric media is that there is no fundamental length scale other than the assumption that the system is macroscopic. In atomic physics, the spatial scale of the potential function is generally set by the fundamental length scale of Bohr radius. So, that is the picture of Niels Bohr who was a Danish physicist and he made foundational contributions to the understanding of atomic structure and quantum theory for which he also got the Nobel Prize in physics in 1922. So, what I mean to say here is that in atomic physics there is a spatial scale of the potential function and it is set by the fundamental length scale of Bohr radius. So, consequently configurations of materials that differ only in their overall spatial scale nevertheless have very different physical properties.



Scaling of Maxwell's Equations : Introduction

- For photonic crystals, there is no fundamental constant with the dimensions of length—the master equation is *scale invariant*.

Master equation:

$$\boxed{\nabla \times \left(\frac{1}{\varepsilon(\mathbf{r})}\nabla \times \mathbf{H}(\mathbf{r})\right) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r})}$$

- This leads to simple relationships between electromagnetic problems that differ only by a contraction or expansion of all distances.

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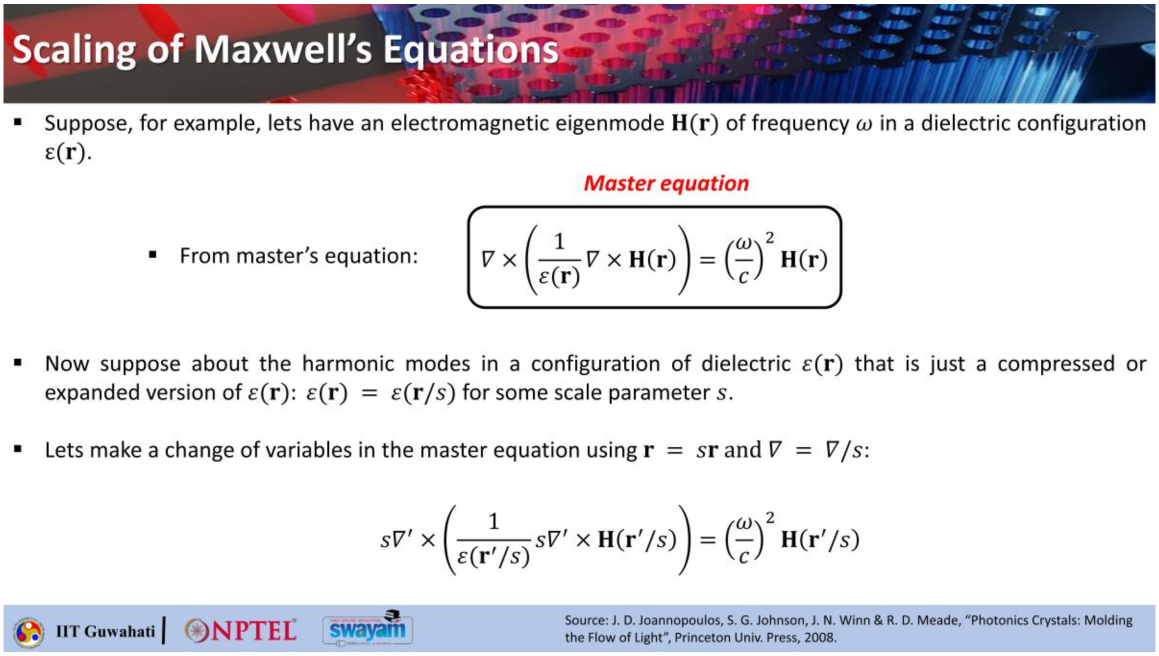
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

But for photonic crystals when you see there is no fundamental constant with the dimensions of length and the master equation is basically scale invariant. So, this is the master equation where you can see we have seen this in the previous lecture how it is obtained. So, it is called of 1 by $\varepsilon(\mathbf{r})$. times you know $\nabla\times\mathbf{H}(\mathbf{r})$ will be equal to (ω/c)^2 $\mathbf{H}(\mathbf{r})$. So, this is the master equation and this leads to simple relationships between the electromagnetic problems that differ only by a contraction or expansion of all distances.



Scaling of Maxwell's Equations

- Suppose, for example, lets have an electromagnetic eigenmode $\mathbf{H}(\mathbf{r})$ of frequency ω in a dielectric configuration $\varepsilon(\mathbf{r})$.

Master equation

- From master's equation: $$\boxed{\nabla \times \left(\frac{1}{\varepsilon(\mathbf{r})} \nabla \times \mathbf{H}(\mathbf{r})\right) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r})}$$

- Now suppose about the harmonic modes in a configuration of dielectric $\varepsilon(\mathbf{r})$ that is just a compressed or expanded version of $\varepsilon(\mathbf{r})$: $\varepsilon(\mathbf{r}) = \varepsilon(\mathbf{r}/s)$ for some scale parameter s.

- Lets make a change of variables in the master equation using $\mathbf{r} = s\mathbf{r}$ and $\nabla = \nabla/s$:

$$s\nabla' \times \left(\frac{1}{\varepsilon(\mathbf{r}'/s)} s\nabla' \times \mathbf{H}(\mathbf{r}'/s)\right) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r}'/s)$$

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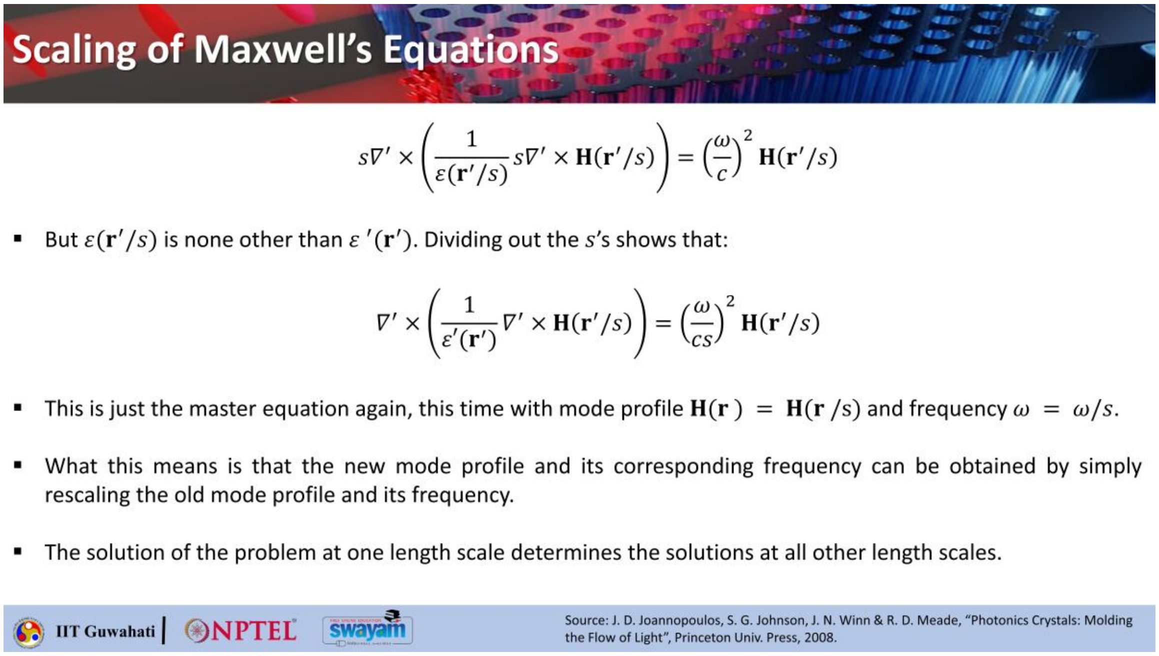
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, we will see that scaling, scaling means you are squeezing the space or expanding the space okay. So, suppose for example we have an electromagnetic eigenmode $\mathbf{H}(\mathbf{r})$ of frequency ω in a dielectric configuration which is given by $\varepsilon(\mathbf{r})$. Again we go back to the master's equation and we put all these parameters here. We know that epsilon is basically $\varepsilon(\mathbf{r})$, the eigenmode is $\mathbf{H}(\mathbf{r})$ and then you have got the frequency ω of the particular eigenmode. Now suppose that the harmonic mode in a configuration of dielectric $\varepsilon(\mathbf{r})$ that is just a compressed or expanded version of $\varepsilon(\mathbf{r})$.

In that case the new one you can say $\varepsilon'(\mathbf{r})$ is basically $\varepsilon(\mathbf{r}/s)$ where s is some scale parameter. So let us make a change of variables in the master equation where we consider $\mathbf{r}' = s\mathbf{r}$ and and ∇' is basically ∇/s. So if you do that you can actually write everything in terms of you know so this nabla or del operator can be written as ∇' times s. So, s ∇' like this okay, this one can be replaced by this okay. So, it is basically epsilon.



Scaling of Maxwell's Equations

$$s\nabla' \times \left(\frac{1}{\varepsilon(\mathbf{r}'/s)} s\nabla' \times \mathbf{H}(\mathbf{r}'/s)\right) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r}'/s)$$

- But $\varepsilon(\mathbf{r}'/s)$ is none other than $\varepsilon\,'(\mathbf{r}')$. Dividing out the s's shows that:

$$\nabla' \times \left(\frac{1}{\varepsilon'(\mathbf{r}')} \nabla' \times \mathbf{H}(\mathbf{r}'/s)\right) = \left(\frac{\omega}{cs}\right)^2 \mathbf{H}(\mathbf{r}'/s)$$

- This is just the master equation again, this time with mode profile $\mathbf{H}(\mathbf{r}\ ) = \mathbf{H}(\mathbf{r}\ /s)$ and frequency $\omega = \omega/s$.
- What this means is that the new mode profile and its corresponding frequency can be obtained by simply rescaling the old mode profile and its frequency.
- The solution of the problem at one length scale determines the solutions at all other length scales.

