Statistical Signal Processing
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Lecture-5
Linear Shift Invariant Systems with Random Inputs

Hello students, in the last lecture, we discussed the basics of random process we showed that is wide sense stationary random process WSS random process can be characterized in terms of its mean mu x and the autocorrelation function R x of m. We also showed the importance of power spectral density and how power spectral density is related with autocorrelation function. In this lecture, we discussed about linear invariant systems with random inputs. Why a linear model?

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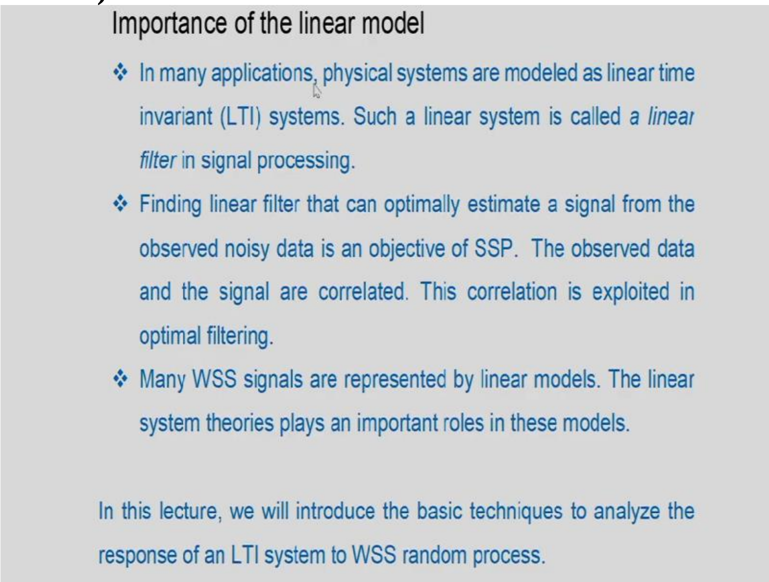

In many applications, physical systems are modeled as linear time invariant systems, we will discuss what is LTI system such a linear system is called a linear filter in signal processing. So, filter and system they are used interchangeably in this course. Finding a linear filter that can optimally estimate a signal from the observed noisy data is an, objective of SSP the observed data and the signal are correlated that is one important observation that observed data and the signal are correlated.

This correlation is exploited in optimal filtering so one application of linear filter is optimal filtering, which exploit the correlation between observed data and the unknown signal. Many WSS signals are represented by linear models. We will discuss about those linear models. The linear system theories play an important role in these models. In this lecture will introduce the basic techniques to analyze the response of an LTI system to WSS random process as an input.

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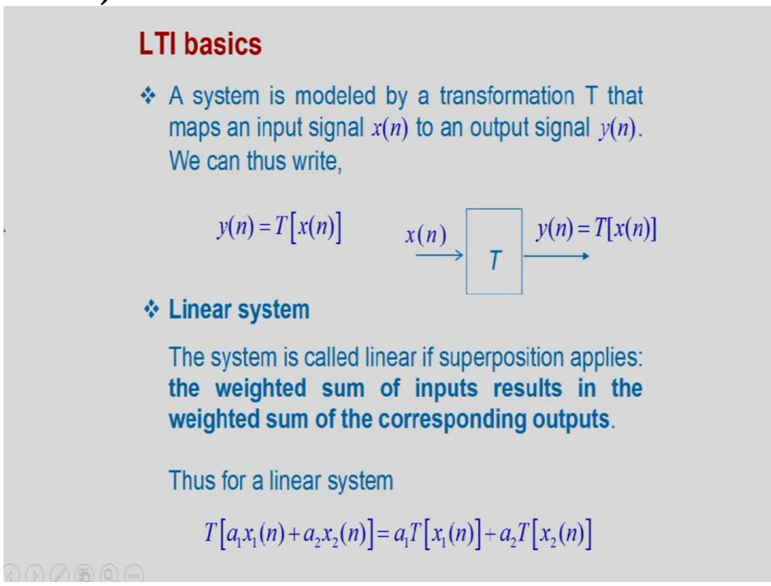

First, I will introduce the LTI basics. A system is modeled by transformation T that maps and input signal xn to an output signal yn. And we write yn = T xn .Thus, if xn is the input the system is modeled by a transformation T and yn is equal to T xn output. So, we know what is a system? System is basically modeled by transform. Now, linear system this system is called linear if superposition principle applies, what does it means?

