Statistical Signal Processing
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Lecture 7
Linear Models of Random Signal

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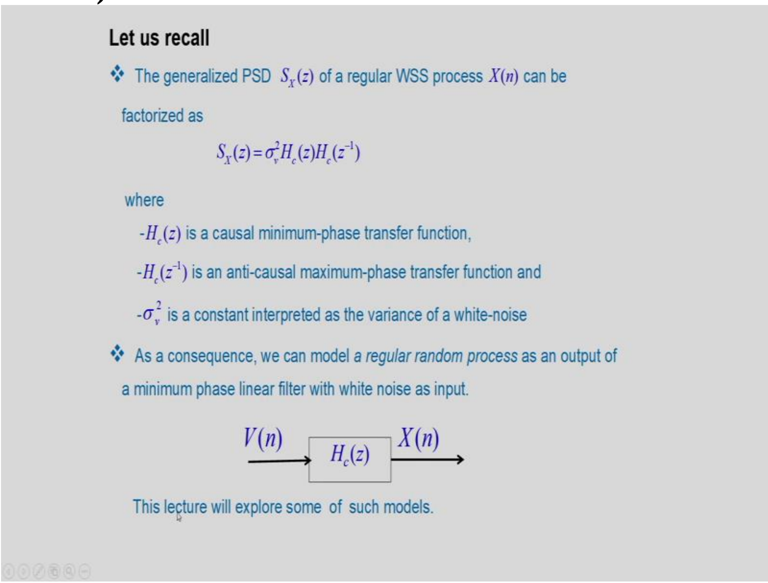

Hello students in this lecture I will talk about linear models of random signals. Let us recall the last lecture. These generalized PSD assets that have a regular WSS process Xn can be factorized as Sx z = sigma v squared into Hc z into Hc z inverse, where Hc z there is a causal minimum phase transfer function Hc z inverse is an anti-causal maximum phase transfer function. And sigma v squared is a constant which can be interpreted as the variance of a white noise.

As a consequence, we can model a regular enough process as an output of a minimum phase linear filter with white noise as input that is this is the white noise Vn appearance z squared this is input to a linear minimum phase filter of transfer function Hc z and output is the WSS process Xn this lecture will explore some of such models.

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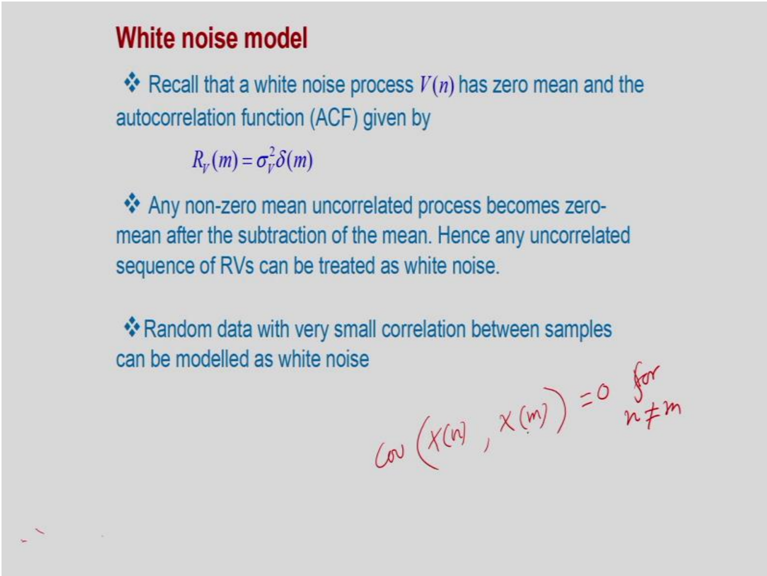

The most elementary model is the white noise recall that a white noise process Vn has zero mean so, there is no DC component it is has zero means and the autocorrelation function ACF is given by this Rv of m = sigma v squared del by m. So, delta m = 1 at m = 0 0 l3. Now about any non-zero mean uncorrelated process suppose, the processes non-zero mean, but it is uncorrelated in that case, the process becomes the domain after the subtraction of the mean.

So, therefore, any uncorrelated sequence of random variables also can be treated as white Noise because any non-zero mean uncorrelated process becomes zero mean after the subtraction of the mean and any uncorrelated sequence uncorrelated means that autocorrelation function is delta function that also after subtraction have been it will become zero mean they are pretty to become a white noise.

So, the definition of white noise include zero mean what any non-zero mean uncorrelated random process also can be modelled as white noise. Now, random data with very small correlation between samples can be modelled as white. That means, we have suppose Xn is 1 sample and Xm is another sample and if the covariance between them = 0 for m not equal to m then we can write this z m is white noise.