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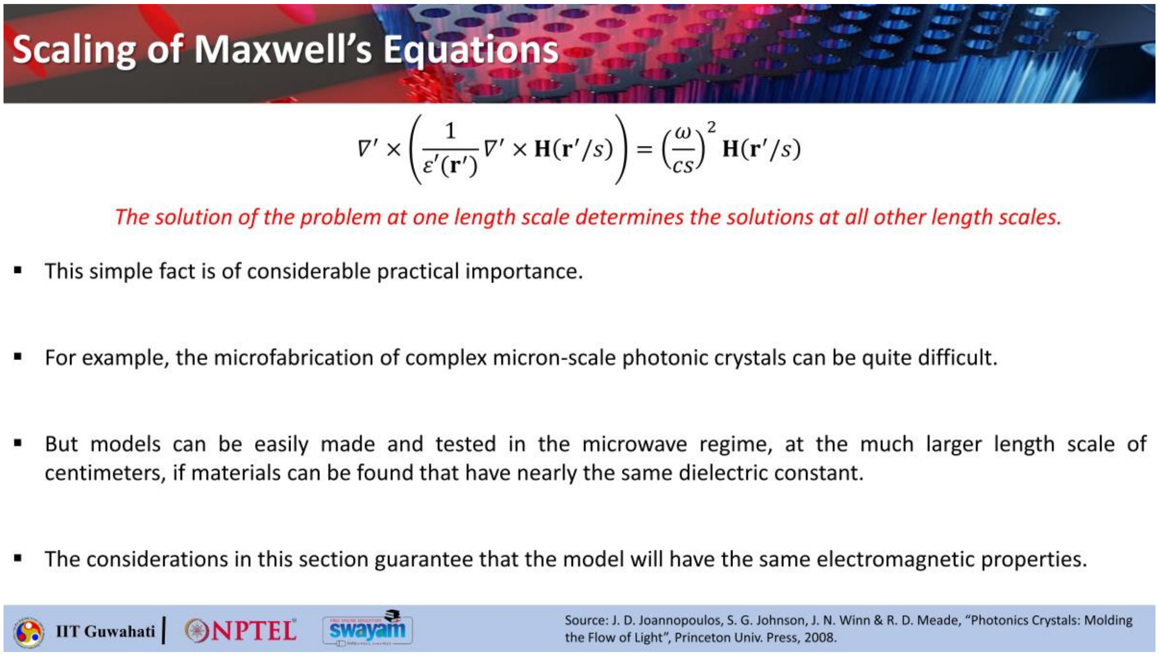
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, r is basically r' by s okay so on. So, this is how the new master equation looks like. which is again reproduced here from the previous slide. So, here you can carefully see that $\varepsilon(\mathrm{r}'/s)$ is none other than $\varepsilon'(\mathrm{r}')$ right. So, if you put that you know dividing out throughout by s's, you will see that you actually obtain this particular equation where s actually enters into this particular ω term.

So there you can rewrite a equation okay in the normal form by considering $\mathbf{H}'(\mathbf{r}'\ )$ as a new parameter which is basically this one okay $\mathbf{H}(\mathbf{r}'/s)$ okay and you consider a new frequency ω' which is basically omega by ω/s . So what it means that the new mode profile and its corresponding frequency can be obtained by simply scaling the old mode profile and its frequency. So this is what you are doing, right? So the solution of the problem at one length scale determines the solution at all other length scale. So that's something amazing.



Scaling of Maxwell's Equations

$$\nabla' \times \left(\frac{1}{\varepsilon'(\mathbf{r}')}\nabla' \times \mathbf{H}(\mathbf{r}'/s)\right) = \left(\frac{\omega}{cs}\right)^2 \mathbf{H}(\mathbf{r}'/s)$$

The solution of the problem at one length scale determines the solutions at all other length scales.

- This simple fact is of considerable practical importance.
- For example, the microfabrication of complex micron-scale photonic crystals can be quite difficult.
- But models can be easily made and tested in the microwave regime, at the much larger length scale of centimeters, if materials can be found that have nearly the same dielectric constant.
- The considerations in this section guarantee that the model will have the same electromagnetic properties.

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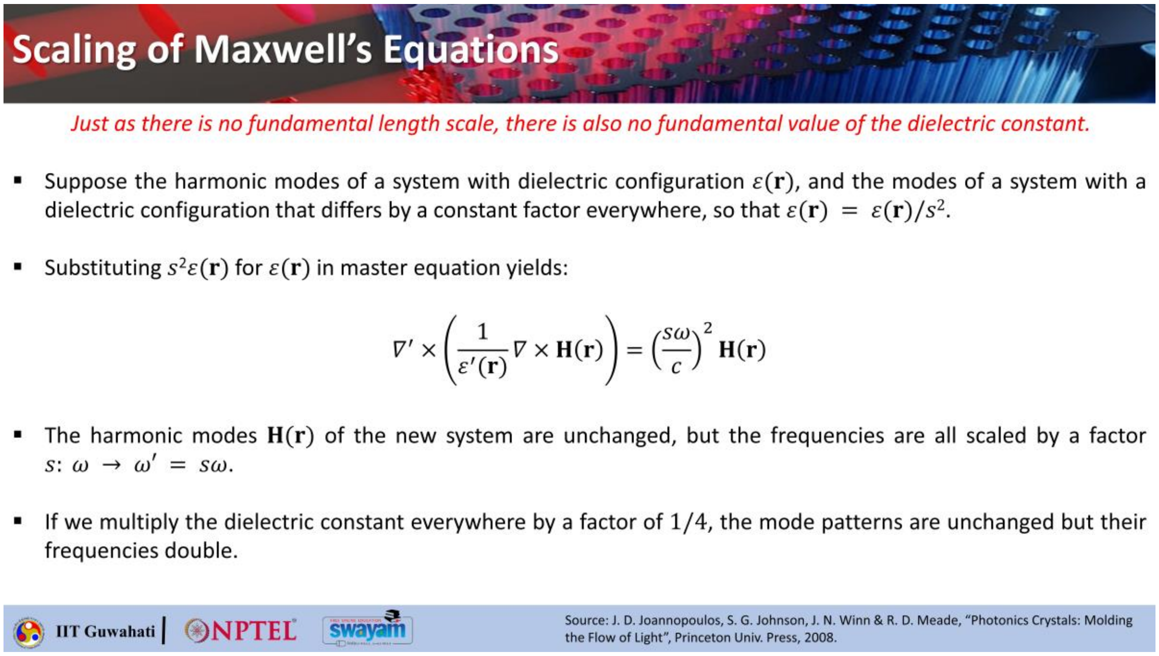
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So we again, you know, put this equation here and we can state this important observation that the solution of the problem at one length scale determines the solution at all other length scales.

This simple fact is of considerable practical importance, right? For example, the microfabrication of complex micron scale photonic crystals can be quite different and difficult. But models can be easily made for testing purpose. And you can test those in microwave regime, which is at a much larger length scale, typically of the order of centimeters. If materials can be found that have nearly same dielectric constant at both those frequency range. And that is where if you remember that the first experimental demonstration of photonic crystal was done in the microwave range.

So, these considerations guarantee that the model will have the same electromagnetic properties.



Scaling of Maxwell's Equations

Just as there is no fundamental length scale, there is also no fundamental value of the dielectric constant.

- Suppose the harmonic modes of a system with dielectric configuration $\varepsilon(\mathbf{r})$, and the modes of a system with a dielectric configuration that differs by a constant factor everywhere, so that $\varepsilon(\mathbf{r}) = \varepsilon(\mathbf{r})/s^2$.
- Substituting $s^2\varepsilon(\mathbf{r})$ for $\varepsilon(\mathbf{r})$ in master equation yields:

$$\nabla' \times \left(\frac{1}{\varepsilon'(\mathbf{r})} \nabla \times \mathbf{H}(\mathbf{r})\right) = \left(\frac{s\omega}{c}\right)^2 \mathbf{H}(\mathbf{r})$$

- The harmonic modes $\mathbf{H}(\mathbf{r})$ of the new system are unchanged, but the frequencies are all scaled by a factor s: $\omega \rightarrow \omega' = s\omega$.
- If we multiply the dielectric constant everywhere by a factor of $1/4$, the mode patterns are unchanged but their frequencies double.

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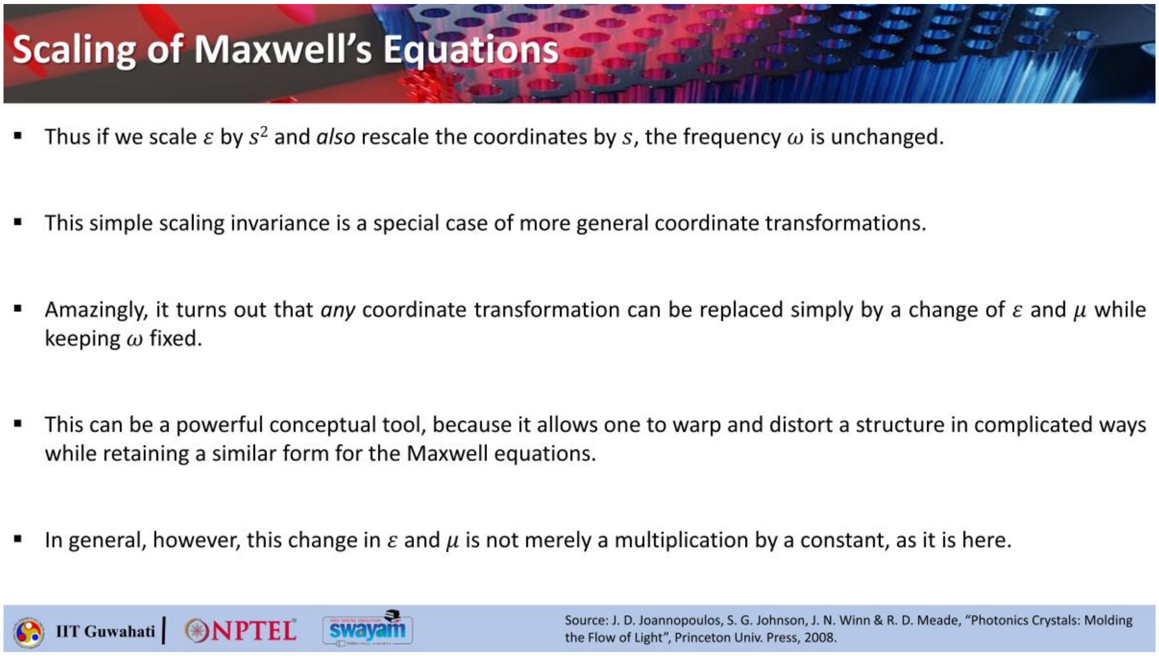
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, just as there is no fundamental length scale, there is also no fundamental value of the dielectric constant. So, suppose the harmonic modes of a system with dielectric configuration $\varepsilon(\mathbf{r})$ and the modes of a system with dielectric configuration that differs by a constant everywhere So, that you can write $\varepsilon'(\mathbf{r})$ is nothing but $\varepsilon(\mathbf{r})/s^2$. So, here the parameter is s^2. So, now if you substitute $s^2\varepsilon(\mathbf{r})$ instead of $\varepsilon(\mathbf{r})$ in the master equation, you will see that you actually obtain this.