The weighted sum of inputs result in the weighted sum of the corresponding outputs. So, in terms of transformation this is the weighted sum Ta 1 x 1n+ a 2 x 2n is same as a 1 Tx 1n + a 2 Tx 2n and so, this is the weighted sum of the output a 1 times output due to X 1n + a 2 times output due to x 2n.

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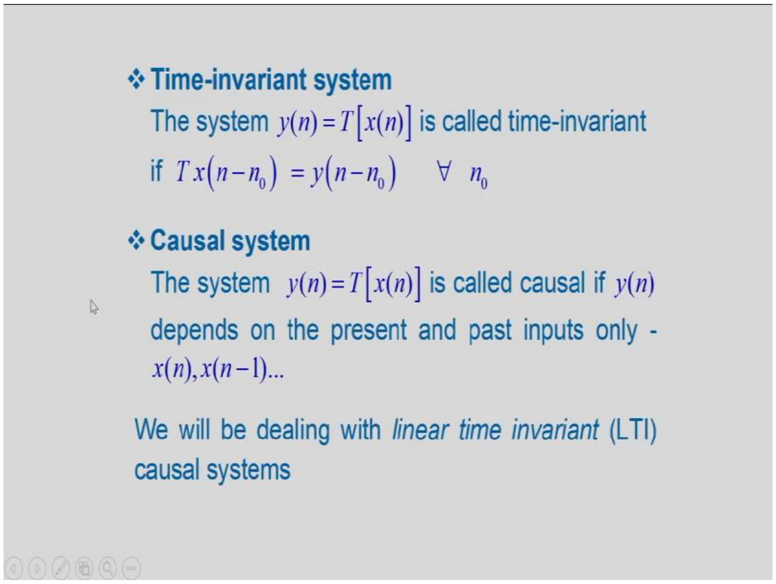

So, we saw what is a linear system. Now, time invariant system, this system yn = T of xn is called time invariant if T of xn - n 0 is same as yn - n 0. So, if we have the delayed input, then the corresponding response will be also delayed. So, this is the time invariant system. Another important concept is causal system. The system yn = T of xn is called causal if yn depends on the present and the past inputs only. So, it does not depend on the present data so that way yn is a function of xn, xn – 1 etcetera. We will be dealing with linear time in radiant causal systems.

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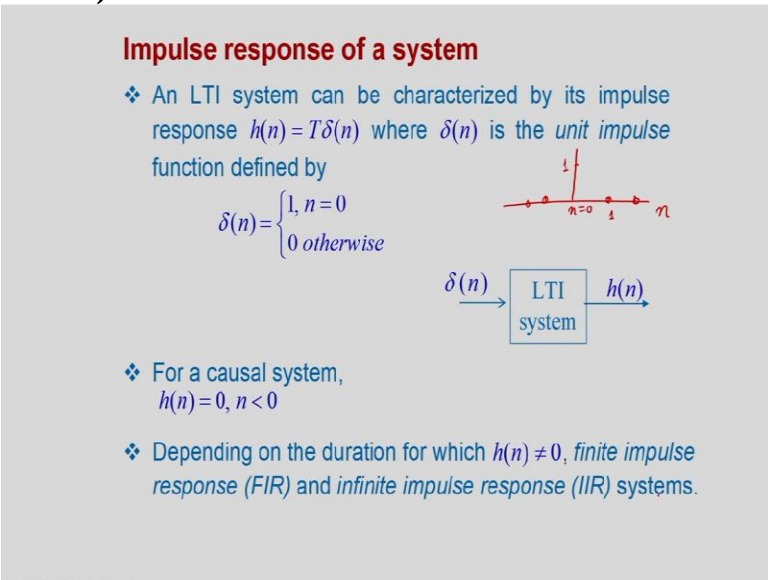

This is very important we will discuss LTI causal systems impulse response of a system now a linear system is characterized by its impulse response. So, we will see what is an impulse response of a system? An LTI system can be characterized by its impulse response hn it is denoted by hn is equal to transform of delta n so output due to impulse where delta n is the unit impulse in signal processing it is defined it delta n is equal to 1, n = 0 0 otherwise, it an only at n = 0 there is an input.