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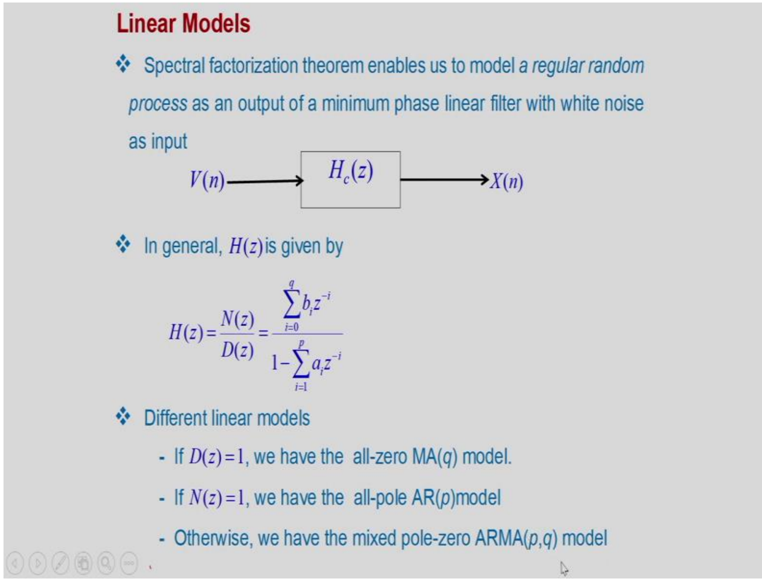

Now, let us go to linear models, spectral factorization theorem enable us to model a regular and random processed as an output of a minimum phase linear filter with white noise as input that is Vn is the input to a linear minimum phase system with function is Hc z and extended the output which is WSS process. In general, now, if we consider the transfer function of a core z LTI system minimum phase LTI system and it will have 1 numerator part and denominator part. So, it is a ratio of and that divided by Dz.

Now, this general part we can write this summation i going from 0 to q bi z to the power - i this is the numerical part. Similarly, denominator part we can write as 1 - summation i going from 1 to p ai z to the power - i. So, this is the general transfer function and depending on special situations will have different models. For example, if these are difficult to 1, if this is 1 then Hz will be simply that numerator part will be there.

In that case we have all 0 moving Ma q is model. So, this is a suppose, because there are q terms i equal to i going from 0 to q, therefore, this is a moving average of order q MA q model. When z = 1 that means only numerator term is there. When and z = 1 then only denominator parties there in that case, we will have poles. So, we have the all pole autoregressive this model is known as the autoregressive or integrated model of order p this is the p parameter.

So, other regressive model of order p and this is because it has only denominator polynomial, so, that will have poles So, this is an all pole model, otherwise the general model will be ARMA autoregressive moving average in terms of parameters p and q model ARMA pq model.

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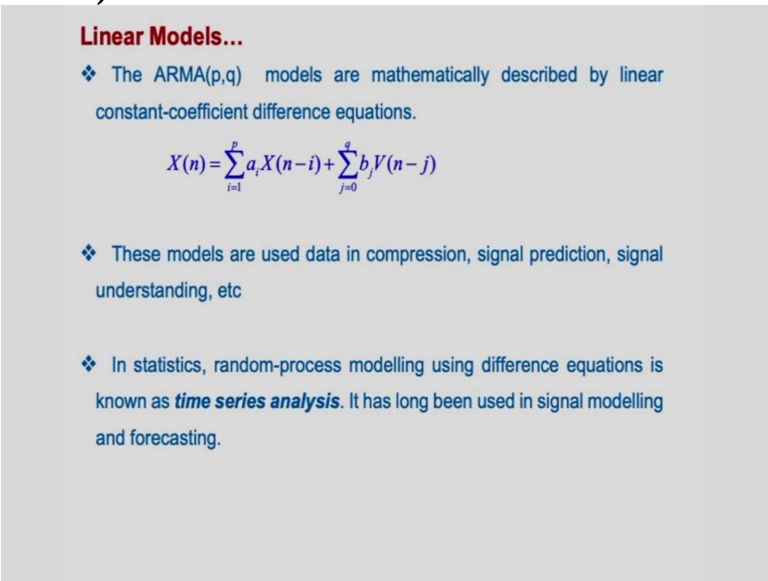

Now, in the time domain, this ARMA pq model can be described by linear constant coefficient difference equation. So, this is a different equation in terms of the difference we are getting the equation. Therefore Xn that is the output variable Xn = summation ai X of n – i, i going from 1 to P that is the autoregressive part + summation b j V of n - j are they going from 1 to q. So, this is the con linear constant coefficient differential equation model for the ARMA pq process.

P is the number of autoregressive part that mesh it tells about the auto regressive part and it tells about the moving awareness part. These models are very useful, they are used data compression signal predicts on 1 of the very important use of this type of model is forecasting or prediction, signal understanding etcetera in statistics random process modeling, using the Frank's equation is known as time series analysis in statistics for example, in an in a stochastic process stochastic finance for example, we have time series analyzes.

Time series analysis means, we analyze the models, the given by this type of different equations such models has long been used in signal modeling and forecasting. So, their main purpose was earlier it was that is signal modeling and forecasting, but nowadays, this models are used in many signal processing application like speech recognition, etcetera.

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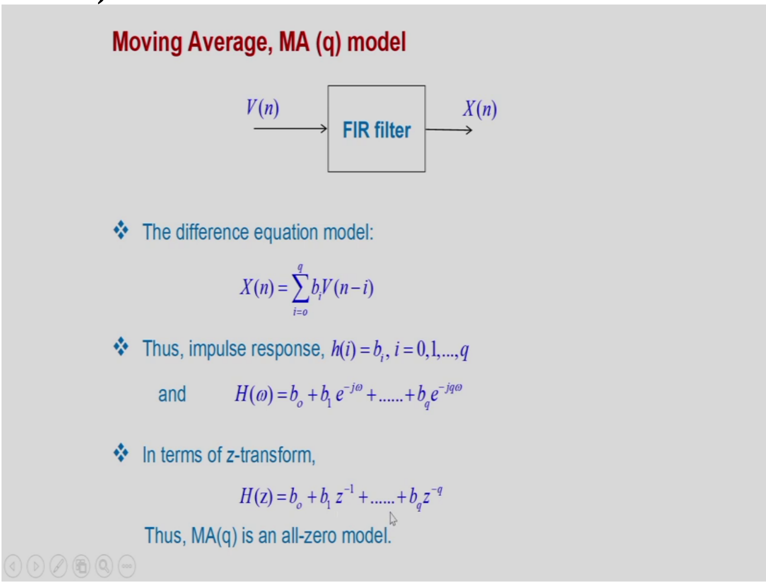