So, what you are doing you actually have this s^2 and you are pushing that into this particular ω term. So, what do you conclude from here? You can see that the harmonic mode $\mathbf{H}(\mathbf{r})$ of this new system remains unchanged, but what is changed? The corresponding frequency got scaled by a factor of s. So, the new frequency you can write as ω' which is basically $s\omega$. And that happens when you know the dielectric constant get a scale by this kind of factor. So, this is what numerically we can understand that if we multiply the dielectric constant everywhere by a factor of 1 by 4.



Scaling of Maxwell's Equations

- Thus if we scale ε by s^2 and *also* rescale the coordinates by s, the frequency ω is unchanged.
- This simple scaling invariance is a special case of more general coordinate transformations.
- Amazingly, it turns out that *any* coordinate transformation can be replaced simply by a change of ε and μ while keeping ω fixed.
- This can be a powerful conceptual tool, because it allows one to warp and distort a structure in complicated ways while retaining a similar form for the Maxwell equations.
- In general, however, this change in ε and μ is not merely a multiplication by a constant, as it is here.

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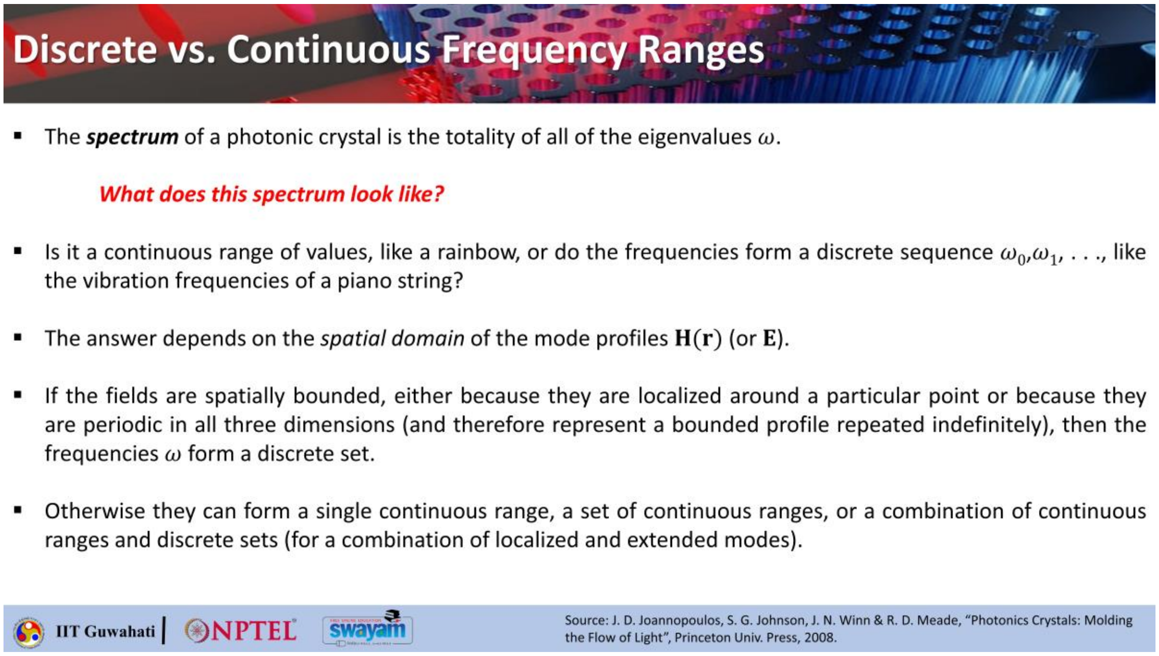
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, what will happen? that is like s^2, right? If you do that calculation from here, the mode patterns will remain unchanged, but their frequency will become 2ω, that is the frequencies will double. Thus, if we scale epsilon by s^2 and also rescale the coordinates by s, so what will happen in that case? In that case, the frequency will remain unchanged. So, this is something like called this simple scaling invariance is a special case of more generalized you know coordinate transformation. So, this is an important observation that if you scale the permittivity by s^2 and then rescale the coordinates by s your frequencies will remain unchanged and we will see you know coordinate transformation later in this course, but you see that in coordinate transformation you can actually you know any material from one coordinate system can be translated or transformed into another shape or coordinate system by replacing the values of ε and μ.

So it turns out that any coordinate transformation can be replaced simply by a change of ε and μ while keeping the ω fixed.

And this can be a very powerful conceptual tool because it allows one to wrap and distort a structure in complicated ways while retaining a similar form of the Maxwell's equation. However, in general this kind of change in ε and μ is not just a mere multiplication rather you have to deal with Jacobian matrix and all those things we will see later on in this course. But just remember that the change is not simply multiplying you know ε and μ by a constant it is slightly more rigorous than that. Now let us look into you know where you see discrete, where you see continuous frequency ranges. So, frequency ranges are important when you want to study the spectrum of a photonic crystal, right? And the spectrum of a photonic crystal is the totality of all the eigenvalues ω.



Discrete vs. Continuous Frequency Ranges

- The ***spectrum*** of a photonic crystal is the totality of all of the eigenvalues ω.

  What does this spectrum look like?

- Is it a continuous range of values, like a rainbow, or do the frequencies form a discrete sequence $\omega_0, \omega_1, \ldots$, like the vibration frequencies of a piano string?
- The answer depends on the *spatial domain* of the mode profiles $\mathbf{H}(\mathbf{r})$ (or $\mathbf{E}$).
- If the fields are spatially bounded, either because they are localized around a particular point or because they are periodic in all three dimensions (and therefore represent a bounded profile repeated indefinitely), then the frequencies ω form a discrete set.
- Otherwise they can form a single continuous range, a set of continuous ranges, or a combination of continuous ranges and discrete sets (for a combination of localized and extended modes).

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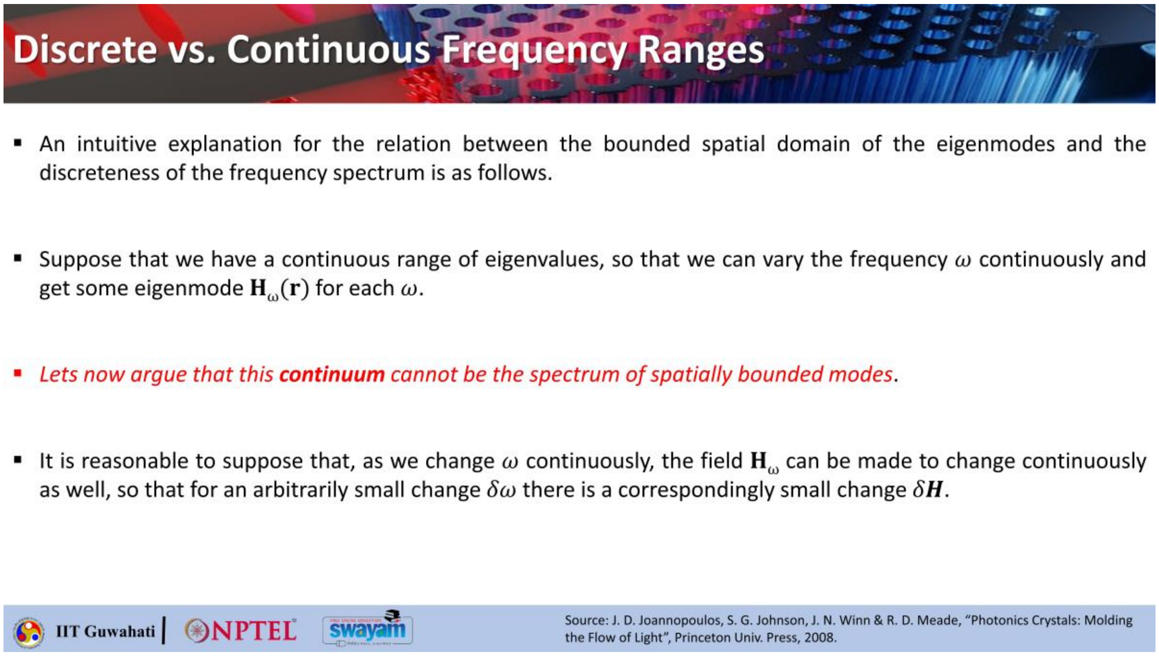
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, what does this spectrum look like? Is it a continuous range of values like a rainbow or do the frequencies form discrete sequence, something like $\omega_0, \omega_1, \ldots$ and so on, okay? Like the vibrational frequencies of a piano string. The answer is it depends on the spatial domain of the mode profile H(r) that is the vector magnetic field or electric field. So, if the fields are spatially bounded either because they are localized around a particular point or because they are periodic in all three dimensions.

okay and therefore represent a bounded profile which is repeated indefinitely then the frequencies ω form a discrete set. Otherwise they can form a single continuous range, a set of continuous values or a combination of continuous ranges and discrete sets something like for combination of localized and extended modes



Discrete vs. Continuous Frequency Ranges

- An intuitive explanation for the relation between the bounded spatial domain of the eigenmodes and the discreteness of the frequency spectrum is as follows.
- Suppose that we have a continuous range of eigenvalues, so that we can vary the frequency ω continuously and get some eigenmode $\mathbf{H}_\omega(\mathbf{r})$ for each ω.
- *Lets now argue that this **continuum** cannot be the spectrum of spatially bounded modes*.
- It is reasonable to suppose that, as we change ω continuously, the field $\mathbf{H}_\omega$ can be made to change continuously as well, so that for an arbitrarily small change $\delta\omega$ there is a correspondingly small change $\delta\boldsymbol{H}$.