So that way, suppose if this is my n axis, then suppose this is n = 0 1 etc. So, at n = 0 the value of the signal is 1 and elsewhere it is 0. So, that way so this is the impulse signals. Now, impulse response is the response due to impulse so here is about the LTI system, your input is delta n corresponding output is called the impulse response and it is denoted by hn now for a casual system hn = 0 for n less than 0 because your delta n is nonzero.

Only at n = 0 therefore, for n < 0 hn must be equal to 0 for a causal system. Depending on the duration for which hn is nonzero finite impulses response FIR and in finite impulse response IIR systems are defined we will call suppose an FIR system for which the impulse response is a finite duration and IIR system for this impulse response is of infinite duration.

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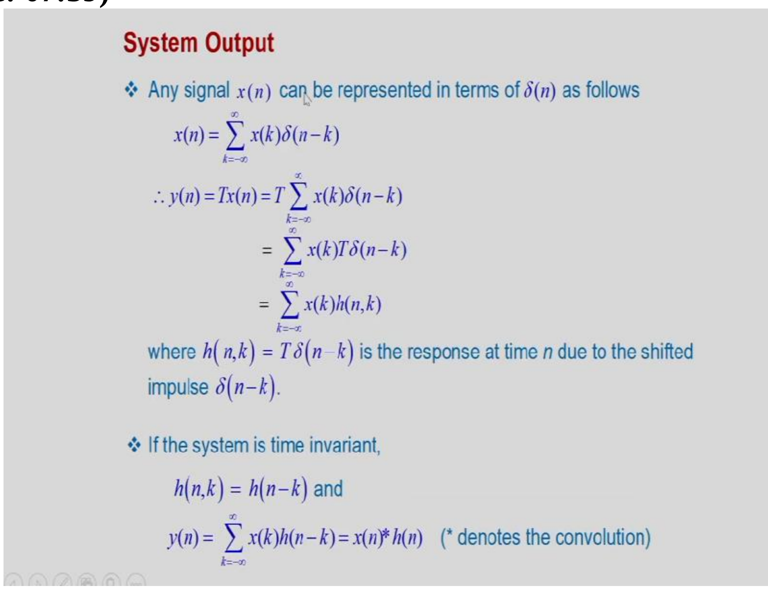

Now how to get the output of an LTI system any signal xn can be represented in terms of delta n as follows that is xn is summation xk into delta n - k, k going from minus infinity to plus infinity because we see that when we put k = n here it will become n - n 0 then it will be delta 0 is equal to 1 and for. So, when it is xn that means this one will be delta 0 that is equal to 1 Therefore, it will be the right hand side will be xn only for remaining values of n delta n - k will become 0.

So, that way xn can be represented in terms of this infinite sum. Now, what will we be the yn will be transforming sum of xn that is equal to the transformation of this sum. Now, we see that xk the constant it is not a function of n. So, therefore, we can write T here so, this summation will be xk into T times delta n - k and T of delta n - k that is the response due to delayed impulse is denoted by h of n k therefore yn for a linear system will be summation xk into h n k, k going from minus infinity to plus infinity.

So, as I said this h n k is the response at time n due to this shifted impulse delta n - k h of n k this is the response at instant n due to the impulse at n - k. Now, support the system is time invariant the linear system is time invariant in that case, because the output due to the delayed impulse will be the will be just delayed impulse response. So, that way h of n k = h of n - k for yet time invariant system.

So, in that case we can write y n is equal to summation xk into h of n - k, k going from minus infinity to infinity and this is the convolution operation. Therefore, we can write xn star hn star is the convolution operation. So, what we have seen, if the system is time invariant system is linear and time invariant in that case, yn output yn , can we express the convolution of xn input xn and the impulse response hn convolution of xn and hn.

