Computer Vision and Image Processing – Fundamentals and Applications
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Lecture – 10
Image Reconstruction from a Series of Projections

Welcome to NPTEL MOOCS course on Computer Vision and Image Processing – Fundamentals and Applications. In my last class, I discussed about the concept of the Radon transform that I have shown the distinction between x-ray imaging and the CT scan a computed tomography. This CT scan is nothing, but the 3D representation of the object from number of projections.

So, I can consider number of projections and from this I can get 3D representation of the object a particular object and I have discussed about how to determine the projection. The projection is determined like this g s theta I can determine from f (x, y), f (x, y) is the object. So, this class the last class I am going to continue for Radon transform and also the inverse Radon transform.

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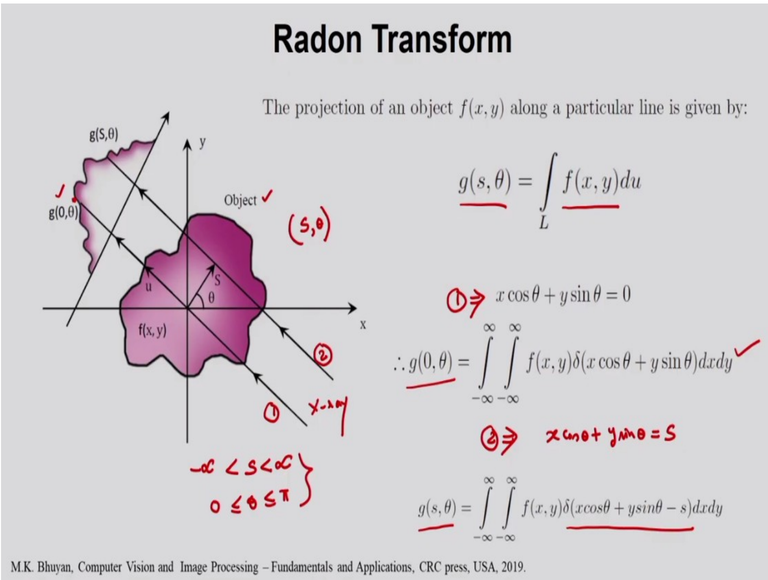

So, what is the Radon transform? If you see the projection of f (x, y) along a particular line that we have define that is g s θ. Now what is s θ? S and theta represent the coordinate of the x-ray so we have the x-ray here coordinate of x-ray relative to the object. So, I have the object and I have the x-ray. So, s θ coordinates the coordinate is s theta that is the coordinate of the x-ray relative to the object.

And in this case the s is defined in the range infinity minus infinity. So, this is the value of s the range and also theta the range of θ is from pi to 0. So, this is the range for s and θ. Now, in this case what I am considering I am considering g s θ g s theta is nothing, but the line integral this is the line integral this I am considering the object function is f x, y. The image reconstruction problem can be defined as the process of determining f x, y from g s theta.

G s theta is the projection so in this case I want to determine f x, y from g s θ. In my last class, I have shown suppose corresponding to this line, line number 1 my equation of the line is 1, that is the equation of the line number is 1, x cos theta plus y sin theta is equal to 0 and corresponding to this my projection g 0 theta. So, if you see here g 0 theta is the this projection at this point.

And after this I am considering g s theta, what is g s theta? The projection of the second line, if I consider the second line, so what is the equation of the second line? Equation of the second line is x cos theta plus y sin theta minus s that is the equation of the second line and corresponding to the second line what is the projection? The projection is g s theta that is the projection of f x, y.

The equation of the second line is x cos theta plus y sin theta is equal to s that is the equation of the second line. So, if you see this expression the g 0 theta, what is the meaning of this? G 0 theta is the integration along the line passing through the origin of x, y coordinate and whose normal vector is in the theta direction and that is given by g 0 theta and like this I am considering g s theta.

And in this case I am considering the direct delta function because I am just determining the line integral that is only along the line particular line, along the line number 1, along the line number 2.

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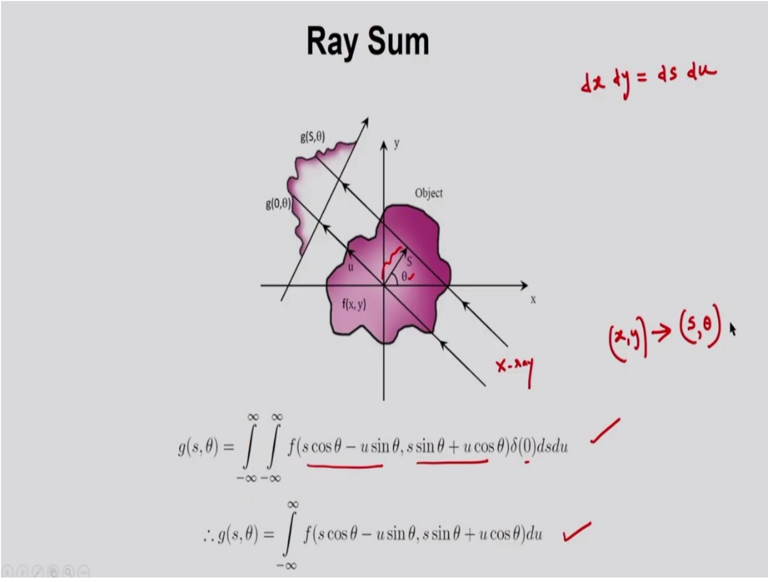

After this I considered the concept of the Ray Sum. Ray Sum expression if you see g s theta we have this expression and after this I am putting the value of x and y that is in terms of s and u. This is x and this is y and I am considering because x cos theta plus y sin theta minus s that is equal to 0 so it is del 0 dsdu and also if you remember that one the dxdy is equal to dsdu because this coordinate transformation from x, y to s, u there is no shrinkage and there is no expansion.

