Statistical Signal Processing
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Lecture - 6
White Noise and Spectral Factorization Theorem

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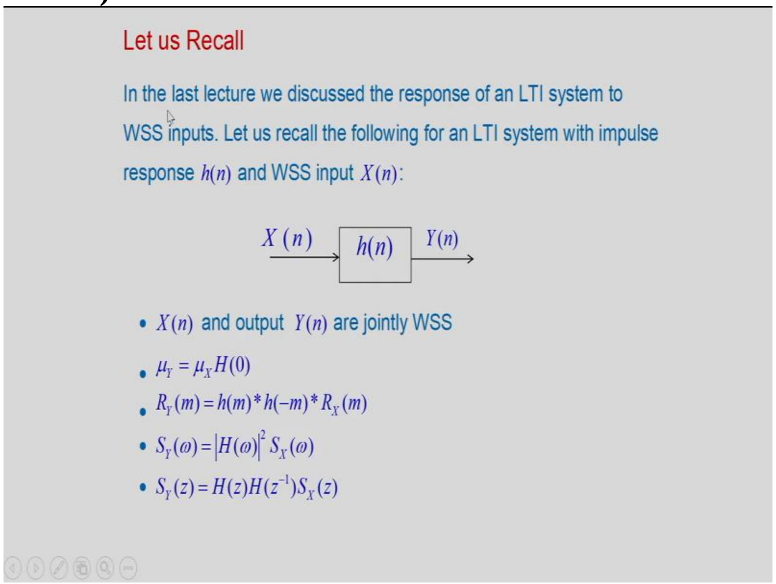

Hello students in this lecture, I will discuss white noise and spectral factorization theorem. Let us recall in the last lecture we discussed the response of an LTI system to WSS input, let us recall the following for an LTI system with impulse response hn and WSS input Xn. So this is the LTI system impulse response hn and input is a WSS process Xn and output its Yn. We established that Xn and output Yn are jointly WSS. Then the process Y has a mean mu y, which is equal to mu x multiplied by H0.

H0 is the frequency response at 0 frequency and the output autocorrelation function R ym is convolution of hm h - m and R x of m. So that way, R ym that it is a function of m only and Yn is a WSS process, the power spectrum of Yn that is S y omega is given by mod of H omega square into S x omega input power spectrum and output power spectrum are related by this transfer function this is power transfer function. In terms of z transform the generalize PSD S yz can be written as Hz Hz inverse into S xz these are the results we established in the last class.

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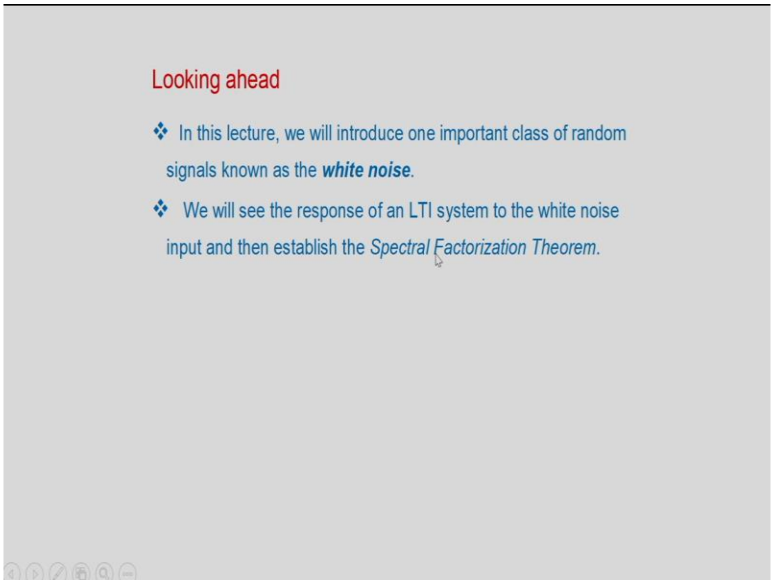

Looking ahead in this lecture will introduce one important class of random signals known as the white noise. So, this is very important building block for statistical signal processing. We will see the response of an LTI system to the white noise input and then establish this spectral factorization theorem. So we will first see what happens when a white noise is passed through or filtered by linear system and uploaded will establish the spectral factorization theorem.

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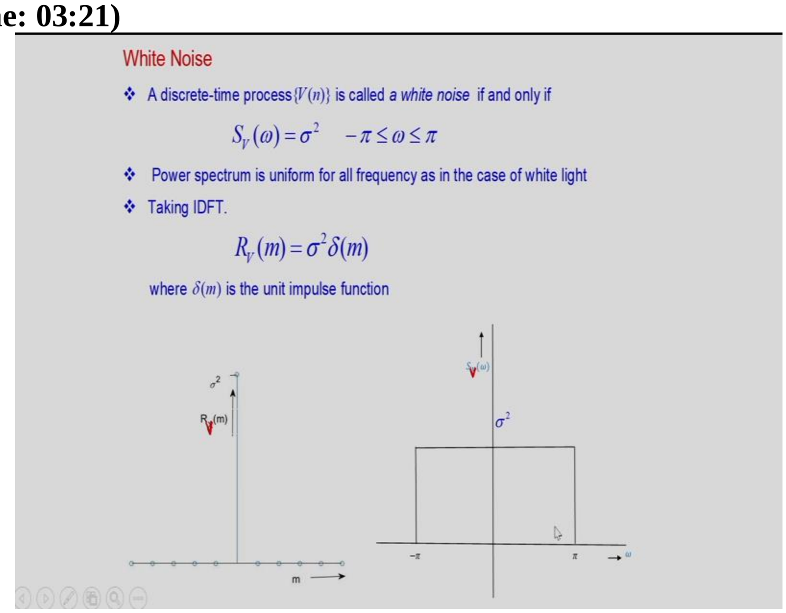

Let us define white noise a discrete time process Vn this is a random process Vn is called white noise if and only if S v omega is equal to sigma square omega line between minus pi and pi because discrete time Fourier transform is uniquely defined in a period. So, that way it is defined in the period minus pi to pi. So it is sigma square that is it is constant for all frequency power spectrums is uniform for all frequency. So, that is why it is called white because as in white light, all frequency components has uniform power.