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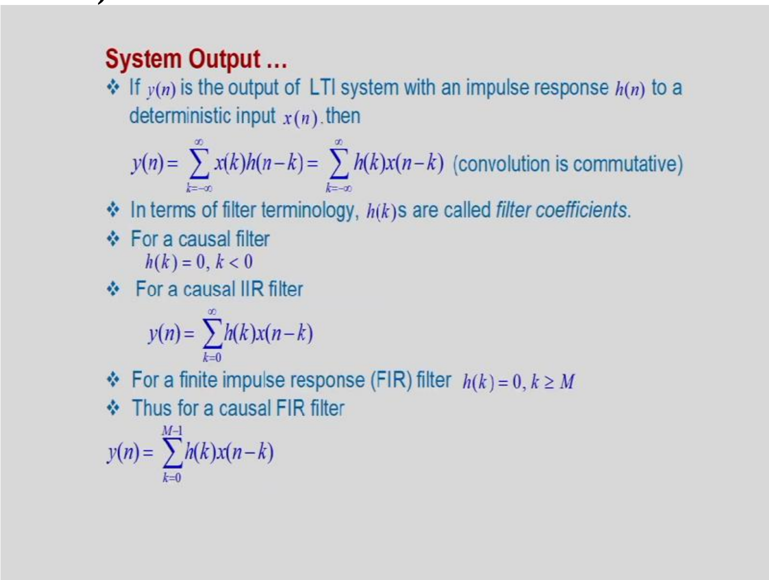

Thus, if yn is the output of an LTI system, which an impulse response hn to deterministic input xn yn is the output of the LTI system to deterministic input xn then yn is equal to in the previous expression so that it is summation xk into h of n - k, k going from minus infinity to infinity. Now, noting that convolution is a commutative operation, we can write this as summation hk into x of n - k, k going from minus infinity to infinity.

So, this is the output of an LTI system to a deterministic input xn. Now, in terms of filter terminology, these hk are called filter coefficients. For a causal filter if we assume that the filter is causal which is realizable hk = 0 for k less than 0. So in that case we can write suppose for a causal IIR filter yn will be equal to summation hk into x of n - k, k going from 0 to infinity. So, if we put hk = 0 for k less than 0.

We get this result upon a finite impulse response a FIR filter, hk 0 for k greater than equal to m for some constant m. So, therefore, what a causal FIR filter we can write yn is equal to summation hk into x of n - k, k going from 0 to m - 1. So, that way we have seen how to represent the output of a linear time invariant system in terms of the input and the impulse response of this system.

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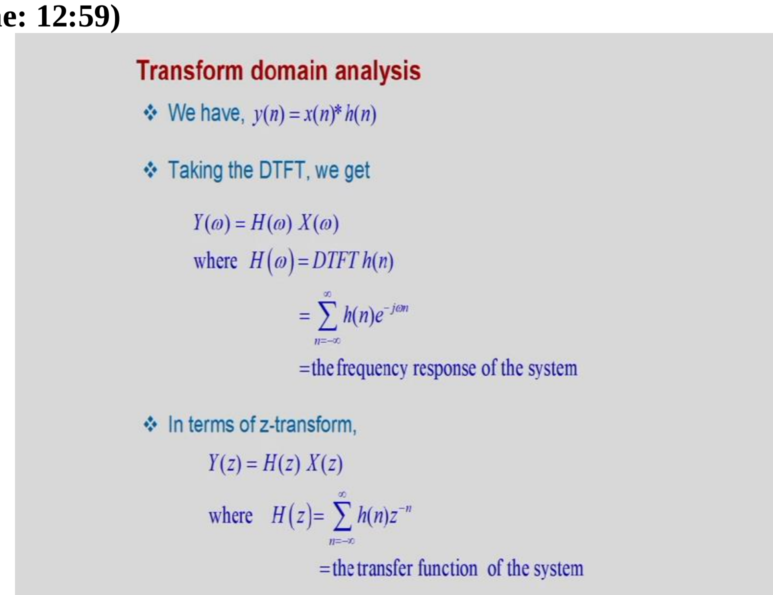

Now, Let us see how we can represent the same result in the frequency domain we have yn = xn star hn variation is the impulse response which will be finite duration or infinite duration taking the DTFT in the, because it is a discrete time signal with take the discrete time Fourier transform taking the DTFT we get y omega that is Fourier transform of yn discrete time Fourier transform of y n is equal to it is convolution.

In time domain therefore, in frequency domain it will be multiplication that is h omega into x omega, where h omega is the DTFT of hn and we can write it as summation hn into e to the power - j omega n , n going from minus infinity to plus infinity. So this is the DTFT of hn sequence. And this term in system theory, we call the frequency response of the system h omega is the frequency response of the system.