So, that is why dxdy is equal to dsdu that we have considered and corresponding to this I am having the Ray Sum equation, this is the Ray Sum equation. So, what is the meaning of this equation? This is the equation that sum of the input function f x, y along the x-ray direction the direction of the x-ray, x-ray is this whose distance from the origin is s so this distance is s if you see this distance is s and whose normal vector is in the theta direction.

So, the normal vector is in the theta direction that is sum of the input function f x, y along the direction of the x-ray part whose distance from the origin is s and whose normal vector is in the theta direction that is the direction of the Ray Sum. So, the Radon transform is mapping the special domain x, y into that s theta domain this is the mapping. So, last class we explain about this concept the concept of the Ray Sum.

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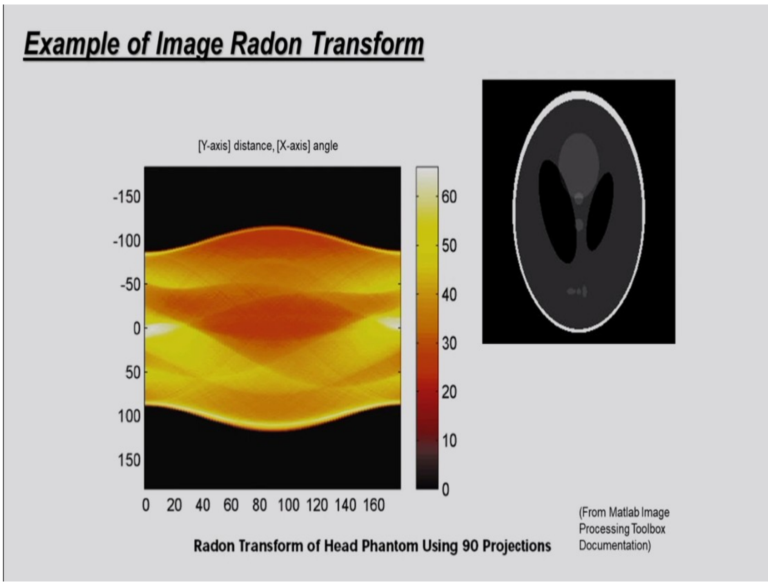

And this is the Sinogram the Sinogram we have explained the Sinogram is nothing, but the Radon transform is displayed as an image.

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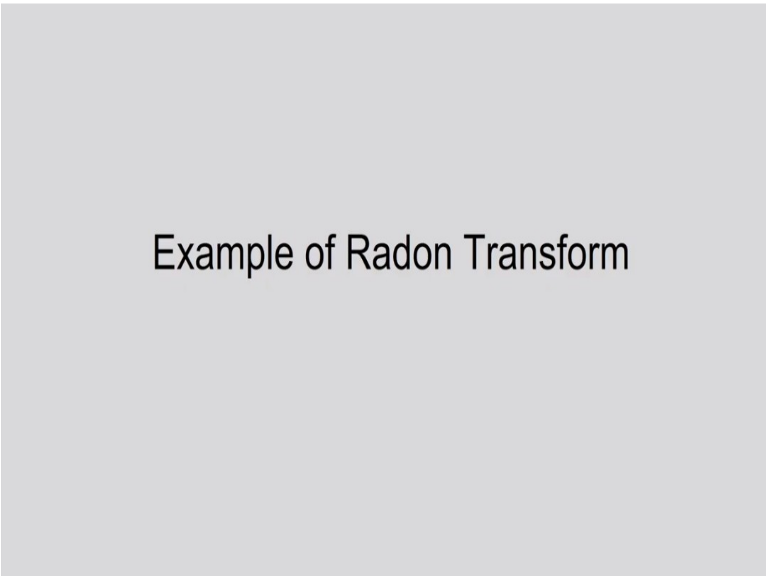

And I have shown some examples of the Radon transform.