So also in white noise all frequency component has uniform contribution to power. Now, if I take the IDFT of this expression, then I will get the autocorrelation function of Vn sequence R vm that is equal to sigma square into delta m, where delta m is the unit impulse function, delta m = 1 for m = 0 is 0 elsewhere. So, this is the spectrum of the white noise. This is uniform between minus pi and pi. And this one is the corresponding autocorrelation function. It has sigma square value at m = 0 and 0 elsewhere.

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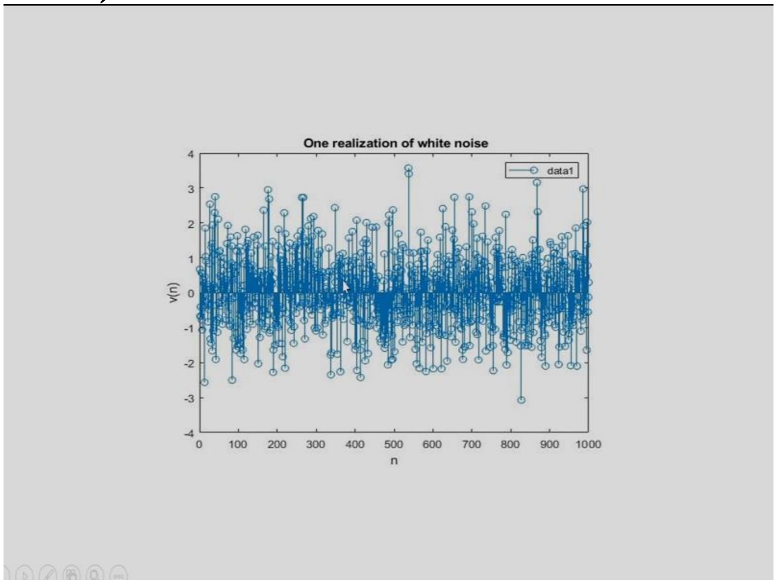

This is a realization of a white noise we see that these observations are pretty irregular and we cannot find a pattern in this realization.

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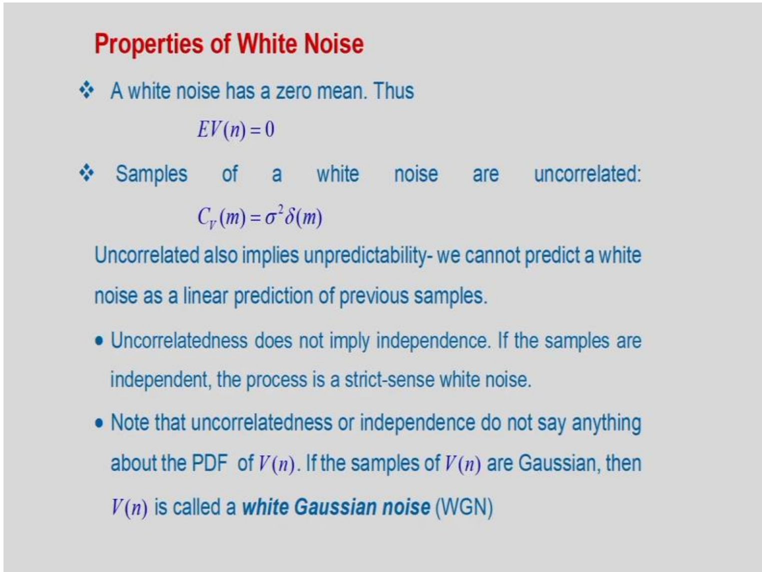

Property of white noise first of all a white noise has a 0 mean thus E of Vn = 0, this follows from the fact that power spectrum is the uniform. Therefore, at 0 frequency also it is uniform there is no spike at the 0 frequency therefore; mean must be equal to 0. Now, samples of a white noise are uncorrelated, why? Because I know that since means is 0 C vm will be equal to R vm. That is the covariance function is same autocorrelation function this is equal to R vm why because it is 0 mean.

Therefore, a white noise Vn is a sequence of uncorrelated random variables that is very important it is C vm is equal to sigma square delta m which is 0 for m not equal to 0 and therefore, samples of a white noise are uncorrelated, uncorrelated also implies unpredictability we cannot predict a white noise as a linear prediction of previous samples. Uncorrelated this does not imply independence only case of Gaussian random variables jointly Gaussian random variables uncorrelated imply independence.

If the samples are independent the process is called a strict sense white noise process. So, that is a very special case generally for a white noise in samples are uncorrelated. Note that uncorrelatedness or independence do not say anything about the PDF of Vn. So, a white noise can have any type of distribution if the samples of Vn are Gaussian in that case, Vn is called a white Gaussian noise. Vn may be a Bernoulli process in that case it will be a white Bernoulli noise.