Therefore, discrete time Fourier transform of the output is equal to frequency response multiplied by the discrete time Fourier transform of the input. So if we write in terms of z transform, this will be yz = hz into xz In fact, if we put that hz is equal to j omega will get this expression. Where is that? It z transform of hn and this is the expression, summation hn into the z to the power - n and going from minus infinity to plus infinity and is that In system theory is called transfer function of the system for a discrete time system is that is the transfer function. Now, we have the background of the linear time invariant system.

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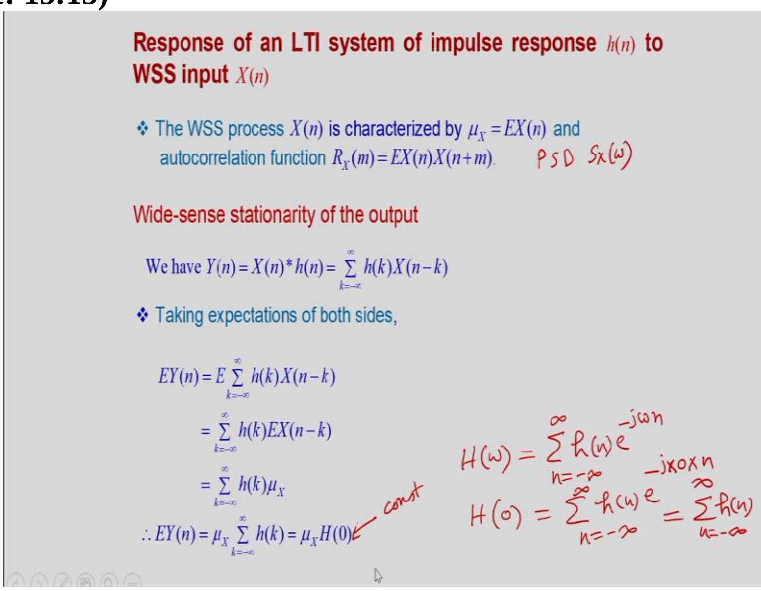

Let us discuss the response of an LTI system of impulse response hn to a WSS input xn. So, impulses response is hn and input Xn is WSS that we assume now you know that the WSS process Xn is characterized by mean mu x that is E of Xn and autocorrelation function R X of m that is E of Xn into X of n + m. And also we can collect as because autocorrelation function had the Fourier transform power spectral density. Therefore, we can also characterize the WSS process in terms of power spectral density sx omega PSD, S x omega.

So, these are the quantities which characterize a WSS random process mean mu x autocorrelation R X of m and PSD S x omega. Now input is a WSS signal whatever the output so will prove the wide sense stationarity of the output we have Yn = Xn convolved with hn that is equal to summation hk X of n - k, k going from minus infinity to plus infinity. Let us take expectations of both sides, then E of Yn will be E of this sum.

This is the expectation of summation and these we can write as summation k going from infinity to plus infinity hk into E of X of n - k because hk is a constant random variable. So, therefore, expectation operation will come here and this we can write E of Y n will be equal to summation hk into mu x because this quantity is constant at every time point therefore, we can write it as mu x that is the h xn is a WSS process this quantity will be mu x.

So, what we conclude E of Yn = mu x, mu x we are taking out into summation of hk, k going from minus infinity to plus infinity. This quantity is the frequency response at 0 frequency or 0 frequency gain how because it can write suppose we have h omega is equal to summation hn into e to the power - j omega n, n going from minus infinity to plus infinity. Now will put omega is equal to 0 is 0 will be equal to summation hn e to the power – j 0, j 0 into n.

And that will be 1 and going from minus infinity to plus infinity. That is equal to summation hn and going from minus infinity to plus infinity, so that way E of Yn is equal to mu x into h of 0 What does it mean? That E of Yn is a constant now, this is a constant, this quantity is a constant so, this is one of the requirements for WSS process that process should have constant mean and we have seen that Yn has constant mean.