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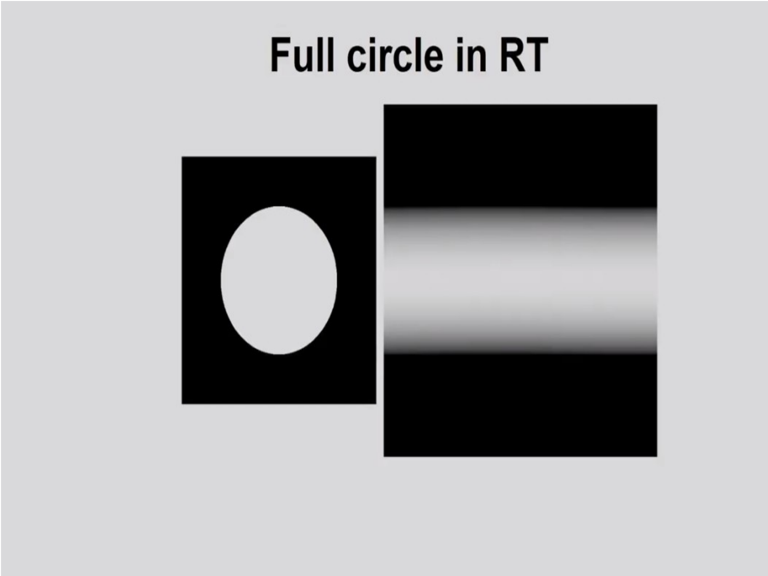

The examples like this you can see one is the full circle in the Radon transform.

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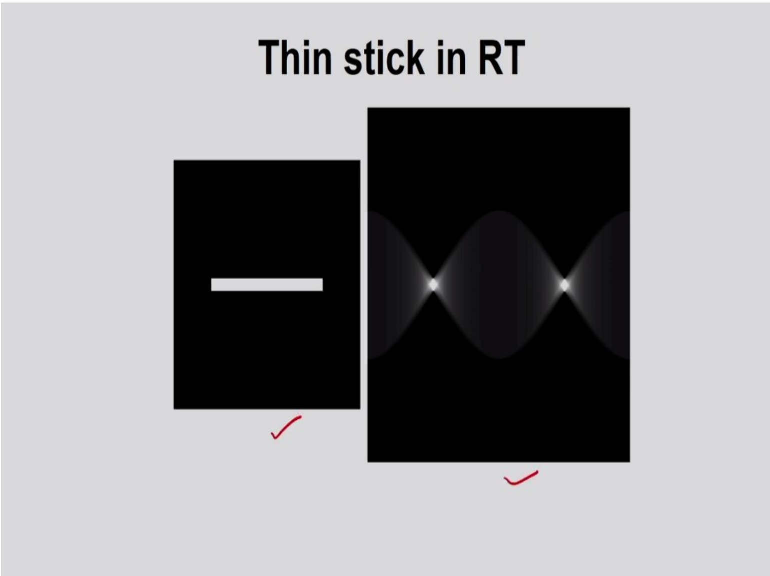

And the second one is the thin stick in the Radon transform. So, my input is this input and corresponding to this, this is my Radon transform.

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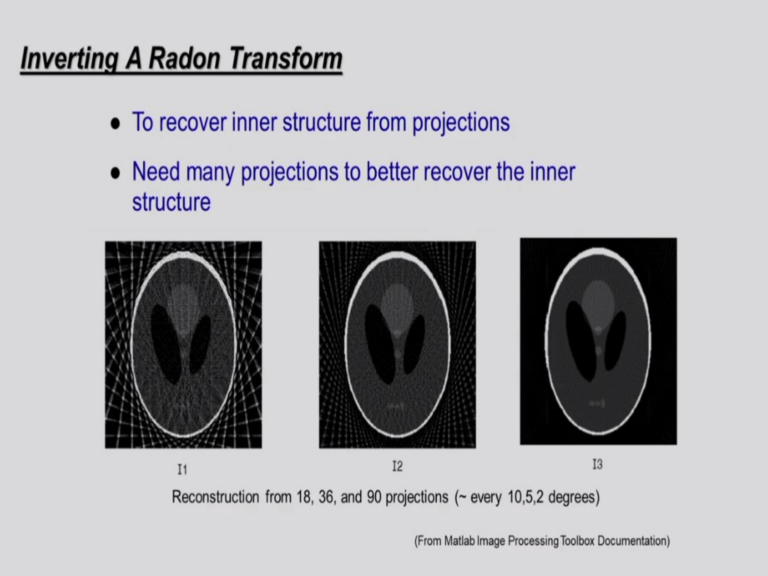

Now the inverting a Radon transform. So, if I want to reconstruct f x, y from g s theta I have to do inverse Radon transform. Now I can get the reconstructed value of f x, y from the projection the projection is g s theta. So, in this example I have shown reconstruction from 18, 36 and the 90 number of projections. So, inverse Radon transform that is the back projection I am considering and based on this back projection I am getting the reconstructed image that is the reconstructed image that is nothing, but the 3D representation of a particular object.

So, to construct a particular image or the object f x, y I can apply two approaches. There are many approaches, but in this case I can consider two approaches one is the back projection method and another one is the Fourier transform method. So, now I am going to explain the concept of the back projection method and after this the Fourier transform method.

So, in this example I have shown here that is just I am doing the back projection and after the back projection the image the f x, y is reconstructed from g s theta. I can give another example how to do the back projection. So, in the next slide I am showing how to do the back projection.