So, therefore, what is white Gaussian noise that noise samples are uncorrelated and secondly, the samples follow Gaussian distribution? So, we have seen what a white noise. Now, let us apply a white noise to a linear time invariant system.

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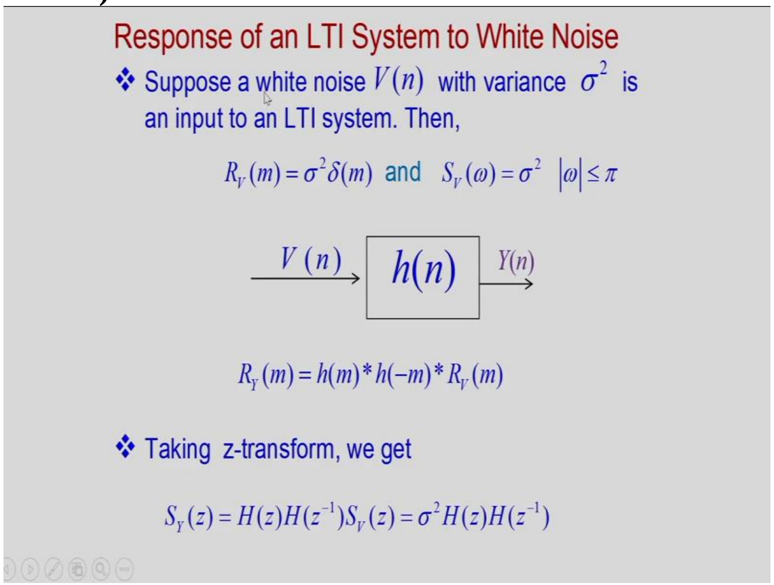

Suppose the white noise Vn with variance sigma square is an input to an LTI system then I know this is white noise R vm is equal to sigma square delta m and Sv omega is equal to sigma square for mod omega less than equal to pi. This is the input Vn is characterized by autocorrelation function and corresponding power spectral density. Now, considering the input output a relation for LTI system will get the autocorrelation function of the output R ym = hm convolve with hm - m convolve with R v of m and now we know what is R vm that is sigma square delta m.

Therefore, it will contribute only sigma square part taking z transform we get S y of z why here S y of z = Hz into H of z inverse into S v of z. Now I know that S v of z is equal to sigma square for all z therefore, it will be simply sigma square. So, therefore, S y of z will be sigma square into Hz into Hz inverse.

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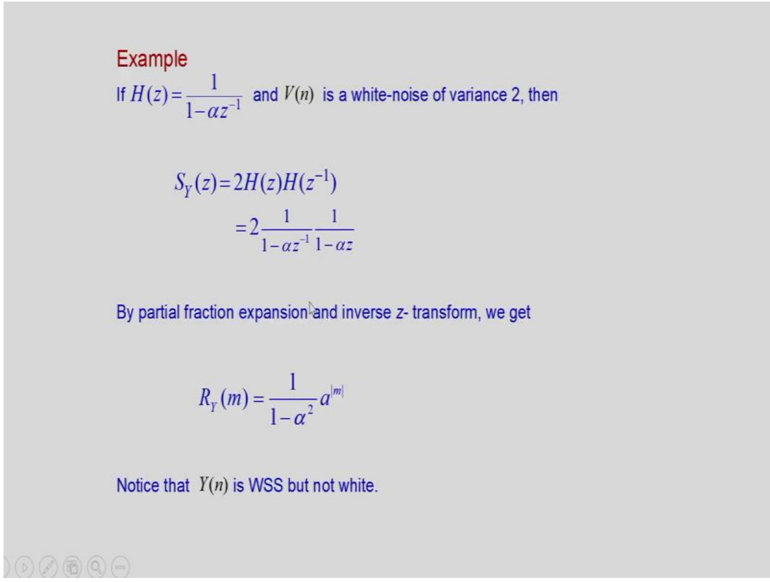

We can give an example, suppose Hz = 1 minus suppose H z = 1 / 1 - alpha inverse and Vn is a white noise of variance 2, then S yz will be 2 times sigma square is 2 into Hz into Hz inverse. So, that will be equal to 2 times 1 / 1 - alpha z inverse and z of inverse will be 1 - alpha z. So, that way, this will be 2 into 1 / 1 – alpha z inverse into 1 / 1 - alpha z. Now, you can do the partial fraction expansion and find the corresponding inverse z transform to get.

So, R ym = 1 / 1 - alpha square into alpha mod m and therefore, R ym = 1 / 1 - alpha square into alpha mod of m. Now, we see that R ym is nonzero for the values of m suppose m = 1 2 3 etc. Therefore, Yn is not white, but it is WSS because it is a function of m only, but it is not white. So that we are from a white noise after passing LTI system we got a WSS signal which is not white.

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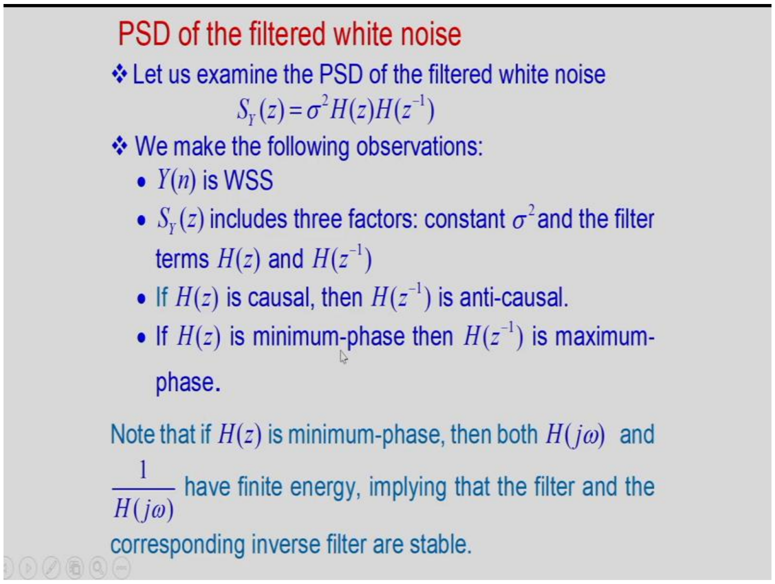