So, we will discuss part of moving average of order q. As I have told earlier, this is a FIR filter model. So, there is the white noise in the input and Xn and that is the WSS process, the output differential equation will be Xn = summation bi V n - i i going from 0 to q this is the MA q process. Thus impulse response hi = bi how many impulse responses are there are q + 1 i = 0 1 up to q. And in the transform domain frequency domain that frequency response h omega is given by b 0 + b1 into e to the power - j omega + b2 into e to the power – j2 omega and so on up to b q into e to the power - j q omega.

This is the frequency response of the FIR filter the present and make you process in terms of Hz transform will have Hz = b 0 + b 1 into z to the power - 1 + up to b q into z to the power - q. Thus MA q is an all 0 model because there is only that transfer function has only the numerator term. Therefore, it is an all 0 model.

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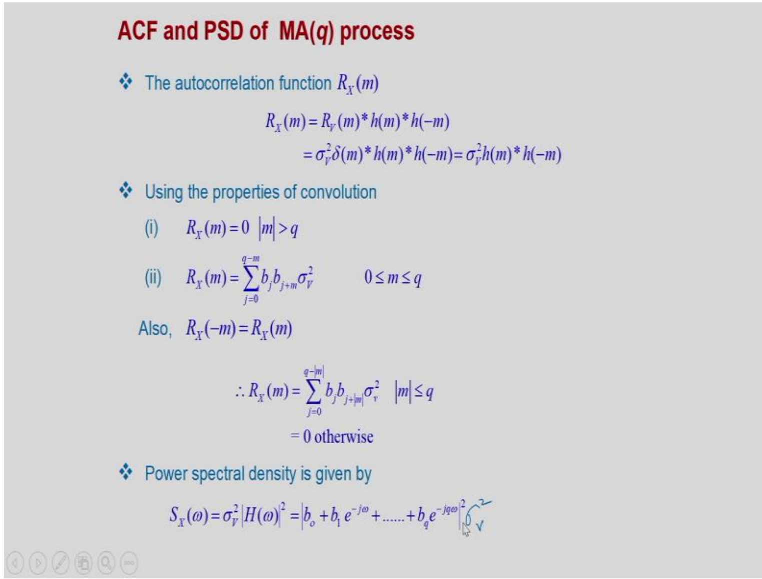

We know that if it is a double exercise process. It is characterized by autocorrelation function. And corresponding power spectral density. We will see what is the ACF autocorrelation function and PSD power spectral density of and MA q process the autocorrelation function Rx of m we know that Rx of m = Rv m convolved with hm convolved with h of - m that we derive this relationship while studying that linear system that responsible in linear system to WSS input.

So, they are we established that output autocorrelation function is the convolution of now, in put auto correlation function hm and h of - m hm is the impulse response of this system. So, now, I know that Rvm because it is the input is white noise Rvm is sigma v squared into delta m. So, therefore, Rx m will be sigma v squared delta m convolved with hm convolved with h of - m, but this is sigma Delta squared m any function consolve will get the same function, therefore the this will be sigma v squared hm convolved with h of - m.

Now, use this properties of convolution. So, you will know because we are to finite the resonant sequence we are convolving therefore, Rx of em will become 0 for mod m greater than q and I put substitute because I know the process RDB parameters. So, if we substitute those and this expression can be simplified to Rx of m = summation j = 0 to q - m of b j into b j + m into sigma b squared for m lying between 0 and q so, this is the expression for autocorrelation function.

And R x of - m = Rxm this is the basic property of autocorrelation function of WSS signal therefore, Rx of - m = Rx of m therefore generalize upward in this 2 expression we can write R x of m = summation j going from 0 to q - mod m of b j into b of j + model m into sigma v squared, where model m is less than equal to q = 0 otherwise this is the expression for the autocorrelation function away and make you process.

Now, the power spectral density PSD is given by Sx omega that is ACF of Sx omega = sigma v squared multiplied by mod H omega squared, that is, we established that relationship. Now, I know H of omega = b 0 + b 1 e to the power - j omega + up to b q e to the power - j omega q. So therefore, mod of this expression squared into sigma v squared. So this is the power expression density of a MA q process. So, we got the autocorrelation function given by this expression and power spectral density given by this expression.