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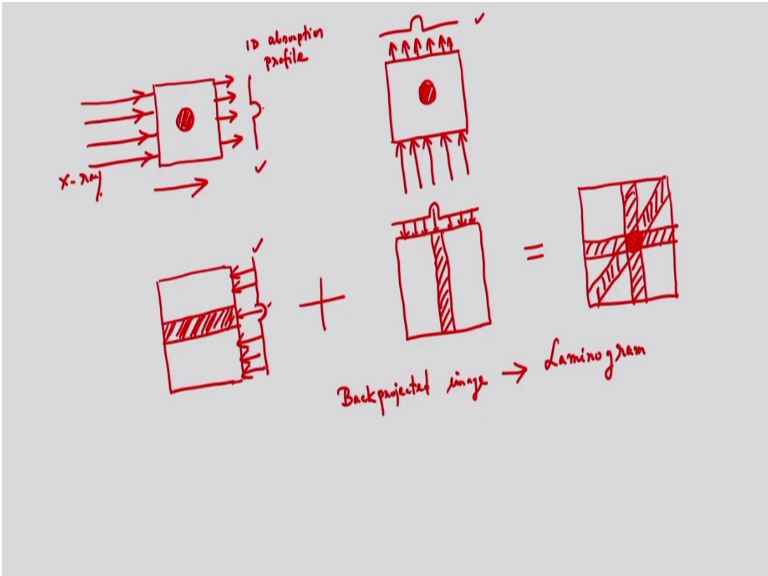

Suppose I have the object like this image suppose this is one internal structure here and suppose the x-ray is coming from this side these are the x-ray. So, corresponding to this I can determine the Ray Sum and because this object is present here I will be getting the 1D absorption profile something like this I will show you. So, I will be getting the profile something like this is the 1D absorption profile.

This actually amplitude depends on the absorption. So, in this case we have the maximum absorption here. So, corresponding to this absorption I have the profile so that is the projection in this direction I am doing the projection along this direction. Now I am considering another projection the angle will be different now. So, from this side I am doing the projection of the same scene this is the object, the same scene I am considering.

But in this case I am getting the projection in another direction. So, the projection will be something like this, this is the 1D projection because in the object this position the absorption will be maximum. So, corresponding to this I have the 1D absorption profile. This amplitude depends on the absorption. Now from this two projections, I have the two projections that is I am determining g s theta.

So, first projection is this and second projections is this. To reconstruct this object I have to do back projection. So, how to do the back projection? So, here you can see I have the profile this profile I am doing the back projection I am doing. So, in this case corresponding to this portion you see duplicating the same 1D signal across the image perpendicular to the direction of the beam that means this is repeated that is duplicating the same 1D signal.

The 1D signal is this duplicating the same 1D signal across the image perpendicular to the direction of the beam. Similarly, corresponding to the second one just I want to do the back projection. So, my 1D profile is this, this is the 1D profile now I am doing the back projection and corresponding to this the same procedure I am applying. So, I have this profile.

Now, if I combine this two the first one and the back projected image I will be getting this I will be getting. So, if you see intensity at this particular point it will be double as compared to this portion. So, at this portion I have the more information so this information I have the most information. Similarly, if I do the projection in another projection then again after this we can do the back projection.

Suppose, in this angle I am doing the projection and back projection then in this case also you will see the information the more information I will be getting at this point as compared to this points. So like this I can do the reconstruction that is nothing, but the 3D representation of the object. So in that case intensity will be twice that of the individual back projected image. This back projected image is called the Laminogram.

So, we have seen that how to do the back projection. So, first I am doing the projection in one direction and again I am doing the projection in another direction. And after this I am doing the back projection and after this I am combining. So, like this I can get the back projected image that means I am reconstructing the object the reconstruction is possible.

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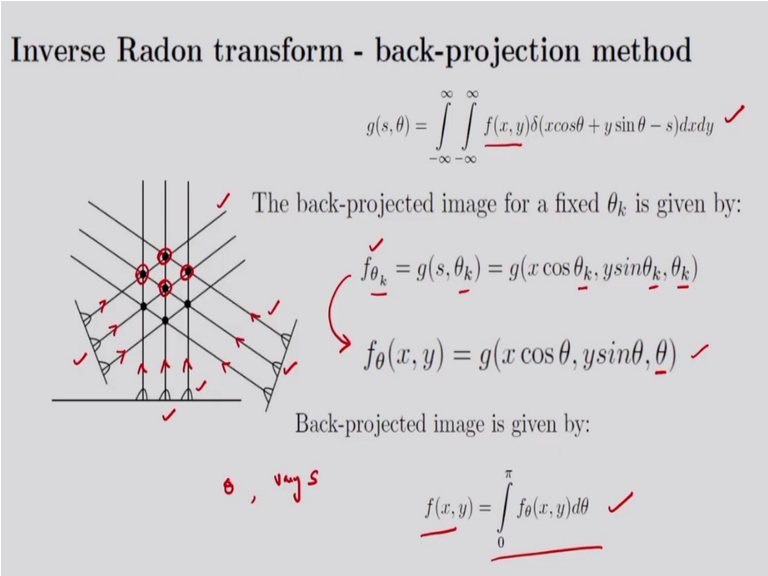

Now, I will discuss the concept of inverse Radon transform and in this case I will discuss the back projection method. So, what is the back projection method? So already physically I have explained what is the back projection? So this expression you know that is the Radon transform g s theta that is the projection g s theta corresponding to the object the object is f x, y.