Let us examine the PSD of the filtered white noise. What do we have? S yz equal to sigma square into Hz into Hz inverse. So, it has 3 components. We make the following observation because we have as already said that autocorrelation function of Y is function of n only therefore Yn is WSS. S yz includes 3 factors constant sigma square and the filter terms Hz and Hz inverse. Now, if Hz is suppose causal then corresponding Hz inverse will be anti causal. Similarly, if Hz is minimum phase, then Hz inverse will be maximum phase.

So, for a minimum appear system for all poles and 0s are inside the unit circle and for a maximum phase systems for all poles and 0s are outside the unit circle note that if it is a minimum phase then both Hj omega and 1 / Hj omega have finite energy implying that the filter and the corresponding inverse filter are stable. So, that is the importance of minimum phase system. Not only filter is stable but its inverse filter is also stable.

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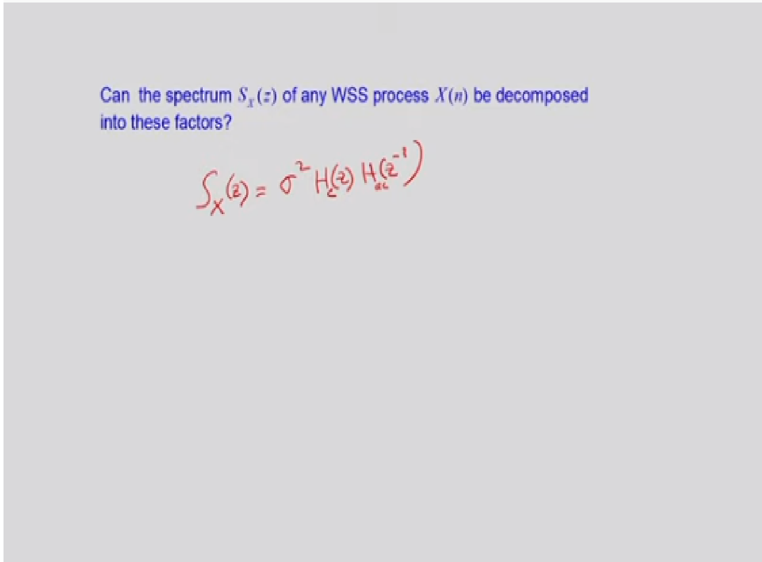

Now, can the spectrum S xz of any WSS process xn be decomposed into these factors that means for a general WSS signal suppose can you write a S xz is equal to some sigma square into Hz into Hz inverse can you write like this and will be preferring if one part is suppose causal then the other part will be anti causal. So, is it possible?

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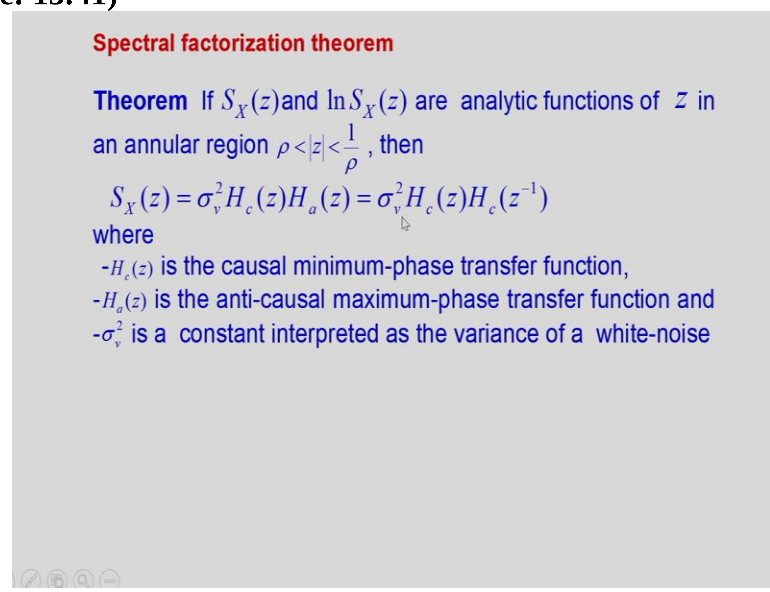

We will the spectrum factorization theorem? Now, if S xz and log of S xz are analytic function analytic function means they are derivatives exist. So if S xz and log of S xz are analytic function of z in an annular region that Rho region of convergence mod of z is greater than Rho and less than 1 / Rho then S xz can be written as sigma v square that constant is sigma v square into H cz that into H cz and this can be part a simplified as sigma v square into H cz that H cz that is nothing but H c of z inverse.