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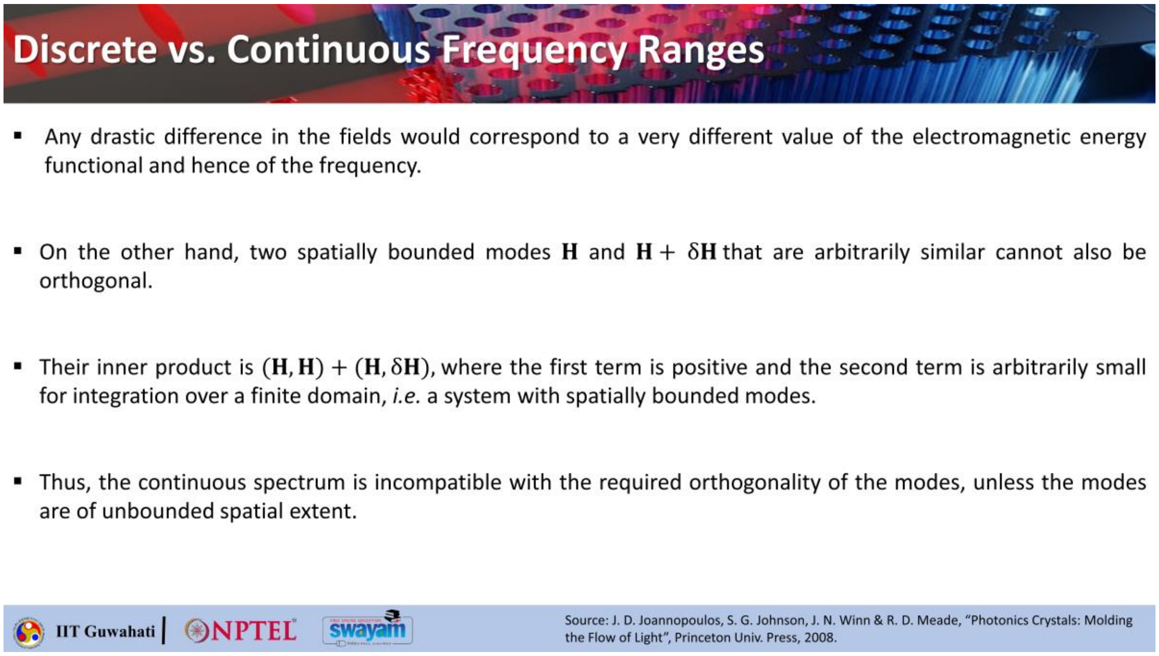
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So an intuitive explanation for the relation between the bounded special domain of the eigenmodes and the discrete nets of the frequency spectrum is discussed here. So, let us suppose that we have a continuous range of eigenvalues. So, that we can vary the frequency omega continuously and get some eigenmode which is expressed as $\mathbf{H}_\omega(\mathbf{r})$ for each of this frequency ω.

So, we can now argue that this continuum cannot be the spectrum of the spatially bounded modes. and it is reasonable to suppose that as we change ω continuously the field H_ω can be made to change continuously as well. So, that any arbitrarily small amount of change that is $\delta\omega$ will also have a corresponding small change in the vector field δH. So, any district difference in the field would correspond to a very different value of the electromagnetic energy functional and hands of the frequency. So, on the other hand you will see that two spatially bounded modes$\mathbf{H}$ and $\mathbf{H}+\delta\mathbf{H}$ they are arbitrarily similar cannot also be orthogonal.



Discrete vs. Continuous Frequency Ranges

- Any drastic difference in the fields would correspond to a very different value of the electromagnetic energy functional and hence of the frequency.
- On the other hand, two spatially bounded modes $\mathbf{H}$ and $\mathbf{H} + \delta\mathbf{H}$ that are arbitrarily similar cannot also be orthogonal.
- Their inner product is $(\mathbf{H}, \mathbf{H}) + (\mathbf{H}, \delta\mathbf{H})$, where the first term is positive and the second term is arbitrarily small for integration over a finite domain, *i.e.* a system with spatially bounded modes.
- Thus, the continuous spectrum is incompatible with the required orthogonality of the modes, unless the modes are of unbounded spatial extent.

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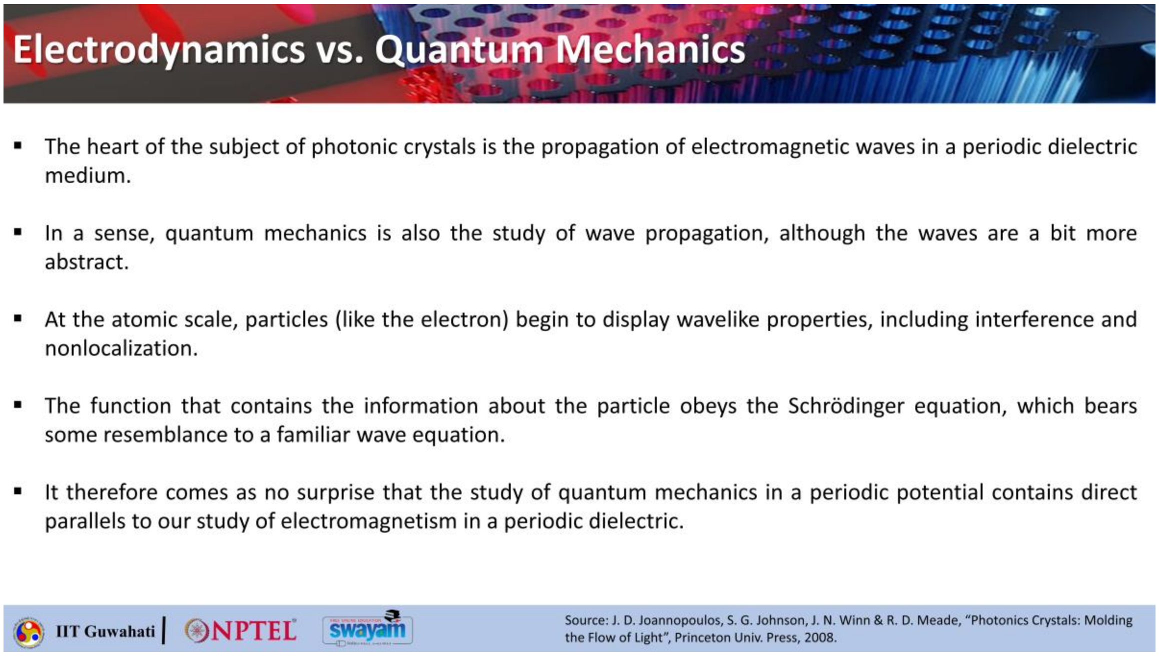
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So if you take the inner product okay which will be inner product of $(\mathbf{H},\mathbf{H})+(\mathbf{H}, \delta\mathbf{H})$. So here the first term is positive and the second term is arbitrarily small for integration over a finite domain that is where a system has got spatially bounded modes. So, thus the continuous spectrum is basically incompatible with the required orthogonalities of the modes unless the modes are unbounded spatial extent.



Electrodynamics vs. Quantum Mechanics

- The heart of the subject of photonic crystals is the propagation of electromagnetic waves in a periodic dielectric medium.
- In a sense, quantum mechanics is also the study of wave propagation, although the waves are a bit more abstract.
- At the atomic scale, particles (like the electron) begin to display wavelike properties, including interference and nonlocalization.
- The function that contains the information about the particle obeys the Schrödinger equation, which bears some resemblance to a familiar wave equation.
- It therefore comes as no surprise that the study of quantum mechanics in a periodic potential contains direct parallels to our study of electromagnetism in a periodic dielectric.