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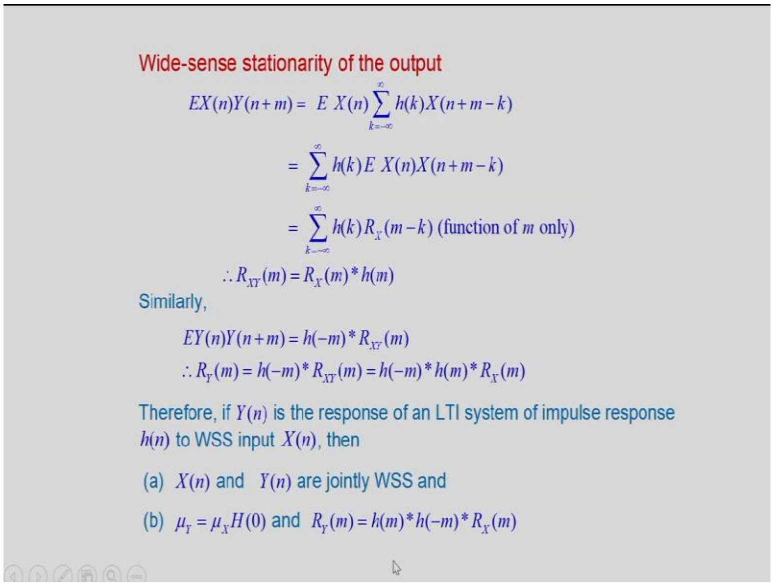

Will continue about to wide sense stationarity of the output. Let us see what happened to the cross correlation function, E of Xn into Y n + m that is equal to E of Xn into and Y of n + m that is summation or convolution of Xn and Y n + m so that we can write a summation hk into X of n + m - k, k going from minus infinity to plus infinity. Now, we can take E inside because these are a constant quantity, so E of Xn into X of n + m - k.

Now, using the WSS concept, this quantity will be a function of the lag n + m - k - n that is the lag that is equal to m - k therefore, this cross correlation function is equal to summation hk into R x of m - k, k going from minus infinity to plus infinity. So, we see that this is a function of lag only, it does not depend on n and therefore, we can write R xy m that is the cross correlation function at lag m is equal to R x of m, R x of m convolved with h of m.

Similarly, we can write E of Yn into Y n + m = h of - m convolved with R xy of m. So, here again if we substitute Yn is equal to summation hk X of n - k, k going from minus infinity to plus infinity, then we will get this result. So, here again we will see that it is a function of lag m only, we can write R y m that is output autocorrelation function is equal to h of - m convolved with R xy m and from this result will write this equal to h of - m convolved with h m convolved with R x of m so, this is the output autocorrelation function.

And what we conclude? We conclude that if Yn is the response of an LTI system of impulse response hn, to WSS input Xn then Xn and Yn are jointly WSS. Because, output yn is a WSS process Xn is the WSS process and they are cross correlation depends only on lag. Therefore, Xn and Yn are jointly WSS and second conclusion is that mu y, is equal to mu x into H0 where H0 is the 0 frequency gain or this gain and R y m is the convolution of hm - m and R x of m so these are the results we established. This is the characterization in time domain. So, these are characterization in the time domain in the frequency domain we will see what happened.

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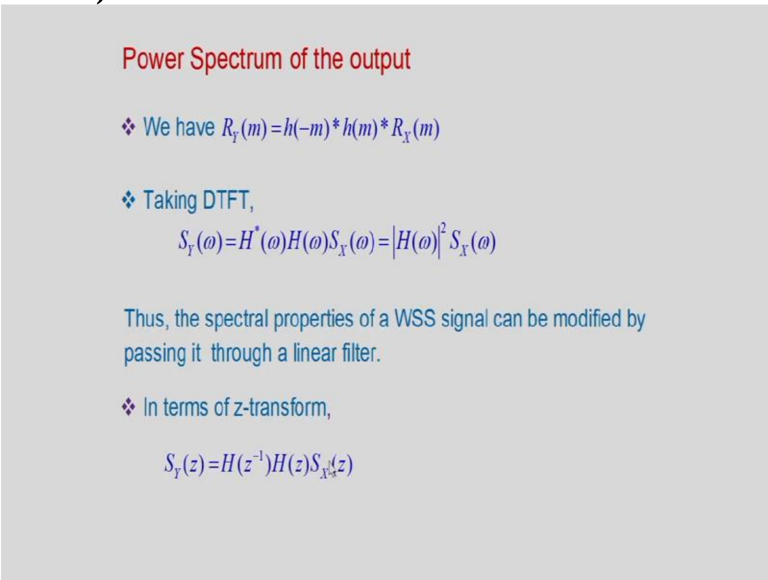

Power spectrum of the output, we have R y m = h of - m star h of m star R x of m Now again, we will take the DTFT so, DTFT of R y m is S y omega. So, that will be equal to now hm - m, if we take the DTFT that will be H omega star complex conjugates of H omega into H omega into S x of omega PSD of xn so, because convolution in time domain will be multiplication in the frequency domain we get this result.