Now, let us consider the back projected image for a particular angle that angle is theta k that is given by the theta k particular theta is fix corresponding to the theta k I am getting f theta k. So, in general I can write like this, this expression I can write like this f theta x, y d x cos theta comma y sin theta comma theta that means the theta is fix corresponding to theta I am getting f theta x, y.

So, in this case how to get a particular projection? So, I am explaining again the particular projection I am getting the theta is fixed and we are varying the variable s that means we simply sums the pixel of f x, y along the line defined by the specified value of this parameters. So, parameters are mainly s and theta so that means we have to first fix theta and we have to vary s so theta is fix and vary s that means we are summing the pixels of f x, y along the line defined by the specified value of these two parameters.

The parameters s and theta. So, I can increment the values of s that means this is to cover the image, but in this case I have to make theta fix. So, if I make theta fix then I will be getting one projection like this I will be getting number of projection. So, in this figure you can see so corresponding to this particular angle I am getting one projection corresponding to this angle I am getting one projection, corresponding to this angle I am getting one projection.

After this what I am doing I am just doing the back projection in this direction the back projection I am doing like this I am doing the back projection. So, if you see the intensity value at this point or this point if you see the information at this points so I will be getting maximum information because of the information from this, information from this, and information from this that means I can get 3D information that is 3D representation of the object.

So, ultimately the back projected image f x, y I can determine by using this expression because the theta varies from 0 to pi. So, for all the angles I am considering and I am determining f x, y.

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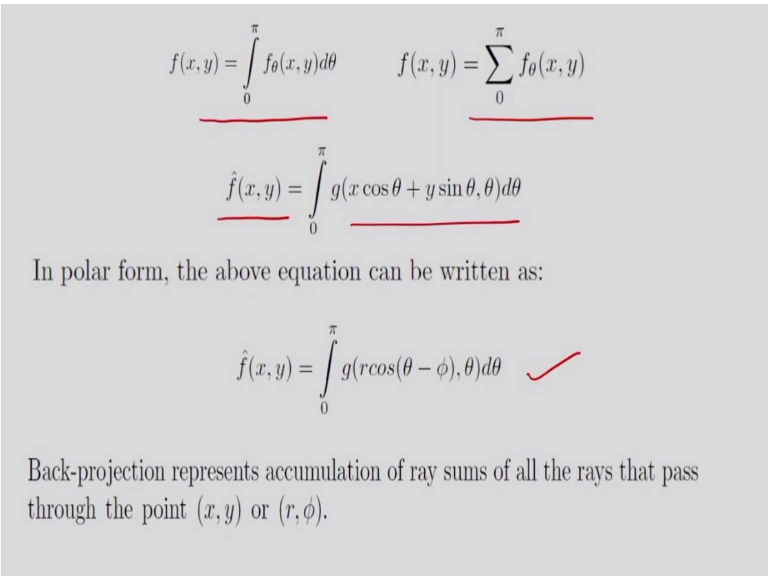

And already I have shown this one and if I consider the discrete value then instead of integration I have to consider summation. So, discrete values if I consider the theta 1, theta 2, theta 3 like this so that in this case instead of considering the integral I am considering the summation. So, if I consider only the discrete directions then approximate reconstruction this approximate reconstruction is given by this expression.

So, it is 0 to pi g s cos theta plus y sin theta comma theta d theta. So, in polar form also we can write this expression that is the approximate reconstructed value of f x, y. So, this concept is quite important that is the concept of back projection. So, how to get a particular projection already I have explained the projection you can get corresponding to a particular theta the theta is fixed and we vary s.

After varying s so that it will cover the entire image f x, y then corresponding to this I am getting one projection. After this I am changing theta and the same procedure is repeated and corresponding to this I am getting another projection so like this I will be getting number of projections. After this I will go for the back projection the back projection of the profile g s theta and from this I am getting f x, y. So, this is the concept of the back projection method by which we can do inverse Radon transform.

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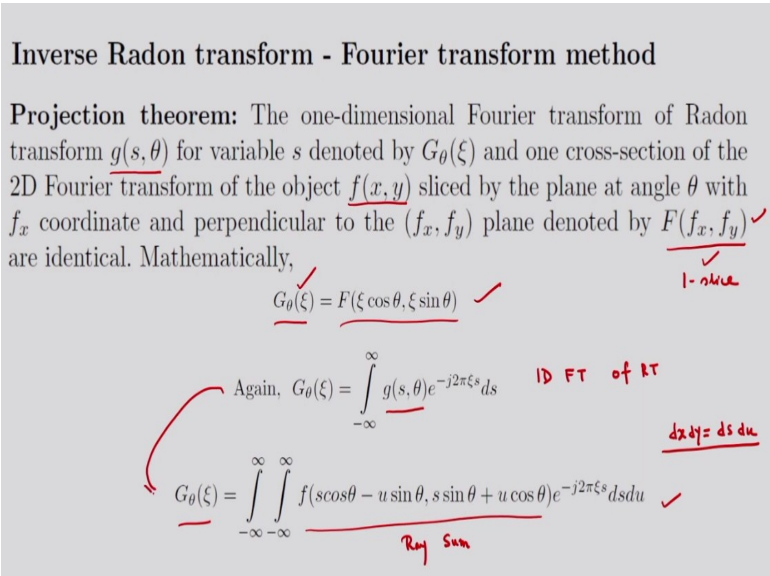

The next method is the Fourier transform method. So, inverse Radon transform by Fourier transform method. So what is the Fourier transform method here? The one dimensional Fourier transform of Radon transform that is g s theta and corresponding to g s theta, I can determine the Fourier transform that is the one dimensional Fourier transform I can determine and that is g theta epsilon.