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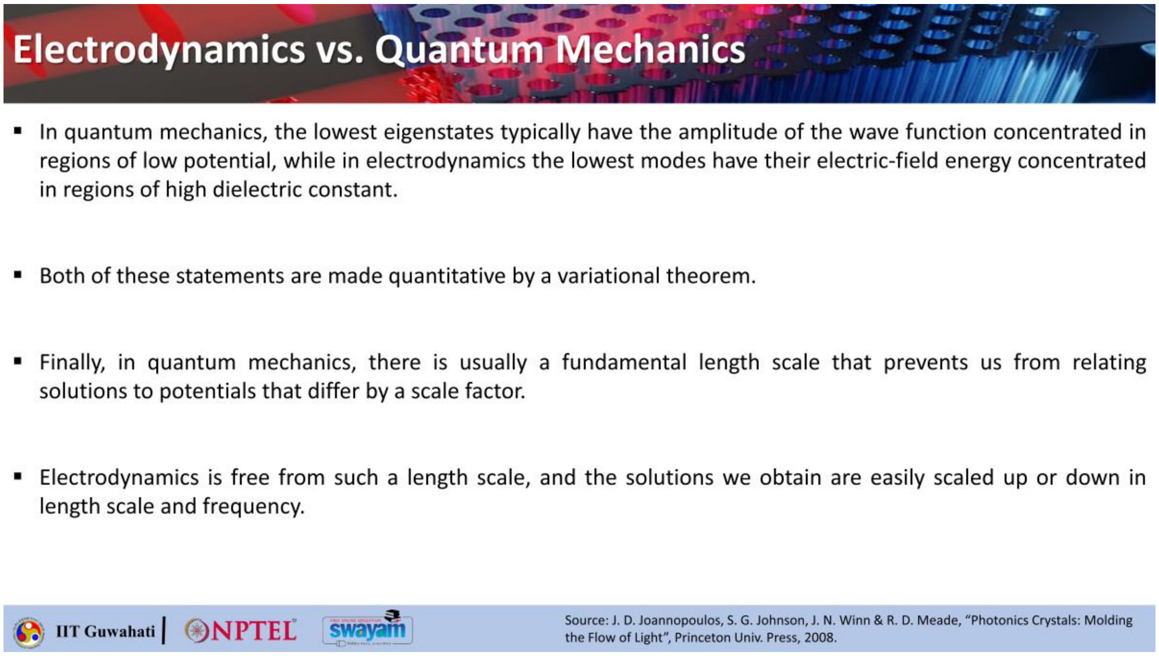
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, “Photonics Crystals: Molding the Flow of Light”, Princeton Univ. Press, 2008.

So, what happens you know when you compute this spectrum we understood that you know you get discrete ranges. So, you try to find out the propagation of electromagnetic modes in photonic crystal and that is where you can also try to draw an analogy between electrodynamics and quantum mechanics.

So, as we mentioned the main soul, hardened soul of the photonic crystal is basically about the propagation of electromagnetic wave in a dielectric medium.



Electrodynamics vs. Quantum Mechanics

- In quantum mechanics, the lowest eigenstates typically have the amplitude of the wave function concentrated in regions of low potential, while in electrodynamics the lowest modes have their electric-field energy concentrated in regions of high dielectric constant.
- Both of these statements are made quantitative by a variational theorem.
- Finally, in quantum mechanics, there is usually a fundamental length scale that prevents us from relating solutions to potentials that differ by a scale factor.
- Electrodynamics is free from such a length scale, and the solutions we obtain are easily scaled up or down in length scale and frequency.

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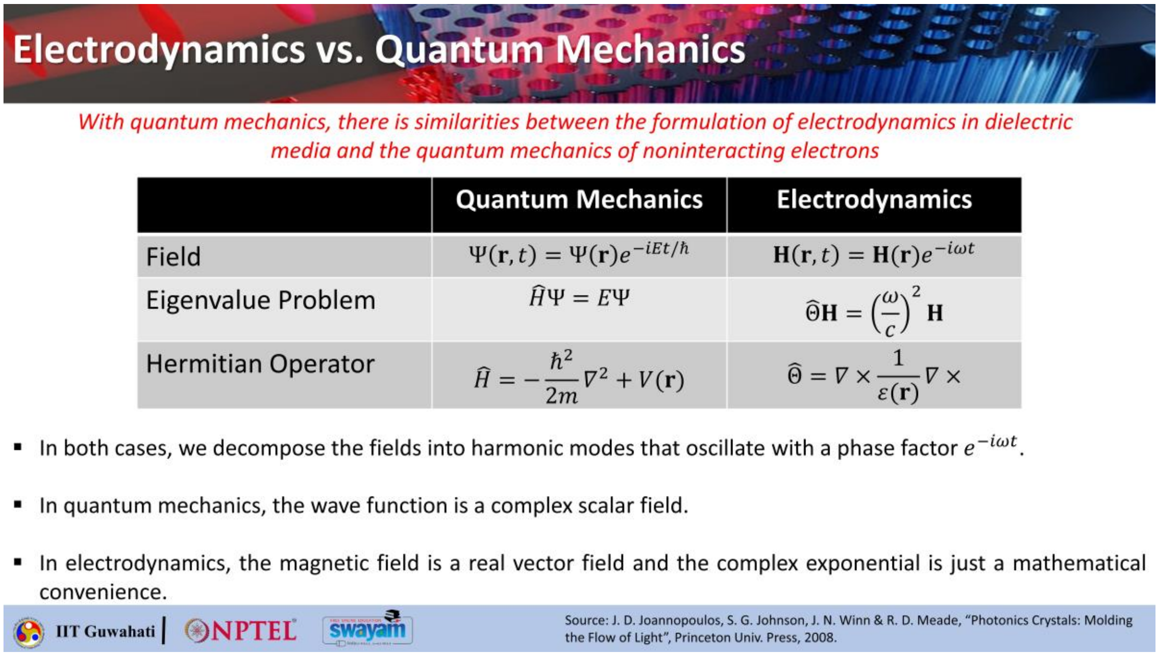
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, “Photonics Crystals: Molding the Flow of Light”, Princeton Univ. Press, 2008.

So, in a sense quantum mechanics is also a study of wave propagation although the waves are a bit more abstract in that case. So, there you know at the atomic scale particles such as electrons began to display wave like properties including interference and non-localization and that function contains the information about the particle okay obeying the Schrodinger's equation which bears some sort of resemblance to the wave equation that is there for the photonic crystal. But it therefore comes as no surprise that the study of quantum mechanics in periodic potential contains direct parallels to our study of electromagnetism in periodic dielectric medium. So since the quantum mechanics of periodic potential is the basic theory of solid state physics, The field of photonic crystal can also inherit some of the theorems and terminologies of the solid state physics in a slightly modified form.

In quantum mechanics, the lowest eigenstates typically have the amplitude of the wave function concentrated in the regions of low potential. While in electrodynamics, the lowest modes have the electric energy concentrated in the regions of high dielectric constant. Both of these statements are made quantitative by a variational theorem which have been discussed also in the previous lecture. So, finally you can see that in quantum mechanics There is usually a fundamental length scale that prevents us from relating solutions to potential that differ by a scale factor. But on the other hand, electrodynamics is basically free from any such length scale.

So, the solutions that we obtain can be easily scaled up or scaled down in length scale and frequency.



Electrodynamics vs. Quantum Mechanics

With quantum mechanics, there is similarities between the formulation of electrodynamics in dielectric media and the quantum mechanics of noninteracting electrons

	Quantum Mechanics	Electrodynamics
Field	$\Psi(\mathbf{r},t)=\Psi(\mathbf{r})e^{-iEt/\hbar}$	$\mathbf{H}(\mathbf{r},t)=\mathbf{H}(\mathbf{r})e^{-i\omega t}$
Eigenvalue Problem	$\hat{H}\Psi=E\Psi$	$\hat{\Theta}\mathbf{H}=\left(\frac{\omega}{c}\right)^2\mathbf{H}$
Hermitian Operator	$\hat{H}=-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})$	$\hat{\Theta}=\nabla\times\frac{1}{\varepsilon(\mathbf{r})}\nabla\times$

- In both cases, we decompose the fields into harmonic modes that oscillate with a phase factor $e^{-i\omega t}$.
- In quantum mechanics, the wave function is a complex scalar field.
- In electrodynamics, the magnetic field is a real vector field and the complex exponential is just a mathematical convenience.

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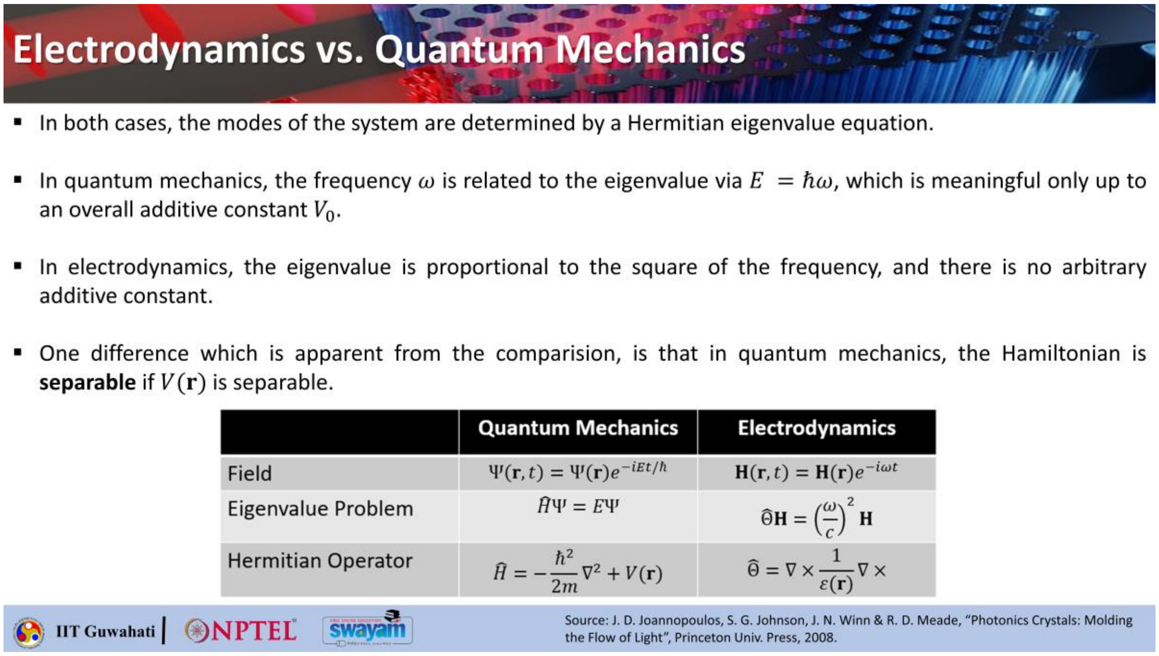
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, these are the you know differences between electrodynamics and quantum mechanics. But then there are some similarities as well because in with quantum mechanics there are similarities between the formulation of electrodynamics in dielectric media and the quantum mechanics of non interacting electrons. So, if you tabulate those you can see that how do you express field in quantum mechanics and this is how you express field in electrodynamics they have very similar form ok this is the wave equation of potential. This is the potential function and this is the magnetic vector field Eigenvalue problem can be written like this.