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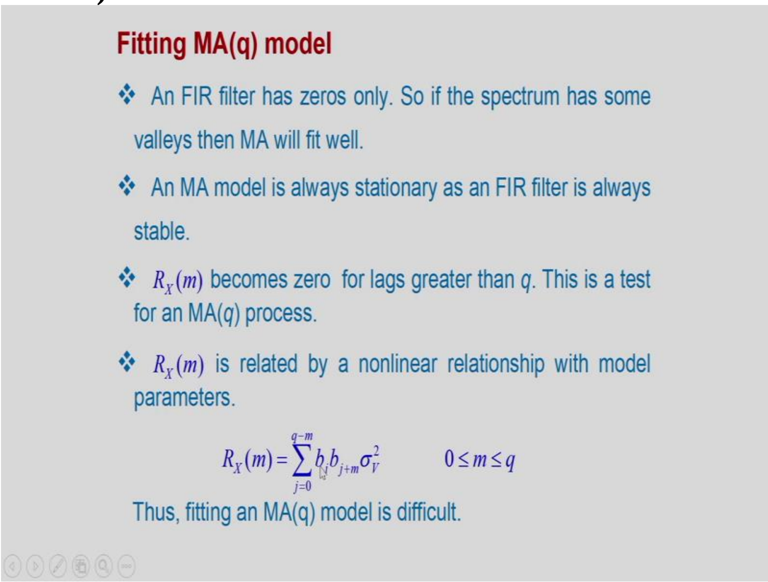

Now, how do I fit an MA q model an FIR Filter has zeros only. So, if the spectrum has some valleys then MA will fit well, because if we consider suppose FIR filter it will have 0 and those zeros will give rise to valleys this spectrum. I show an example later on and MA model is always stationary why because FIR filter is always stable and correspondingly, Hm will be also WSS because Hm is double and Hm is the output of this double system and therefore R to a white noise this output will be also stable and stable in the sense of random process it will be stationary.

So therefore, an MA model is always stationary WSS in this case, Rx of m becomes 0 for lags greater than q, this is a test for an MA q process suppose in an MA q process if we plot the autocorrelation function after some lag, this autocorrelation function will drop down to 0 and that point is the order of the an MA q process. Next point is Rx of m is related by a nonlinear relationships.

This is the trouble would be model parameters b parameters the relationship is like this Rx of m = summation b j b j + m sigma v squared j going from 0 to q - m. So, this is the relationship between autocorrelation function and the model parameters. Therefore, model parameters are related with autocorrelation function in a nonlinear manner, and finding out these coefficients by solving nonlinear equations, it went suppose that is noisy this will be a complex problem, therefore, fitting an MA q model is difficult.

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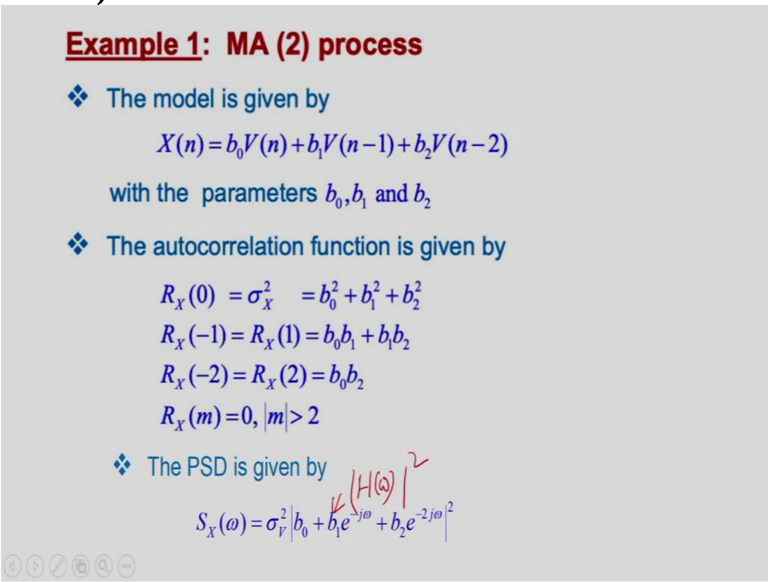

We will give an example, MA 2 process the model is given by Xn this is the WSS signal = b 0 into V n + b1 into V n - 1 + b 2 into V n - 2, therefore 2 is the below q. So 3 parameters are there b 0, b 1 and b 2, b n can be considered as a unit variance white noise because any variances is there can be considered in terms of b 0 you know the formula for autocorrelation function of a moment MA process.

So we will get R x of 0 = sigma x squared = b 0 squared + b 1 squared + b 2 squared R x of - 1, that will be also equal to Rx of 1 is equal to now it will put the formula R formula here. This formula, if we put that will get R x of 1 = b 0 into b 1 + b 1 into b 2 similarly, R x of - 2 = R x of 2 = b 0 into b 2 these are the autocorrelation nonzero autocorrelation values and R x of m will be equal to 0 for MA 2 the power spectral density will be given by sigma v squared into because this transfer function is b 0 + b 1 into e to the power - j omega + b 2 into e to the power – j 2 omega mod of debt whole squared.

So, that will be the Sx mod of omega squared, this quantity the mod of S omega squared so this part is the mod of S squared and therefore, this is the expression part a power expression density away moving average process of order 2.

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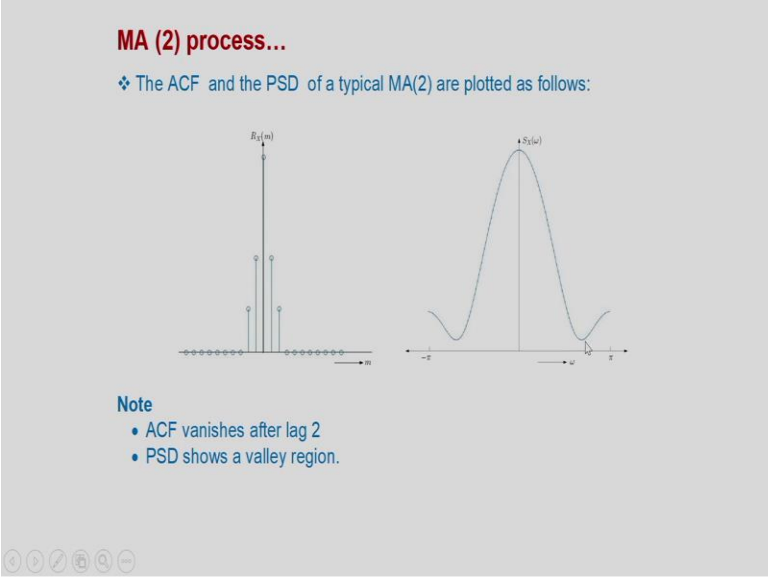