So, under the condition that S x of z and log of S xz are analytic within a region, then S xz can be written as a multiplication of key factors that is sigma v square H cz and H cz inverse. H cz is the causal minimum phase transfer function and H cz that is equal to H cz is inverse is the anti causal maximum phase transfer function and sigma v square is a constant interpreted as the variance of a white noise. So, this is this statement, what is the consequence of

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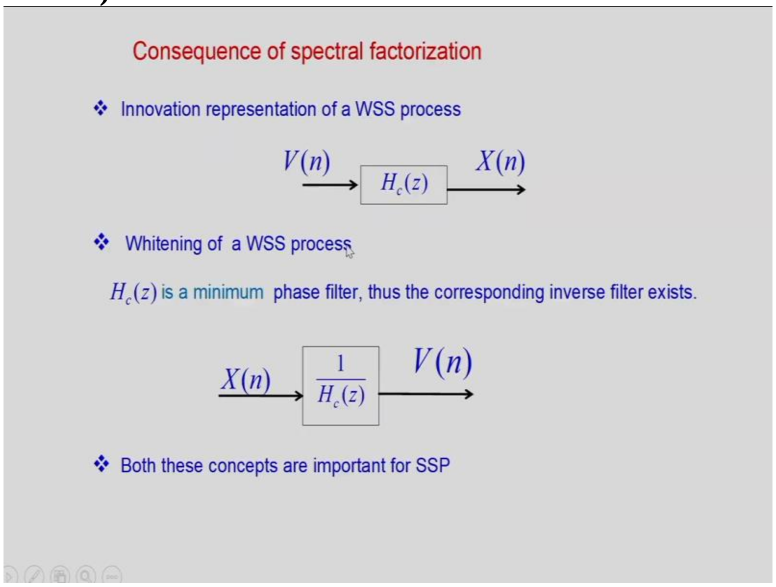

This because power spectral density S xz is sigma v square into H cz into H cz inverse therefore, we can write Xn as the output of a linear time invariant filter or system with input as the white noise Vn. So, because of this spectral factorization, we can write WSS process as the output of a white noise sequence. So, this representation is known as innovation representation of WSS.

So, given Xn this is the unknown thing or informative thing about the signal so that Vn is called an innovation representation of Xn. Now, this H cz that is invertible because of the minimum Facebook party. Therefore, given Xn we can get back Vn by means of Vn inverse filter that is known as the whitening of a WSS process. If H cz is a minimum phase filters thus corresponding inverse filter exist therefore, you can write suppose this Vn is the output of the filter which is transfer function 1 / H cz to an input Xn.

If Xn is passed to a filter which transfer function 1 / H cz that we will get back Vn. So, this is the whitening filter this filter is the whitening filter because Xn after passing to the filter it will become a white noise sequence. Now, both this concept that is innovation representation and whitening away WSS process are important in statistical signal processing.

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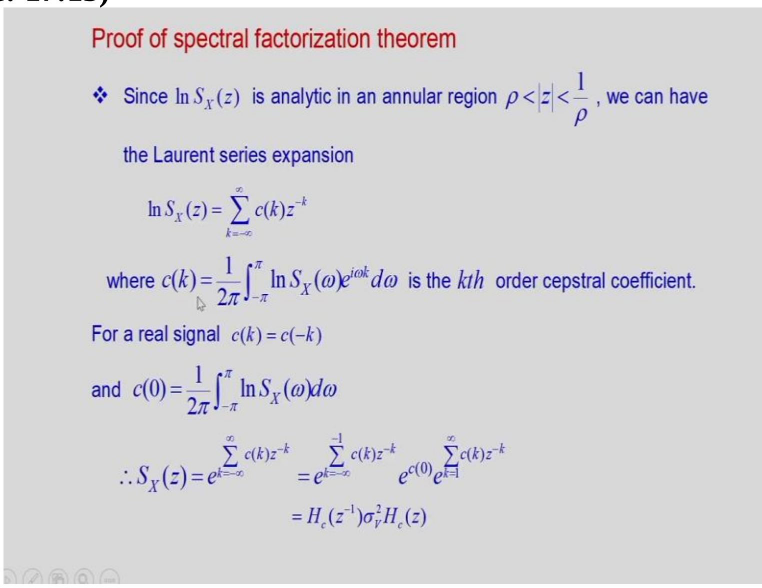

Let us prove the spectral factorization theorem. Since log of S xz is the analytic in an annular region mod of z lies between Rho and 1 / Rho. We can have Laurent series expansion. So, this requirement is for Laurent series expansion. So, we can write log of S xz is equal to summation it is an infinite series k going from minus infinity to infinity ck into the z to the power - k. Now, how do I get ck 1 / 2pi integration from minus pi to pi of log S x omega e to the power i omega k, d omega. So, this is the expression for ck. In fact, this is the Fourier series expansion.

So, from that we get the Fourier coefficient like this and this coefficient is known as the Kth order cepstral coefficient. It has important role in speech signal processing for a real signal because S x omega is equal to S x minus omega we will get that ck = c of - k log of S x omega. This is an even function because of that ck will be equal to c of - k and C0 is equal to if I put omega is equal to 0 and then this exponential term will be equal to 1 therefore, C0 will be equal to 1 / 2pi integration minus pi to pi log S x omega d omega.

And now this is the expansion for log on S xz. Now if I take the anti log we can find back S xz therefore, S xz can be written as e to the power this quantity that is summation k from minus infinity to infinity ck into z to the power - k this in infinite summation we can write into 3 parts, so, first part will be e to the power summation ck z to the power - k, k going from minus infinity to minus 1 that is the prospector and e to the power c0 this will be anywhere constant into e to the power summation k going from one to infinity of ck into that z - k.

So, we have expressed this entire summation into 3 parts and this part is a constant. This will call sigma v square and this part is a causal part because it is positive power of z. so that way it will be H cz and this part is H cz inverse of the anti causal part. Let us see how to find out H cz.