And after this that is for the variable s denoted by g theta and one cross section of the 2D Fourier transform of the object. So, I can determine the 2D Fourier transform of the object f x, y sliced by the plane at an angle theta with f x coordinate f x is the frequency coordinate and perpendicular to the plane the plane is fx, fy that is denoted by this F fx minus fy. So this will be identical so this will be identical.

So I am considering the one slice of the 2D Fourier transform of the object. So, pictorially I am going to show this projection theorem so then you can understand what is the projection theorem. So, mainly for the time being you can see so I am determining g s theta and after this I am taking the 1D Fourier transform of g s theta that is 1D Fourier transform of Radon transform I am determining.

After this, that is defined by G theta 1 and after this I am determining the 2D Fourier transform of the object, the object is f x, y I am determining the 2D Fourier transform of the object the object is F fx, fy that is the 2D Fourier transform of the object. After this we are considering one slice of this Fourier transform, one slice I am considering.

The slice that is the slice by the plane at an angle theta with fx coordinate fx coordinate is the frequency coordinate and perpendicular to the plane fx, fy. So, I am considering one slice of this. So then G theta is equal to F that epsilon cos theta epsilon sin theta so this is identical. So, how to prove this one?

So, first what I have to do I have to determine the Fourier transform that is 1D Fourier transform of the Radon transform. So, first I have to determine this so this is my g s theta that is the Radon transform and this is the expression for the Fourier transform 1D Fourier transform of Radon transform I am determining.

After this, what I am considering you have the G theta this one G theta I have and in place of G s theta that I have the equation for the Ray Sum equation that already I have defined the Ray Sum equation I am just putting here the Ray Sum equation. So, just putting the Ray Sum equation in this expression then I will be getting this one and also I am considering dx dy is equal to ds du that already you know this. So, this condition I am considering so this is the Ray Sum equation already I have defined. So from this we are getting this.

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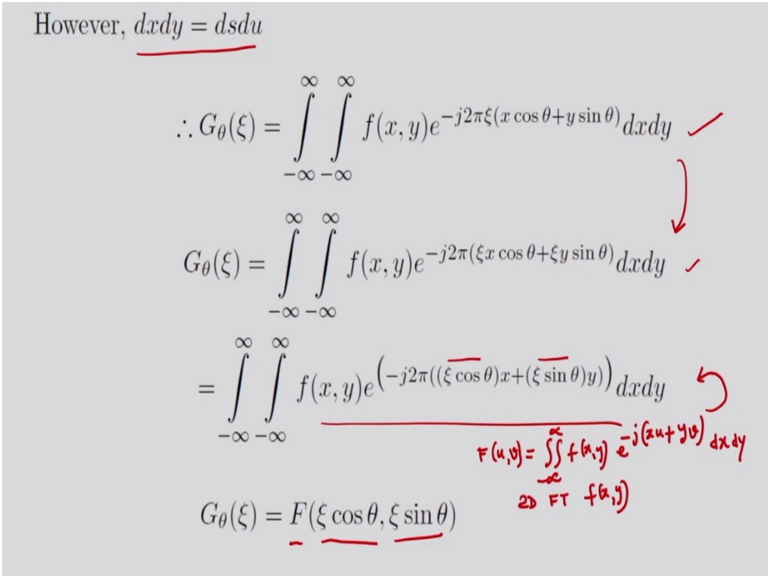

Now what is G theta? G theta is f x, y e to the power this expression I have this expression if you see the previous slide this s cos theta minus u sin theta that is nothing, but x and s sin theta plus u cos theta that is nothing, but y. So, corresponding to this I have f x, y e to the power minus j twice pi epsilon x cos theta plus y sin theta dxdy so I have this expression and after this expression I can write in this form.

You can see just minus j twice pi epsilon x cos theta epsilon y sin theta dxdy I will be getting this one and finally this can be written like this just combining this one. So, one portion I am combining like this another one is epsilon sin theta. So, this expression is very much similar to this expression if you see F u, v minus infinity to infinity f x, y e to the power e to the power minus j xu plus yv and dx dy this is v.

So, this is nothing, but the 2D Fourier transform of f x, y this is the expression for 2D Fourier transform of the f x, y. So, if I compare this is the 2D Fourier transform and this one you can see by comparison of this you will be getting F u, v what is F u, v what is u here? U is epsilon cos theta what is v? Epsilon sin theta so you are getting this one G theta epsilon is equal to this one. So that means the projection theorem is proved.