If we brought this autocorrelation function will get a plug like this auto correlation function after lag row this is 0 1 and 2 after that autocorrelation on sum have become 0 and if we consider the plot of Sx omega versus omega between - pi and + pi, because it is periodic with PSD to pi, so, uniquely defined only between - pi and pi and we see that there is a valley portion here. So, that is a characteristic of a moving average spectrum ACF when it says after lag 2 PSD shows a Valley region so, this type of if suppose your observe spectrum has this type of behavior then we can model it by an MA q process.

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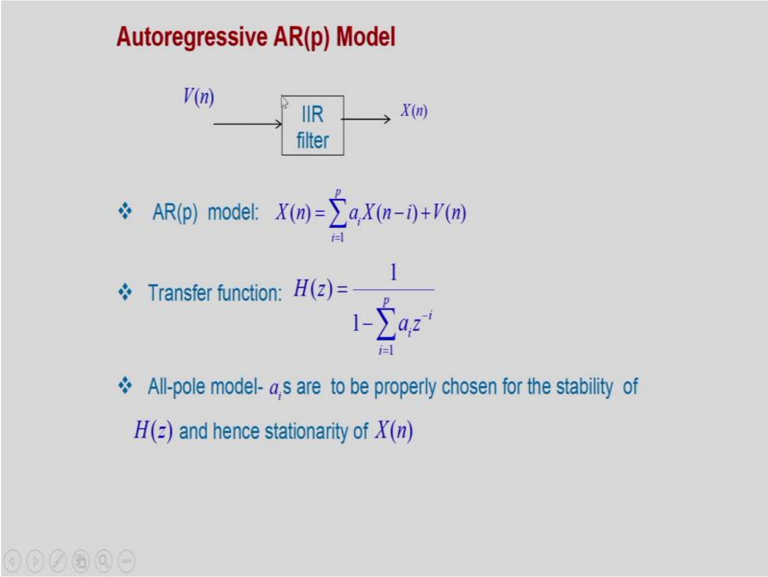

So, we discussed about moving average process and now let us see what is an autoregressive model autoregressive AR p autoregressive model of order p in this case the filter defining the random process is an IIR filter and all pulled filter that is important. So, this is the Vn is the white noise and Xn is the WSS process AR p model is therefore, Xn = summation ai x of n - i i going from 1 to p + V1. So, this present output suppose Xn is consider as a linear combination of past output + this is the input term that is white noise is the input term.

And corresponding transfer function is given by 1 by 1 - summation ai z to the power – i, i going from 1 to p. So, this is the transfer function of this IIR filter. Now, in this case it is an IIR filter, so, there are poles this expression will result in poles therefore, this is an all pole model ai s are to be properly chosen for the stability of Hz and hence stationarity of Xn if z is double then only this Xn sequence will be stationary. So, therefore, we have to select ai such that the poles of this filter lies inside the unit circle. So, we have discussed what is an AR p model?

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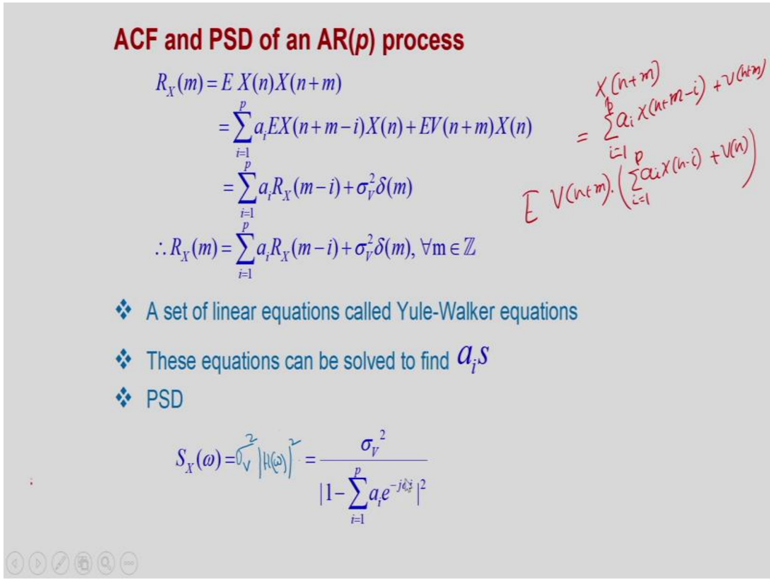

Now, let us see what is ACF autocorrelation function and PSD of an AR p process. We have to find the ACF autocorrelation function and the power spectral density PSD of an AR p process know that RX of m autocorrelation at lag m = e of Xn into X of n + m. Now, x of n + m can be considered in terms of its previous output that way, x of n + m we are writing is X of n + m = summation ai X of n + m - i + b of n + m i going from 1 to p.