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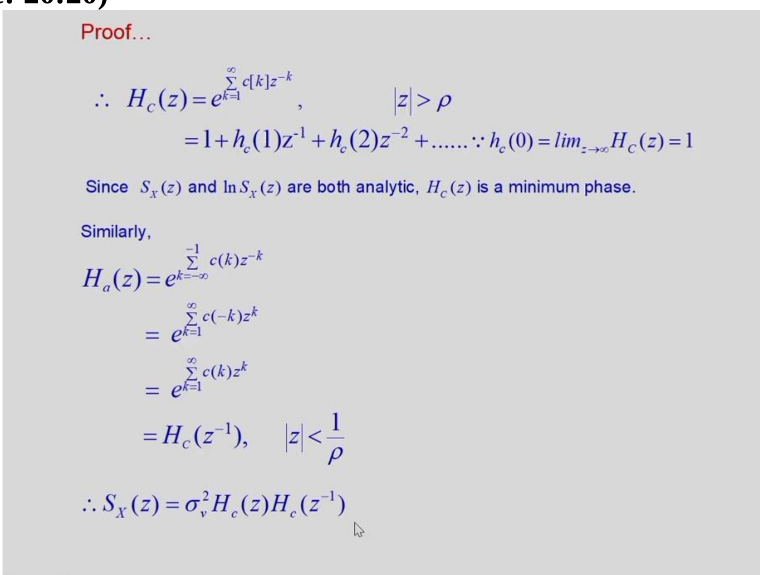

So, H cz by the definition now, we have written e to the power summation ck that 1 – k, k going from 1 to infinity. So, this we can now expand 1 + h c1 into z inverse because exponential part. So, first term will be there the inverse h c2 into z to the power - 2 like that and here h c0 = 1. How do I get h c0 is equal to limit of is the transfer function z tends to infinity in that case, this term will become z to the power minus infinity will become 0. So, it will be 0 this will be equal to 1 therefore first term of this expansion is equal to 1.

So, because both S xz and log of S xz are analytic, therefore, H cz must be a minimum phase filter. What does it means all its poles are inside the unit circle 0s are also inside unit circle. So, we have found out the causal part similarly; anti causal part will be because it is a negative exponent and k from minus and minus 1. So, that way this sequence must be anti causal sequence.

Therefore, H cz will be given by this. Now we know that c of k = c of - k we can write like this and improve substitute this is the expression H cz = e to the power summation k going from minus infinity to minus 1 into ck into z to the power - k and putting k suppose k = - k just we are expressing in terms of positive as you know exponent then this quantity will be c of - k into z to the power k and we know that c - k = ck.

That is from because z omega is even function improve that c of - k = ck. Therefore this quantity will be nothing but e to the power summation k going from 1 to infinity of ck into that z to the power k. So, this is the expansion for H a of z. So, this is same as this quantity H cz except that this z to the power in place of z to the power - k we have that z + k therefore, this quantity must be H c of z inverse.

So, only suppose in place of z we have z inverse. So, this is the now this is a because this powers of data positive so, this is the anti causal sequence and its transfer function is H cz inverse and Rho is mod of z is less than 1 / Rho. Therefore, what do we get S x of z that is power spectrum is equal to sigma v square into H cz into H cz inverse. So, that way we have proved this spectral factorization theorem that is S xz is equal to sigma v square multiplied by H cz into H cz inverse.

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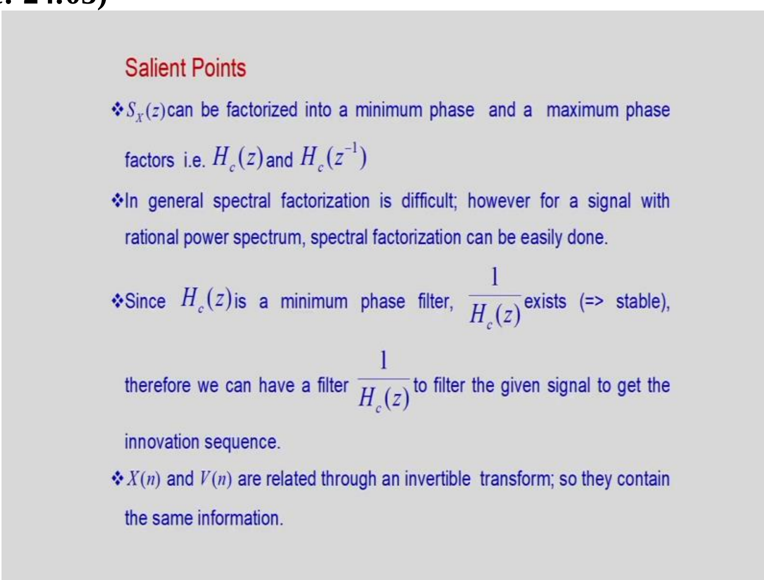

Salient points of spectral factorization theorem S xz that can be factored into a minimum phase and a maximum phase factor that is H cz and H cz inverse. In general spectral factorization is difficult however for a signal with rational power spectrum spectral factorization can be easily done. Since H cz is a minimum phase filter 1 / H cz exist means it is stable. Therefore, we can have a filter will have transfer function 1 / H cz to filter the given signal to get the original innovation sequence. Xn and Vn are related through an invertible transform. So they contain the same information.