Here it is written as this one that is also a eigenvalue problem. The Hermitian operator is basically the Hamiltonian in the case of quantum mechanics and this is the form of this operator Θ cap in case of electrodynamics. So, in both cases we decompose the field into harmonic modes that oscillate with the phase factor of $e^{(-i\omega t)}$ that is correct. In quantum mechanics, the wave equation, the wave function in quantum mechanics, the wave function is a complex scalar field. Whereas in electrodynamic the magnetic field is a real vector and the complex exponential is just a mathematical convenience.



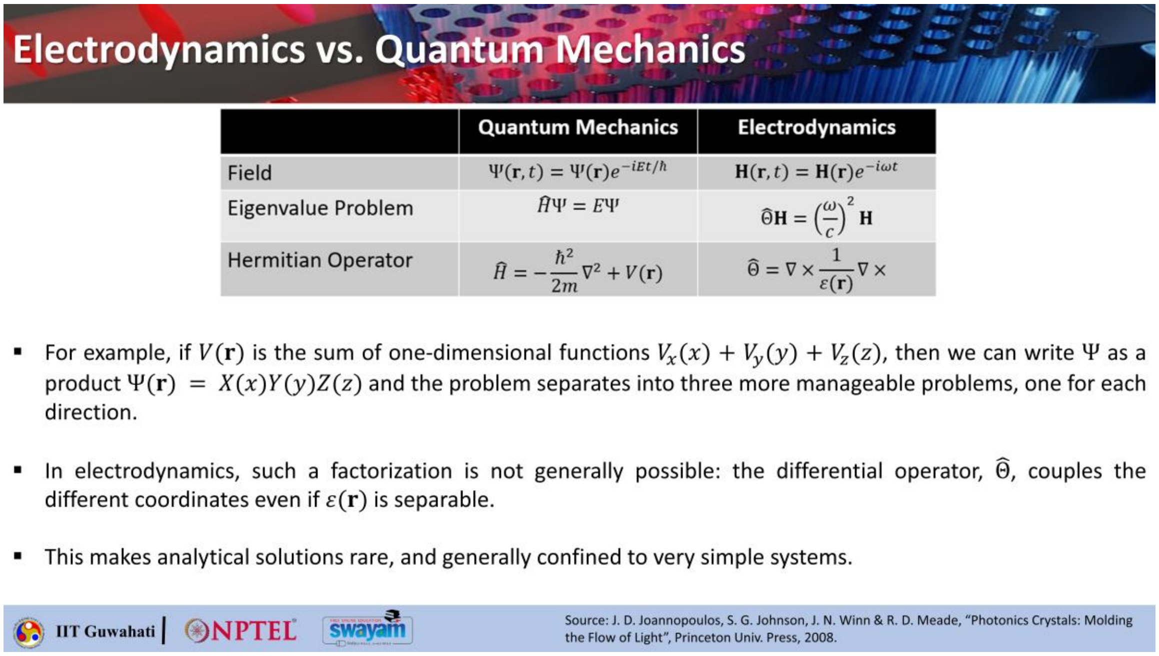
Electrodynamics vs. Quantum Mechanics

- In both cases, the modes of the system are determined by a Hermitian eigenvalue equation.
- In quantum mechanics, the frequency ω is related to the eigenvalue via $E = \hbar\omega$, which is meaningful only up to an overall additive constant V_0.
- In electrodynamics, the eigenvalue is proportional to the square of the frequency, and there is no arbitrary additive constant.
- One difference which is apparent from the comparision, is that in quantum mechanics, the Hamiltonian is **separable** if $V(\mathbf{r})$ is separable.

	Quantum Mechanics	Electrodynamics
Field	$\Psi(\mathbf{r},t) = \Psi(\mathbf{r})e^{-iEt/\hbar}$	$\mathbf{H}(\mathbf{r},t) = \mathbf{H}(\mathbf{r})e^{-i\omega t}$
Eigenvalue Problem	$\hat{H}\Psi = E\Psi$	$\hat{\Theta}\mathbf{H} = \left(\frac{\omega}{c}\right)^2 \mathbf{H}$
Hermitian Operator	$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})$	$\hat{\Theta} = \nabla \times \frac{1}{\varepsilon(\mathbf{r})}\nabla \times$

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, in both the cases the modes of the systems are basically determined by a Hermitian eigenvalue equation. quantum mechanics the frequency ω is related to the eigenvalue that is H equals E equals $\hbar\omega$ which is meaningful only up to an overall additive constant of V_0 that is the potential okay. However, in electrodynamics the eigenvalue is proportional to the square of the frequency and there is no arbitrary additive constant. So, one difference which is apparent from this comparison is that in quantum mechanics the Hamiltonian is separable if V(r) is separable.



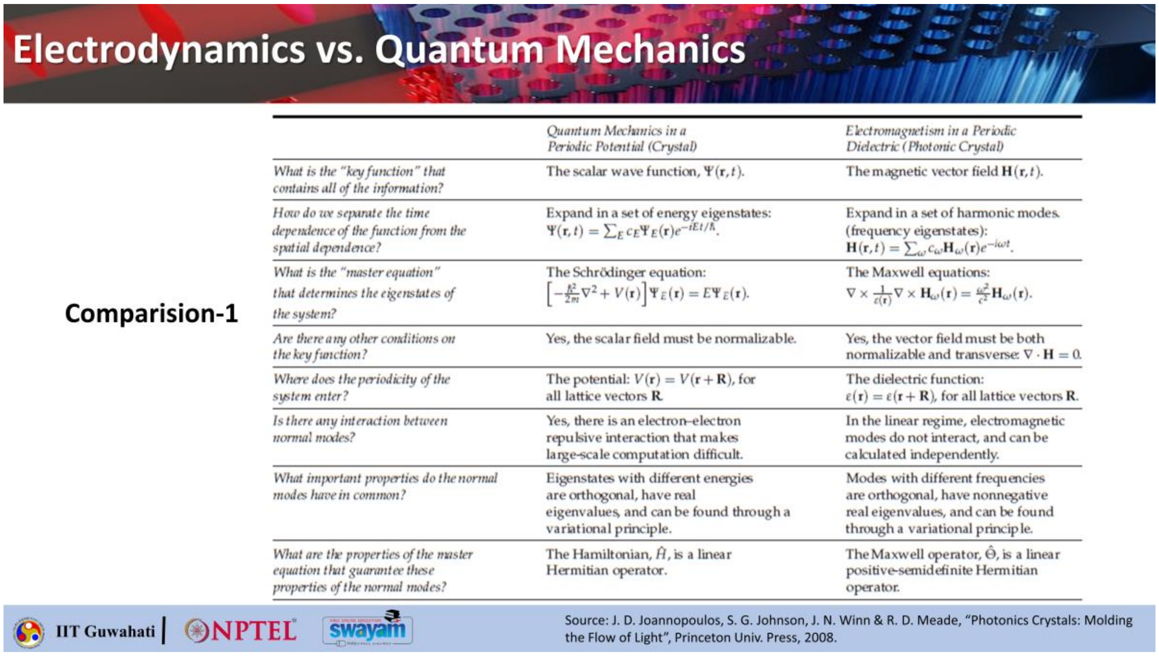
Electrodynamics vs. Quantum Mechanics

	Quantum Mechanics	Electrodynamics
Field	$\Psi(\mathbf{r},t) = \Psi(\mathbf{r})e^{-iEt/\hbar}$	$\mathbf{H}(\mathbf{r},t) = \mathbf{H}(\mathbf{r})e^{-i\omega t}$
Eigenvalue Problem	$\hat{H}\Psi = E\Psi$	$\hat{\Theta}\mathbf{H} = \left(\frac{\omega}{c}\right)^2 \mathbf{H}$
Hermitian Operator	$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})$	$\hat{\Theta} = \nabla \times \frac{1}{\varepsilon(\mathbf{r})} \nabla \times$

- For example, if $V(\mathbf{r})$ is the sum of one-dimensional functions $V_x(x) + V_y(y) + V_z(z)$, then we can write Ψ as a product $\Psi(\mathbf{r}) = X(x)Y(y)Z(z)$ and the problem separates into three more manageable problems, one for each direction.
- In electrodynamics, such a factorization is not generally possible: the differential operator, $\hat{\Theta}$, couples the different coordinates even if $\varepsilon(\mathbf{r})$ is separable.
- This makes analytical solutions rare, and generally confined to very simple systems.

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, “Photonics Crystals: Molding the Flow of Light”, Princeton Univ. Press, 2008.

So, it comes here ok fine. So, rest all form it looks very much similar in quantum mechanics and electrodynamics. if you take an example of V(r) as a sum of one-dimensional vectors okay or functions sorry for example if you consider V(r) as sum of one-dimensional functions like $V_x(x) + V_y(y) + V_z(z)$ then we can write $\Psi(\mathbf{r})$ the wave function as a product of $\Psi(\mathbf{r})$ given as $X(x)Y(y)Z(z)$ okay and the problem separates into three more manageable problem one for each direction. right. In electrodynamics such a factorization is generally not possible because the differential operator Θ cap couples the different coordinates even if you know $\varepsilon(\mathbf{r})$ is separable. So, this makes you know analytical solutions pretty rare in case of electrodynamics and generally those analytical solutions are confined to very very simple systems.