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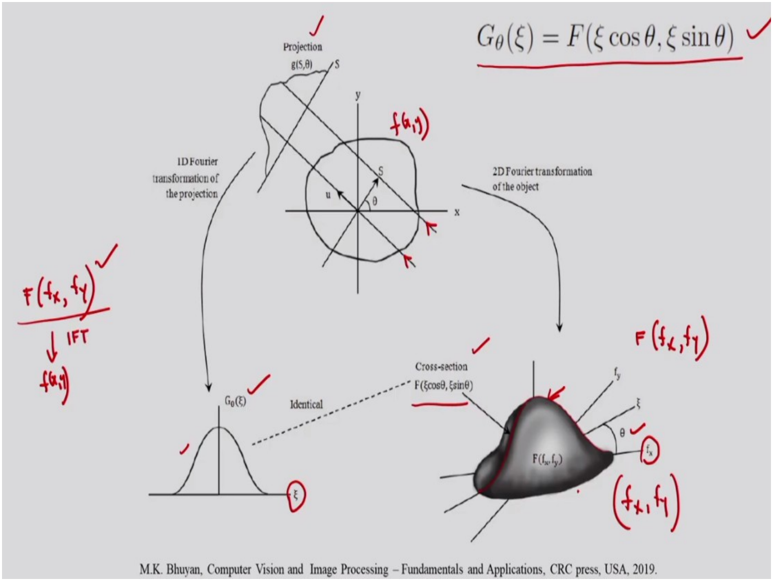

So, pictorially I want to explain this one. So, what is the meaning of this projection theorem? So, this is the projection theorem that is one dimensional Fourier transform of the Radon transform. So what is this? So, if you see here I am showing the projection here. The projection is g s theta. So, projection along this direction this is the projection I am considering.

I can determine 1D Fourier transform of g s theta so 1D Fourier of g theta is g theta I am getting corresponding to the frequency variable epsilon corresponding to this. So, first I am calculating the 1D Fourier transform of the Radon transform. Now suppose corresponding to this object, the object f x, y I am determining the 2D Fourier transform of the object. So the 2D Fourier transform of the object is F fx, fy.

So, this is the 2D Fourier transform of the object. Now I am considering the one cross section of this the cross section is I am considering this red line I am considering this is the cross section I am considering. The cross section of the 2D Fourier transform of the object I am considering F epsilon cos theta epsilon sin theta so one cross section I am considering.

This cross section is mainly the cross section of the 2D Fourier transform of the object slice by a plane the plane I am considering you can see by a plane at an angle theta with respect to the coordinate fx. And this plane is perpendicular to the plane the plane is fx, fy so I am getting one slice.

So, slice means the cross section of the 2D Fourier transform of the object sliced by a plane at an angle of theta with respect of fx axes and perpendicular to the plane a plane is fx, fy so that is the plane I am considering. So, this is the one cross section of this. This cross section and the 1D Fourier transform of the Radon transform that is identical.

So, if you see the expression that is the 1D Fourier transform of the Radon transform I am getting this profile and if I consider the one slide of the 2D Fourier transform of the object and I am taking one slice then it will be identical. So, projection at an angle theta generates one cross section of F fx, fy. So, the projection of theta at a particular angle theta it will generate the one cross section of fx, fy.

So, I am repeating this so projection at an angle theta generates one cross section of F fx, fy that is the Fourier transform of the original object. So, if I consider the projection for all the thetas so for all the angles they will generate the entire profile the profile is this profile F fx, fy and after this if I take the inverse Fourier transform of this the inverse Fourier transform of this that will give f x,y. So, if I take the inverse Fourier transform that will give f x,y that is the reconstructed f x, y.

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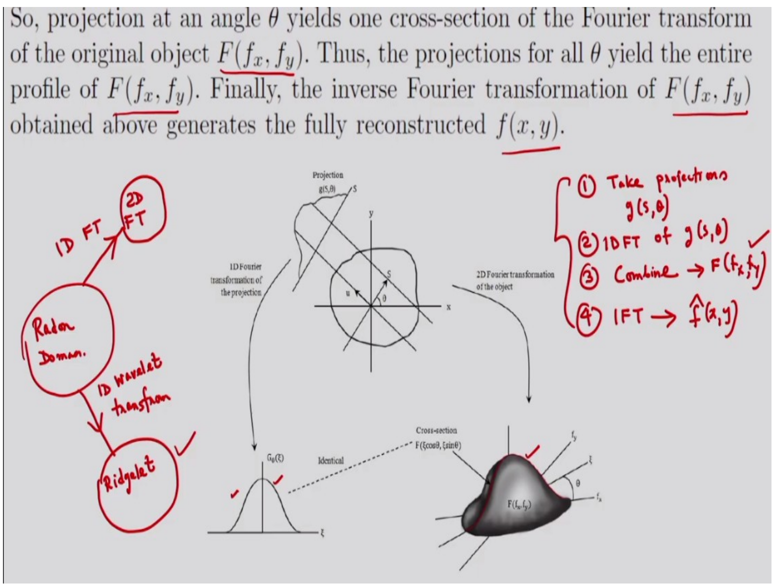

So, this projection theorem I have explained. I am again explaining this so projection at an particular angle theta generates one cross section of the Fourier transform. So one cross section of the original transform of the original object the original object is the Fourier transform of the original object is this and the projection for all the theta it will generates the entire profile of this, entire profile of this and finally I can determine the inverse Fourier transform of this Fourier transform to get the reconstructed f x, y.