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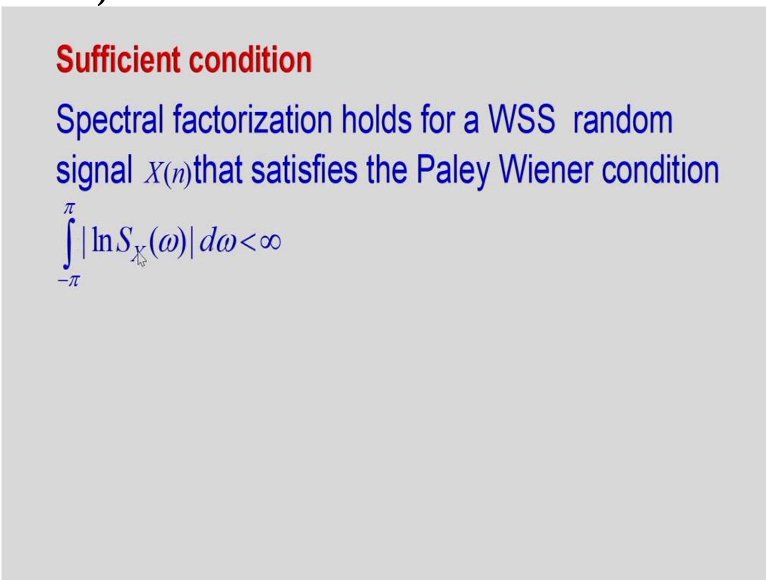

So, that we saw the necessary and sufficient condition, but there is a sufficient condition spectral factorization holds for a WSS random signal Xn that satisfies the Paley Wiener condition. What is that, integration of absolute value of S x omega d omega from minus pi to pi is less than infinity is finite, this if this condition is satisfied, then we can apply this spectral factorization theorem.

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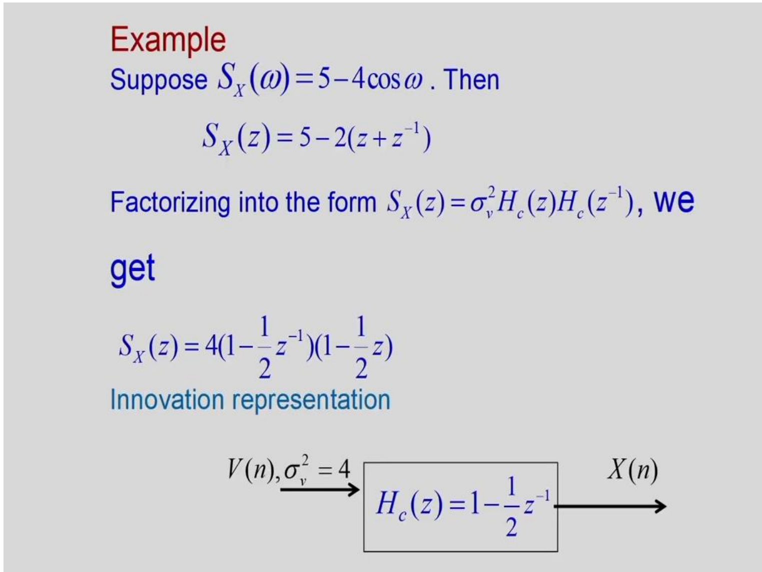

We will consider one example. Suppose S x omega is equal to 5 – 4 cos omega, then S xz will substitute twice cos omega is equal z+ z inverse. So, S xz this is the generalized S xz = 5 – 2 z + z inverse so this is the generalized power spectrum of the signal. Now we want to factorizing into, and that is the sigma v square H cz into H cz inverse in this form we have to do it and then we will find this we can factorize by ordinary method of factorization and we get a S xz = 4 into 1 minus half z inverse into 1 minus half z.

This is the causal part this is the anti causal part. Therefore, we have the innovation representation. This is Vn input its variance sigma v square. When it is passed through the filter, that is transfer function 1 minus half z of that inverse, we get excellent. This is the innovation representation of the signal was power spectrum is given by this expression, similarly, we can get the corresponding whitening operation also that whitening filter output will be given by suppose, we can have suppose whitening how do I have whitening.

So will have Xn and it will pass through a filter that 1 / H cz that is equal to 1 / 1 - half of z inverse and then we will get Vn. So, we can get Vn from Xn. So that way we see the spectral factorization theorem and how we can apply this to factorize some power or some power spectrum like this spectral factorization is not possible for all WSS signal.

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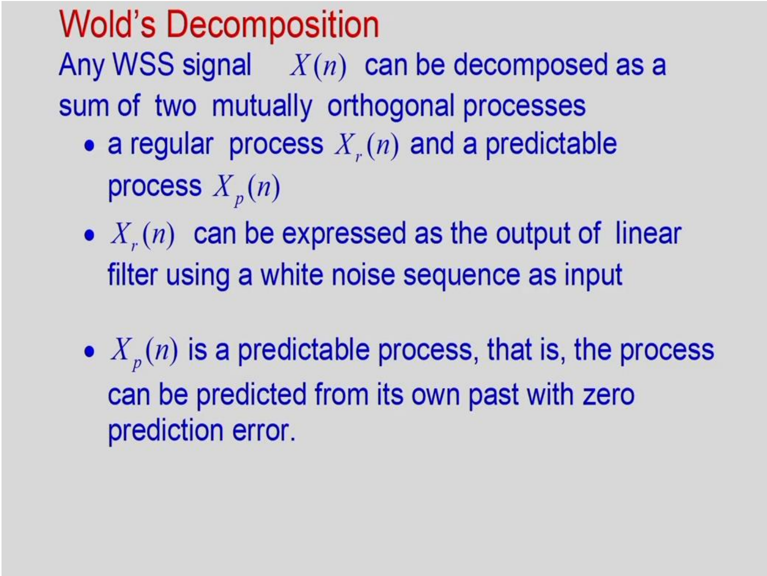

So, prove depth that one important result is there that is world's decomposition will not prove depth any WSS signal Xn can be decomposed as a sum of 2 mutually orthogonal process that with Xn can be expressed as a sum of 2 mutually orthogonal process one is known as the regular process X rn where you can apply spectral factorization and the other one is predictable process X pn both are orthogonal Europe X rn into X pn will be equal to 0. X rn can be expressed as the output of a linear filter using a white noise sequence as input.