Electrodynamics vs. Quantum Mechanics

Comparision-1

	Quantum Mechanics in a Periodic Potential (Crystal)	*Electromagnetism in a Periodic Dielectric (Photonic Crystal)*
What is the "key function" that contains all of the information?	The scalar wave function, $\Psi(\mathbf{r},t)$.	The magnetic vector field $\mathbf{H}(\mathbf{r},t)$.
How do we separate the time dependence of the function from the spatial dependence?	Expand in a set of energy eigenstates: $\Psi(\mathbf{r},t)=\sum_E c_E \Psi_E(\mathbf{r})e^{-iEt/\hbar}$.	Expand in a set of harmonic modes. (frequency eigenstates): $\mathbf{H}(\mathbf{r},t)=\sum_\omega c_\omega \mathbf{H}_\omega(\mathbf{r})e^{-i\omega t}$.
What is the "master equation" that determines the eigenstates of the system?	The Schrödinger equation: $\left[-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{r})\right]\Psi_E(\mathbf{r})=E\Psi_E(\mathbf{r})$.	The Maxwell equations: $\nabla\times\frac{1}{\varepsilon(\mathbf{r})}\nabla\times\mathbf{H}_\omega(\mathbf{r})=\frac{\omega^2}{c^2}\mathbf{H}_\omega(\mathbf{r})$.
Are there any other conditions on the key function?	Yes, the scalar field must be normalizable.	Yes, the vector field must be both normalizable and transverse: $\nabla\cdot\mathbf{H}=0$.
Where does the periodicity of the system enter?	The potential: $V(\mathbf{r})=V(\mathbf{r}+\mathbf{R})$, for all lattice vectors $\mathbf{R}$.	The dielectric function: $\varepsilon(\mathbf{r})=\varepsilon(\mathbf{r}+\mathbf{R})$, for all lattice vectors $\mathbf{R}$.
Is there any interaction between normal modes?	Yes, there is an electron–electron repulsive interaction that makes large-scale computation difficult.	In the linear regime, electromagnetic modes do not interact, and can be calculated independently.
What important properties do the normal modes have in common?	Eigenstates with different energies are orthogonal, have real eigenvalues, and can be found through a variational principle.	Modes with different frequencies are orthogonal, have nonnegative real eigenvalues, and can be found through a variational principle.
What are the properties of the master equation that guarantee these properties of the normal modes?	The Hamiltonian, $\hat{H}$, is a linear Hermitian operator.	The Maxwell operator, $\hat{\Theta}$, is a linear positive-semidefinite Hermitian operator.

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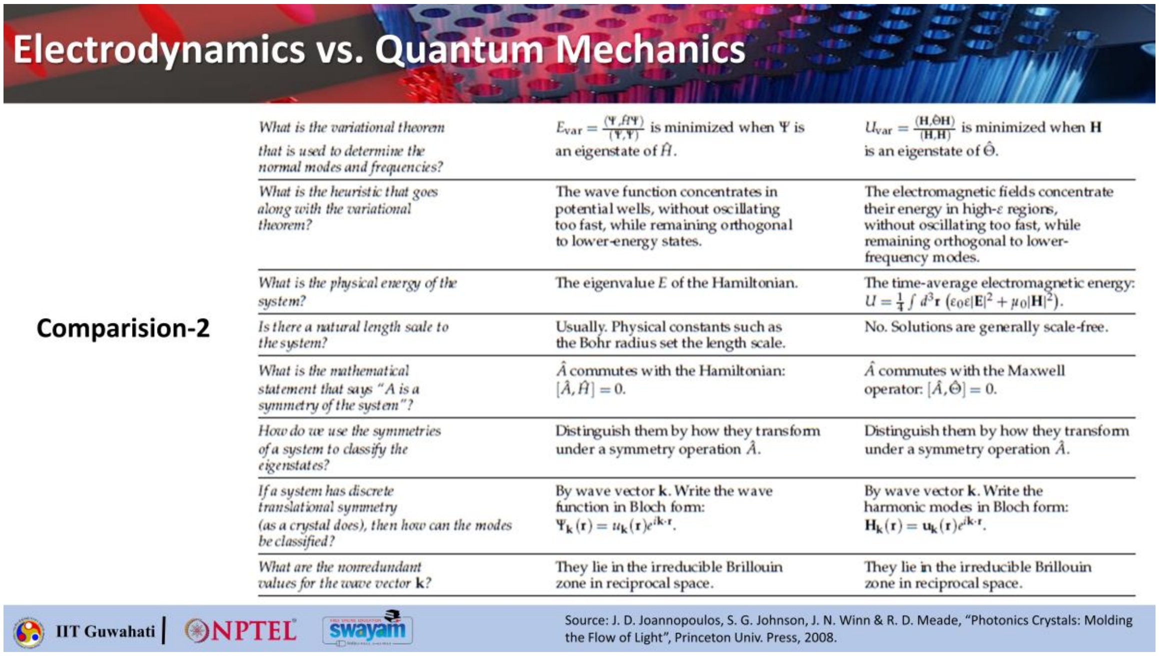
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, let us compare electrodynamics and quantum mechanics in more details in different aspects. So, what is the key function that contains all of the information in quantum mechanics in a periodic potential which can be thought of a crystal? The answer is the scalar wave function $\Psi(\mathbf{r},t)$. And if you consider electromagnetism in a periodic dielectric, which is basically a photonic crystal, the key function that contains all the information is magnetic vector field, which is given as $\mathrm{H}(\mathbf{r},t)$. How do you separate the time dependence of the function from the spatial dependence? So, the way in quantum mechanics is expand in a set of energy eigenstates. So, you can write $\Psi(\mathbf{r},t)$ in this particular form where the spatial and time dependence can be separated.

In electromagnetism also you can expand in a set of harmonic modes which are basically frequency eigenstates and you can have $\mathbf{H}_\omega(\mathbf{r})$ though that is for particularly for particular frequency okay that is why they are called frequency eigenstates and then you have the time dependence separated out. What is the master equation that determines the eigenstates of the system? In the case of quantum mechanics, it is the Schrodinger's equation and this is how it is written. So, this is the H cap parameter in Schrodinger's equation and this is the Maxwell's equation in case of electromagnetism and this is a master equation that you have all seen. Are there any other condition on the key function? Yes, the scalar wave field must be normalizable. And in case of electromagnetism, yes, the vector field must be both normalizable and transverse.



Electrodynamics vs. Quantum Mechanics

Comparision-2

What is the variational theorem that is used to determine the normal modes and frequencies?	$E_{\text{var}} = \frac{(\Psi,\hat{H}\Psi)}{(\Psi,\Psi)}$ is minimized when Ψ is an eigenstate of $\hat{H}$.	$U_{\text{var}} = \frac{(\mathbf{H},\hat{\Theta}\mathbf{H})}{(\mathbf{H},\mathbf{H})}$ is minimized when $\mathbf{H}$ is an eigenstate of $\hat{\Theta}$.
What is the heuristic that goes along with the variational theorem?	The wave function concentrates in potential wells, without oscillating too fast, while remaining orthogonal to lower-energy states.	The electromagnetic fields concentrate their energy in high-ε regions, without oscillating too fast, while remaining orthogonal to lower-frequency modes.
What is the physical energy of the system?	The eigenvalue E of the Hamiltonian.	The time-average electromagnetic energy: $U = \frac{1}{4}\int d^3\mathbf{r}\,(\varepsilon_0\varepsilon\lvert\mathbf{E}\rvert^2 + \mu_0\lvert\mathbf{H}\rvert^2)$.
Is there a natural length scale to the system?	Usually. Physical constants such as the Bohr radius set the length scale.	No. Solutions are generally scale-free.
What is the mathematical statement that says "A is a symmetry of the system"?	$\hat{A}$ commutes with the Hamiltonian: $[\hat{A},\hat{H}] = 0$.	$\hat{A}$ commutes with the Maxwell operator: $[\hat{A},\hat{\Theta}] = 0$.
How do we use the symmetries of a system to classify the eigenstates?	Distinguish them by how they transform under a symmetry operation $\hat{A}$.	Distinguish them by how they transform under a symmetry operation $\hat{A}$.
If a system has discrete translational symmetry (as a crystal does), then how can the modes be classified?	By wave vector $\mathbf{k}$. Write the wave function in Bloch form: $\Psi_{\mathbf{k}}(\mathbf{r}) = u_{\mathbf{k}}(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}$.	By wave vector $\mathbf{k}$. Write the harmonic modes in Bloch form: $\mathbf{H}_{\mathbf{k}}(\mathbf{r}) = \mathbf{u}_{\mathbf{k}}(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}$.
What are the nonredundant values for the wave vector $\mathbf{k}$?	They lie in the irreducible Brillouin zone in reciprocal space.	They lie in the irreducible Brillouin zone in reciprocal space.

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Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, ∇.H should be equal to 0. What does the periodicity, where does the periodicity of the system enter? So in quantum mechanics, it enters through the potential. So V(r) is basically written as V(r + R), where this is for all lattice vectors at position R. And in case of photonic crystal, electrodynamics, the periodicity enters through the dielectric function. So you can write ε(**r**) is basically ε(**r**+R) for all the lattice vectors R. Is there any interaction between normal modes? Yes.

In quantum mechanics, there is an electron-electron repulsive interaction that makes large-scale computation difficult. And in electromagnetism, in the linear regime, the electromagnetic modes do not interact, so you can calculate them independently. What important properties do the normal modes have in common? In quantum mechanics, eigenstates with different energies are orthogonal. They have real eigenvalues and can be found through a variational principle. Whereas in case of electromagnetism in photonic crystal, you can say the modes with different frequencies are also orthogonal.

They have non-negative real eigenvalues and they also can be found through variational principle. What are the properties of the master equation that guarantee that these properties of the normal modes? So, in quantum mechanics, it is basically the Hamiltonian H cap is a linear Hermitian operator. and in the case of electromagnetism the Maxwell operator which is Θ cap this part okay Θ cap okay is a linear positive semi-definite Hermitian operator. So, you can actually see what is you know Maxwell operator Θ cap. Going further, what is the variational theorem that is used to determine the normal modes and frequencies? So you can see this is E_var which is the inner product of Ψ(H cap Ψ) over (Ψ,Ψ) is minimized when Ψ is the eigenstate of H cap.