So, this model we are putting here and then we are taking the expression inside. So, we will get summation i going from 1 to p of ai into e of x of n + m - i x of n + m - i into Xn this Xn + now, next is this term E of V of n + m into Xn. Now, this term, because extended WSS process therefore, this quantity will depend only on the lag n + m - i - n that will be simply equal to m - i. So, summation ai into Rx m - i i going from 1 to p. So, R x of m = summation ai R x of n - i i going from 1 to p + now, this quantity expected value of b of n + m into Xn that will give us these expression sigma v squared into delta m.

How because E of V of n + m into Xn we can write us suppose summation ai X of n – i, i going from 1 to p + Vn. Now, these are the past signals and this is the future input. So, they are uncorrelated because this exercise will depend on the previous inputs on late will not depend on these are input therefore, this and this expression will be uncorrelated and here E of V of n + m will be uncorrelated but there will be a finite value that various corresponding to m = 0.

Therefore, this expression E of V of n + m into Vn can be written as sigma v squared into delta m. Because when m = 0 there will be a value E of V of n + m that is your Vn into Vn, that way sigma v squared will come otherwise it will become 0. Therefore, we have now that Rx of m = summation i going from 1 to p ai Rx of m - i all this previous autocorrelation value + sigma V squared into del time this is the equation this is for all in this are values of m.

So, that we get a set of linear equations in terms of the autocorrelation function. This set of linear equation are known as Yule-Walker equations and they can be solved to find out the values of ai now how to find out the power spectral density Sx omega = sigma v squared into mod of H omega squared and depressed substitute a mod of H omega squared we will get sigma v squared divided by mod of 1 – summation ai e to the power – j omega i, i going from 1 to p whole squared mod of this quantity whole squared.

So, this is the power spectral density of the AR p process. So, this is the denominator term because it is an all pole filter these are the filter coefficients So, that we now will get this expression mod of 1 - summation ai e to the power – j omega i, i going from 1 to p whole squared so, we see what is the autocorrelation functions and how they are related with the model parameters and our power spectral density is related with the model parameters.

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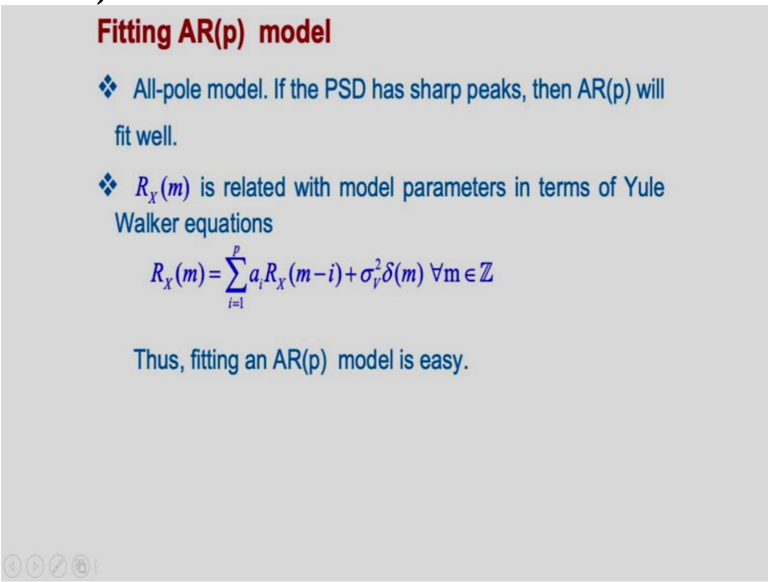

Fitting an AR p model, because AR p model is an all pole model, if PSD has topics, then AR p model will fit it well, Rx of m is related with model parameters in terms of Yule Walker equations. So that also we know this is a linear setup equation there for solving this set of equations is AR p thus fitting an AR p model is EV we will consider 1 example AR 1 process.

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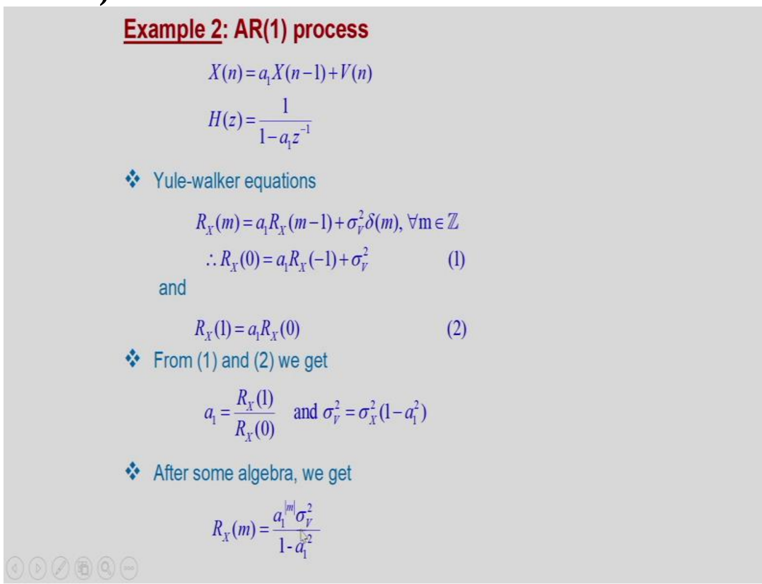

This dependency equation is Xn = a 1 X of n - 1 + Vn. And corresponding, Hz will be = 1 by 1 - a1 into z inverse. So this is the only 1 pole is there at z = a1 Yule walker equation if we substitute p = 1 in the Yule Walker equation will get the Yule Walker equation that is Rx of m will be = a1 into Rx of n - 1 + sigma v squared into delta, we know this expression Rx of m = this from this might p is 1 therefore, only 1 term will be there and this term will be there.