So, because factorization theorem is valid indicates of X rn. So, X rn can be expressed as the output of linear filter linear time invariant filter in a white noise sequence as the input. Now X pn is a predictable process for example, it is a harmonic process that is the process can we predict from its own past with 0 prediction error. For example, in the case of a science of course, if I properly pass if we get page amplitude then we can predict this process without any error.

So, therefore, X pn is predictable process that is the processes that can be predicted from its own pass with 0 prediction you know, and basically spectral factorization holds for x rn the suppose a random process for whose spectral factorization holds is a regular random process.

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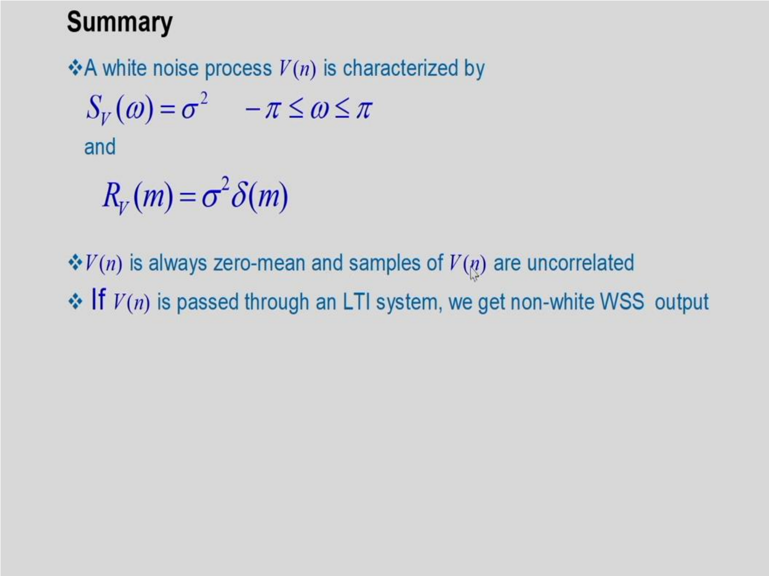

Let us summarize we defined it a white noise process Vn is characterized by its power spectral density which is uniform it is equal to sigma square for omega lying between minus pi and pi and corresponding auto correlation function is equal to sigma square del m sigma square delta m. Delta m is unit impulse function. So, Vn is always 0 mean because of its power spectrum. Vn is already 0 mean and samples Vn are uncorrelated.

Because 0 mean therefore, in this expression we can replace R vm by C vm therefore, that covariance will be 0 for m not equal to 0 and therefore samples Vn are uncorrelated. If Vn is passed through an LTI system, we get a nonwhite WSS input. In the general case we get a nonwhite WSS.

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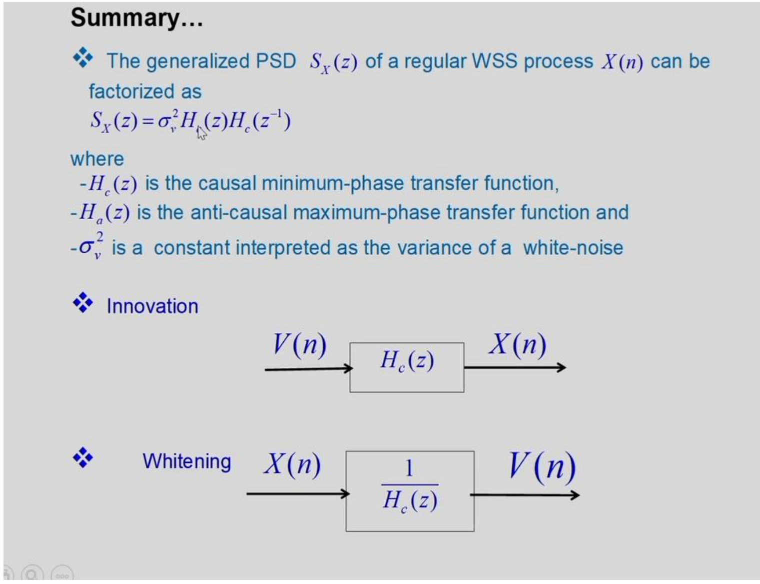

The generalized PSD S xz of a regular WSS process xn can be factored as that we sought that S xz can be sigma v square. This is the variance of the white noise process into H cz. That is the causal filter H cz inverse general decision causal filter. So, H cz is casual minimum phase transfer function that is minimum phase important, H cz is an anti causal maximum phase transport function and sigma v square is the constant interpreted.

As the variance of the white noise from this spectral factorization and we have the innovation representation of WSS random process or regular WSS random process, where Xn is obtained as the output of a filter of transfer function. H cz that with the innovation white noise being at the input that is the innovation representation of a random signal. Similarly, we can whiten a random signal Xn suppose if we pass through the inverse filter, 1 / H cz we will get Vn this is the whitening of Xn to Vn thank you.