So this is how you can actually use the variational theorem. So, for electromagnetism it is basically you know the electromagnetic energy U_var can be obtained as inner product of (H, Θ cap H) over (H,H) inner product of H is minimized when this H is in eigen state of Θ cap. What is the heuristic that goes along with the variational theorem? So, the wave function concentrates in potential well without oscillating too fast while maintaining orthogonal to lower energy states. In case of electrodynamics, electromagnetic fields concentrate their energy in high permittivity region again without oscillating too fast while maintaining orthogonal to the lower frequency modes. So, what is the physical energy of the system? Eigen value E of the Hamiltonian that gives you the physical energy of the system.

In case of electrodynamics the time average electromagnetic energy U is obtained like that ok. So, you take the modular square of the electric field and the magnetic field and you know take the volume integration and that you get it. Is there a natural length scale to the system? Usually in case of quantum mechanics there is the physical constraints such as Bohr radius will set that length scale and in photonic crystal in electrodynamics there is no such length scale. So, the solutions are generally scale free. So, what is the mathematical statement that says A is the symmetry of the system? Okay, you can say that A cap commutes with the Hamiltonian.

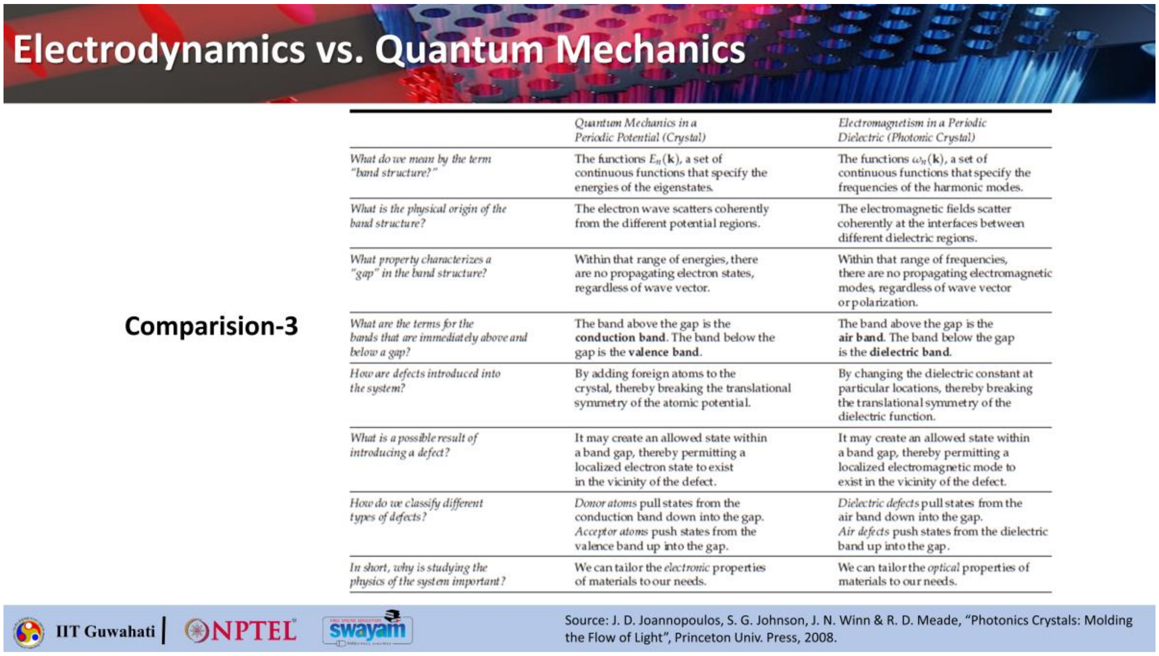
So, you can write this as 0 and in electrodynamics also you can say A commutes with the Maxwell's operator. So, this also becomes 0. How do you use the symmetries of the system to classify the eigenstates? In quantum mechanics, you can distinguish them by how they transform under a symmetry operation of A cap and the same happen in case of electrodynamics as well. Now, if the system has a discrete translational symmetry as a crystal does, how can the modes be classified? Now that is something interesting and that is applicable for both electromagnetics as well as quantum mechanics by wave vector k you can do it.

So, you write the wave function in Bloch form. So, you can write $\Psi_k(\mathbf{r})$ equals $u_k(r)$ e to the power ikr. So, this is where the periodicity comes in. You can also write the harmonic modes. in case of electrodynamics as $H_k(r)$ equals $u_k(r)$ e to the power ikr. So, it is very very similar what you do in quantum mechanics and what you do in electrodynamics.

So, what are the non redundant values for the wave vector k? They basically lie in the irreducible Brillouin zone in reciprocal space and the same concept is applied here. So you only need to actually study the irreducible Brillouin zone to obtain the information of the band diagram and all. So that will drastically minimize your computational requirements. You can also further compare them in terms of what do you mean by the term bench structure. So, the functions e and k which is a set of continuous function that specify the energies of the eigenstates.

So, band structure basically tells you about the energies of the eigenstates. In electromagnetism, you can use the function ω and k, a set of continuous functions that also tells you about the frequencies of the harmonic modes. So, there is eigenstates, energies of the eigenstates, here is the frequency of the harmonic modes. What is the physical origin of the band structure? quantum mechanics in normal crystal is the electron wave scatters coherently from different potential regime right. In electromagnetic it is basically the electromagnetic fields scatter coherently at the interface between different dielectric interface or dielectric region.

So, that is how here what different potential region is doing here different dielectric region is emulating the same. what properly characterizes a gap in the band structure. So it tells you that within that range, of energies, there are no propagating electron states regardless of the wave factor. So in any direction, propagation is not allowed. In electromagnetism, in photonic crystal, it also means the same that within that range of frequencies, there are no propagating electromagnetic modes regardless of the wave factor or polarization.



Electrodynamics vs. Quantum Mechanics

Comparision-3

	Quantum Mechanics in a Periodic Potential (Crystal)	*Electromagnetism in a Periodic Dielectric (Photonic Crystal)*
What do we mean by the term "band structure?"	The functions $E_n(\mathbf{k})$, a set of continuous functions that specify the energies of the eigenstates.	The functions $\omega_n(\mathbf{k})$, a set of continuous functions that specify the frequencies of the harmonic modes.
What is the physical origin of the band structure?	The electron wave scatters coherently from the different potential regions.	The electromagnetic fields scatter coherently at the interfaces between different dielectric regions.
What property characterizes a "gap" in the band structure?	Within that range of energies, there are no propagating electron states, regardless of wave vector.	Within that range of frequencies, there are no propagating electromagnetic modes, regardless of wave vector or polarization.
What are the terms for the bands that are immediately above and below a gap?	The band above the gap is the **conduction band**. The band below the gap is the **valence band**.	The band above the gap is the **air band**. The band below the gap is the **dielectric band**.
How are defects introduced into the system?	By adding foreign atoms to the crystal, thereby breaking the translational symmetry of the atomic potential.	By changing the dielectric constant at particular locations, thereby breaking the translational symmetry of the dielectric function.
What is a possible result of introducing a defect?	It may create an allowed state within a band gap, thereby permitting a localized electron state to exist in the vicinity of the defect.	It may create an allowed state within a band gap, thereby permitting a localized electromagnetic mode to exist in the vicinity of the defect.
How do we classify different types of defects?	*Donor atoms* pull states from the conduction band down into the gap. *Acceptor atoms* push states from the valence band up into the gap.	*Dielectric defects* pull states from the air band down into the gap. *Air defects* push states from the dielectric band up into the gap.
In short, why is studying the physics of the system important?	We can tailor the *electronic* properties of materials to our needs.	We can tailor the *optical* properties of materials to our needs.

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So there is something additional over here. So, that makes this vector and this is a scalar thing. We are talking about the wave potential which is scalar, you are talking about magnetic vector potential that is where the polarization gets into picture. What are the terms for bands that are immediately above and below a band gap? for semiconductor crystal you know that those are conduction band which is above the gap and the one below is valence band okay in case of photonic crystal the band above is called air band and the band which is below the gap is called the dielectric band so how are the defects introduced in the system so in normal crystal in quantum mechanics you can see that By adding foreign atoms to the crystals, you can break the translational symmetry of the atomic potential and that is how you can introduce defects. And in case of photonic crystal, by changing the dielectric constant of a particular location, you are again breaking the translational symmetry of the dielectric function and that is how you can introduce a defect. What is the possible result of introducing a defect? So, in crystals, it may create an allowed state within a band gap.

Therefore, permitting a localized electron state to exist in the vicinity of the defect. And it is very similar what happens in photonic crystal also. So, there also it actually creates an allowed state within the gap. So, you are actually permitting a localized electromagnetic mode to exist in the vicinity of the defect. So how do you classify different types of defects? So in crystals you can think of donor atoms which pull states from conduction band down into the gap and you can also have acceptor atoms which push states from the valence band up into the gap and that is how you know defects effect your crystal band diagram.

In case of photonic crystal, if you have dielectric defects, they also do the same thing. They pull states from the air band into the gap and if you have air defects, they push states from the dielectric band up into the So this is very important. What do you want to do? Okay. Always remember the band that is above the gap is called air band and the band that is below the gap is called the dielectric band. So in short, what is studying the physics of the system? Why is in short, why is studying the physics of the system important? So we need to know the system well to tailor the electronic properties of the materials to our need.

And we do the same thing in case of photonic crystal so that we can tailor the electromagnetic property or optical property of the material to our needs. So, with that we come to an conclusion to this lecture.

Send your queries at deb.sikdar@iitg.ac.in

So, we will start the discussion on symmetries for classification of EM modes in the next lecture. If you have got any queries regarding this particular lecture, you can always drop an email to me mentioning MOOC and photonic crystal on the subject line and this is my email address deb.sikdar@iitg.ac.in. Thank you