So, that way Rx of m = a1 times Rx of n - 1 + sigma v squared into delta m for all m belonging to z set of integers and Rx of 0 therefore, if we put m = 0 that will be = Rx of - 1 + sigma v squared and second equation Rx of 1 = a1 into Rx of 0. That is the when we put m = 1 when m = 0 we get this but we know that Rx - 1 is also = Rx of 1 from 1 and 2 will get a1 = Rx of 1 divided by Rx of 0 this is the value of AR coefficient a1 then a1 = Rx of 1 divided by Rx of 0.

And we can determine also see that relation, we have to find out what is the variance of white noise suppose that is sigma v squared will be equal to sigma x squared into 1 – ai squared. So this is 1 equation. This is another equation to unknown there a1 and sigma v squared. So a1 sigma v squared can be found out from Rx 1 and Rx 0. So we have found out what is a1 that = Rx of 1 divided by Rx of 0 and sigma v squared this sigma v squared = sigma x squared into 1 –

a1 squared. After now we have a differential equation, Rx of m = a1 Rx of m - 1 + sigma v squared into delta m this is the differential equation.

And I know what is Rx of 0 suppose it is related with sigma v squared, that the sum is Rx of 0 sigma v squared = R of 0 into 1 - a squared, therefore Rx of 0 = sigma v squared divided by 1 - a squared sigma v squared divided by 1 - a1 squared. And if I sold this set of differentiation with the initial condition, and we will find out this solution for the autocorrelation function. So, it will be exponentially decreasing function. So, we can plot this autocorrelation function.

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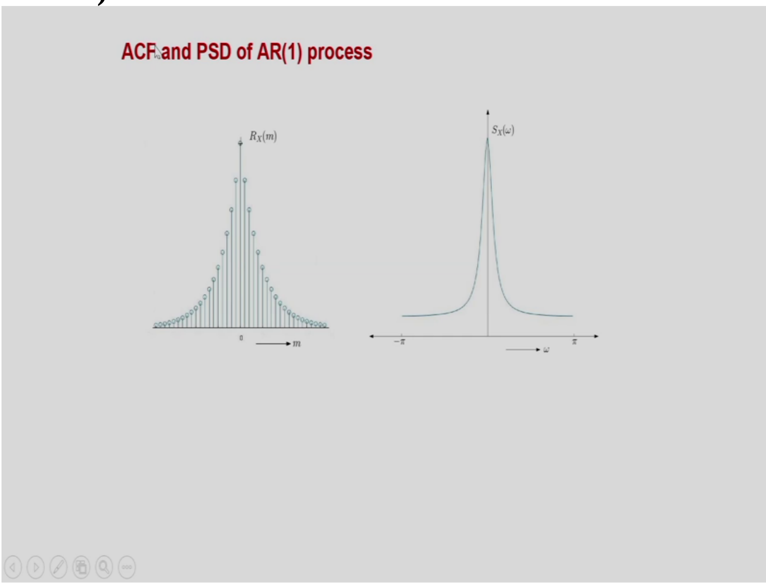

This is the plot of autocorrelation function for different will have m so, it will gradually going down and similarly this power spectral density in this case, pole is z at that is real axis pole is a real number. Therefore, it will have a sharp pickup originally 0 frequently on the this is the power spectral density plot of a typical AR 1 process and this is the typical autocorrelation plot.

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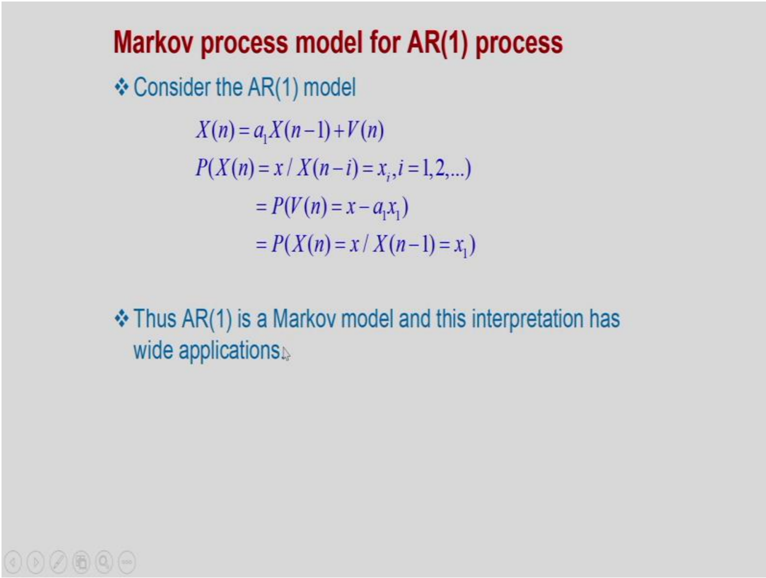

Now, 1 important interpretation of AR process as the Markov process consider the AR 1 process that is Xn = a1 x of n - 1 + V n then probability of X n = z given suppose X of n - i = x i, i = 1,2 etcetera this all the term given then I know that is probability of Xn = x that the same X Vn will be equal to we can add the probability of V n = x - a1 x1 because when I put x of n - 1 that will be = x 1.