So this procedure I can consider how to actually reconstruct f x, y. So, the steps I can consider the first step may be take projections. So, at different, different angles I can take the projections that means in this case I am considering g s theta. After this what I have to do, I have to determine the Fourier transform of the Radon transform, Fourier transform of g s theta that is the 1D Fourier transform.

So this Fourier transform is nothing, but the 1D Fourier transform of g s theta. So, 1D Fourier transform I am getting this profile so like this I have to consider all the 1D Fourier transform and I have to combine. After combining, I will be getting F fx, fy I will be getting that is number three step is this and number four I have to do inverse Fourier transform of this inverse Fourier transform that will give the approximate value of f x, y that will give the approximate value.

So, this steps this is the steps of the projection theorem that is by using the Fourier transform method we can determine the reconstructed f x, y and one thing is important so in Radon domain suppose that means if I take the 1D Fourier transform that means I will be getting 2D Fourier transform from 1D Fourier transform I can get the 2D Fourier transform because you can see this is the 1D Fourier transform.

So, if I combine all the 1D Fourier transform then I can get the 2D Fourier transform that is 2D Fourier transform is this and also I can determine 1D wavelet transform that wavelet transform I am going to discuss in my next classes. So, 1D wavelet transform and that means in this case this transformation is called a very important transformation the ridgelet transformation.

So, how to get the ridgelet transform? First, I have to determine the Radon transform and after this I have to consider 1D wavelet transform then in this case I will be getting the ridgelet transform. So one is the 2D Fourier transform another one is the ridgelet transform I will be getting.

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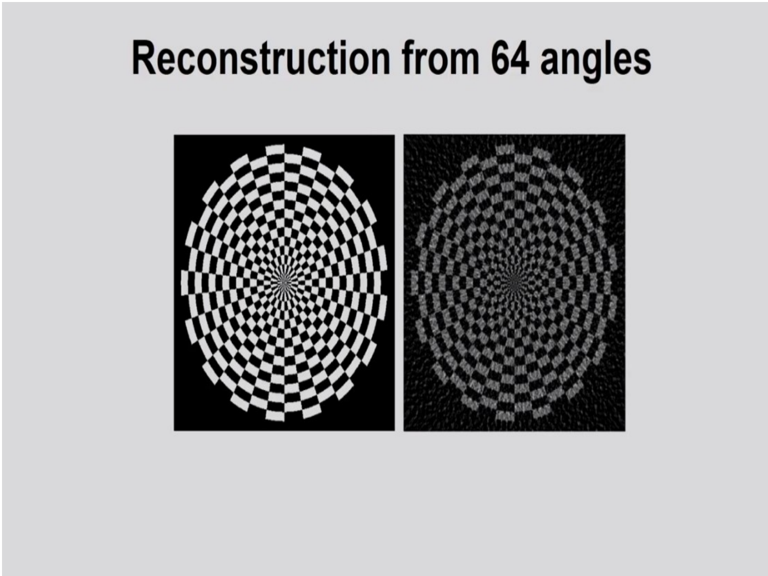

So, here I have shown some reconstruction by considering the back projection method. Reconstruction from 32 angles and another example I am considering reconstruction from 64 angles. So, up till now I discussed the concept of the Radon transform and also the inverse Radon transform. So, Radon transform is nothing, but the g s theta from g s theta I can reconstruction f x, y.

There are two methods I have explained the two approaches one is the back projection method and another one is the Fourier transform method. In the back projection method, I have already explained so how to get a particular projection you make theta fix and vary s the variable s and corresponding to this I will be getting one projection the projection is g s theta. Like this I will be getting number of projections at the particular time the theta is fixed and varying s and I will be getting one projection like this I will be getting number of projections.

So, from all the projections the g s theta I am doing the back projection and by using the back projection I can reconstruct the image the image if f x, y this is one popular technique that is image reconstruction by the back projection technique. The second technique is the Fourier transform technique. In the Fourier technique, what we have to consider 1D Fourier transform of the Radon transform I have to determine.

And also you can see the 2D Fourier transform of the object I can determine that is f x, y is available so from this I can determine the 2D Fourier transform. So, in this method in the Fourier transform method what I have to determine all the 1D Fourier transform of the Radon transforms. So, I have the g s theta because I have the number of projections. For all this projections, I can determine the 1D Fourier transform of the Radon transform I can determine.

After this I can combine all this 1D Fourier transform then I will be getting the 2D Fourier transform of the object after this I have to apply the inverse Fourier transform technique to reconstruct the original f x, y. So, this technique is called the Fourier transform technique. So these two techniques, the two approaches are very important the back projection approach another one is the Fourier transform approach.

So, I think we have covered the concept; the concept is the 3D representation of a particular object from number of projections. So, let me stop here today. Thank you.