So, if x of n - 1 = x1 x of n - 1 = x 1 then what will be Vn. Vn will be Xn = x therefore Vn will be = x – a1 x1. So, this conditional probability is same as probability of Vn = x – a1 x1 and the same result will get if you only consider 1 term probability of Xn = x given that x of n - 1 = x and then also will get the same expression. Therefore, this conditional probability given all the previous value is same as the conditional probability given the immediate past x of n - 1 = x 1. So, same model is known as the Markov model that AR 1 is a Markov model and this interpretation has wide application. Later on we will see.

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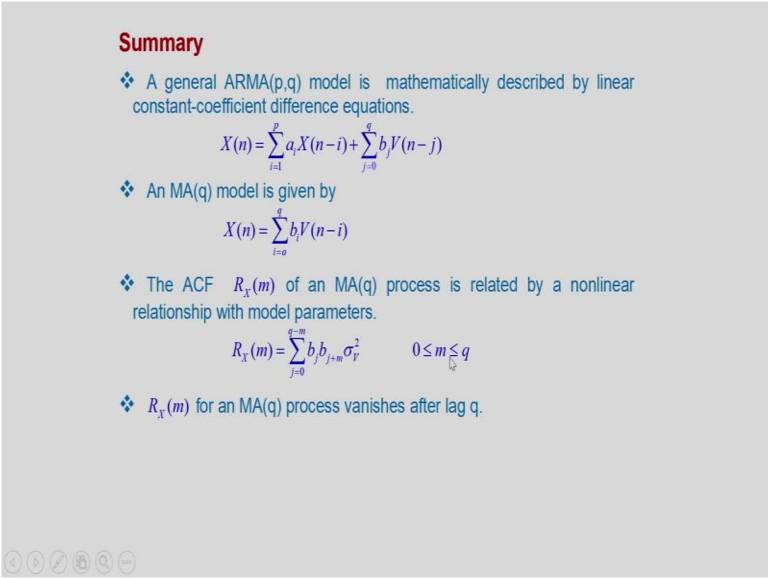

Let us summarize the lecture is general ARMA of pq model is mathematically described by a linear constant coefficient difference equation. So, Xn = summation i going from 1 to p of ai x of n - i + there we moving a various parts summation they are going from 1 to q of b j into V of n - j. This is generalize expression for ARMA of p, q model, which we derived from this spectral patriotism theorem. Then we considered moving MR 1 process which has only numerator part in the filter.

So an MA q model is given by Xn = summation i going from 0 to q bi into V of n - i this is the model for the MA q process moving a whereas process of order q the ACF autocorrelation function Rx m moving MA q process is related by a nonlinear relationship with the model parameters, what are the model parameters, this bi V are the model parameters, they are related with the autocorrelation function we can find out them from the autocorrelation function.

That is Rx of m is given by summation j going from 0 to q - m b j into b j + m into sigma v squared where m lies between 0 and q and Rx m for n and MA q process when i says after like q this is these 2 factors are important.

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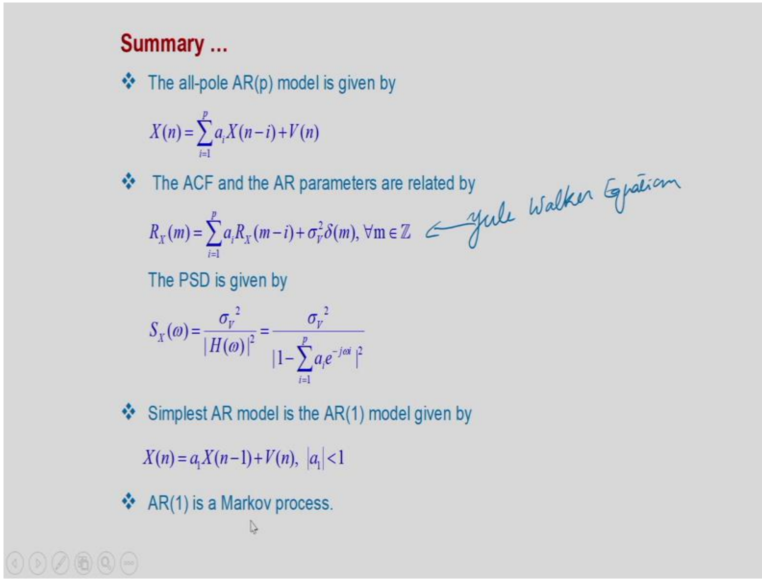

And this is a nonlinear relationship because of depth putting it putting in MA q model is difficult. The All-Pole model AR p is given by AR p autoregressive process of order p is given by Xn = summation i going from 1 to p ai into x of n - i + Vn. So, here i in the filter model only denominator part will be there that why it is an all-pole model the ACF and the AR parameters ai already AR parameters auto regression parameters this ACF and the AR parameters are related by the Yule Walker equation these are the Yule Walker equation.

Because for different values of m will get a set of equation with a different equation in terms of the autocorrelation function. So, that we will have a number of because then we can suppose there are p parameters are there on 1 more parameter is sigma v squared to solve them, we can consider p + 1 equation out of them and we can find out those value. And solving those equations are not difficult.

Because they are linear equations and the PSD is given by this Sx of omega = sigma v squared divided by mod of H omega squared that = sigma v squared divided by mod of 1 - summation i going from 1 to p ai into e to the power - j omega i whole squared simplest AR model we discussed that is AR 1 model which is given by Xn = a1 into x of n - 1 + Vn, where mod of a1 = is less than 1 that is this requirement for stationarity AR 1 is a Markov process that also we established. Thank you.