Now H star omega into H omega that is mod of H Omega Square therefore, S y omega will be equal to mod of H omega square into S x of omega. So, this is the relationship between PSD of the output and PSD of the input. Thus this spectral properties of a WSS signal can be modified this is modified because this function is multiplying S x omega. So, S x omega will be now modified can be modified by passing it through a linear filter.

So, this power spectrum we can also express in terms of z transform. So, that S y z will be equal to H and inverse that transform of this will be Hz inverse into Hz into S x of z this is S y z will be generalized PSD of the output process S x z their generalized PSD of the input process. Will consider one example,

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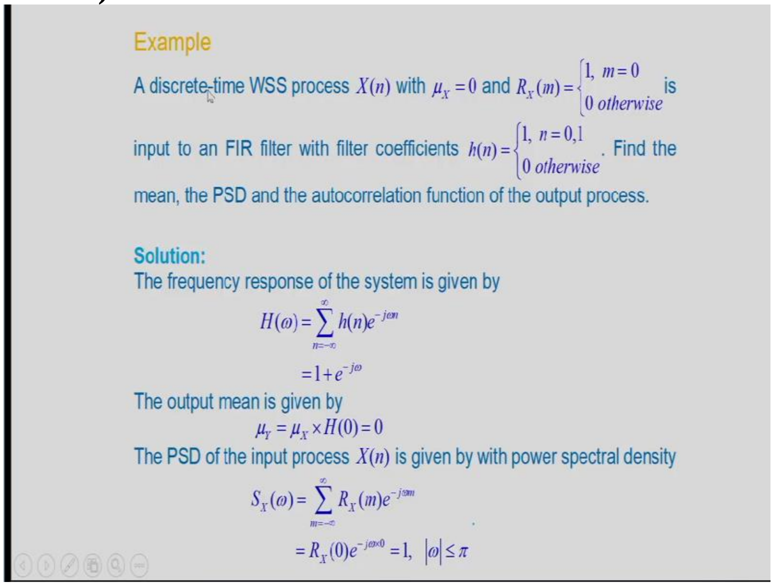

A discrete time WSS process Xn with mu x = 0 and the autocorrelation function R xm. Which is equal to 1 at a mu = 0 and 0 otherwise, input to FIR filter with filter coefficient hn is equal to 1 for n = 0 and n = 1 and 0 otherwise it is the 2 type filter, only 2 values h0 = 1 and h 1 is also equal to1. Find the mean the PSD and the autocorrelation function of the output process will find the solution here.

The frequency response of the system because this is the impulse response, we have to take the DTFT. So frequency response of the filter is given by summation hn e to the power - j omega n and n going from minus infinity to plus infinity but here for all other value of n hn = 0 only at n = 0 and 1 will get hn = 1 therefore will have this is h0 that is 1 into e to the power j omega 0 that will be equal to 1+ h1 that is 1 into e to the power - j omega into 1 that is equal to - j omega. So that way h omega will be 1 + e to the power – j omega.

Once we know h omega we can find out the output means that is equal to mu y into mu x is H of 0 and we know that mu x = 0 therefore this will be equal to 0. So, therefore output of this filter is 0 mean. Now, we have to find out the output power spectral density before that we have to find out what is input power spectral density, that is S x of omega is equal to summation R x of m into e to the power - j omega m, m going from minus infinity to infinity.

So, here only one value of R x of m is non 0 that is at time is equal to 0 it is 1 otherwise it is 0 therefore, it will be equal to R x of 0 into e to the power - j omega 0 that is equal to 1 for omega anyway power spectral density is uniquely defined because it is a discrete time fourier transform it is in an uniquely defined in a period from - pi to pi. So, that way S x omega = 1 par mode of omega less than equal to pi.

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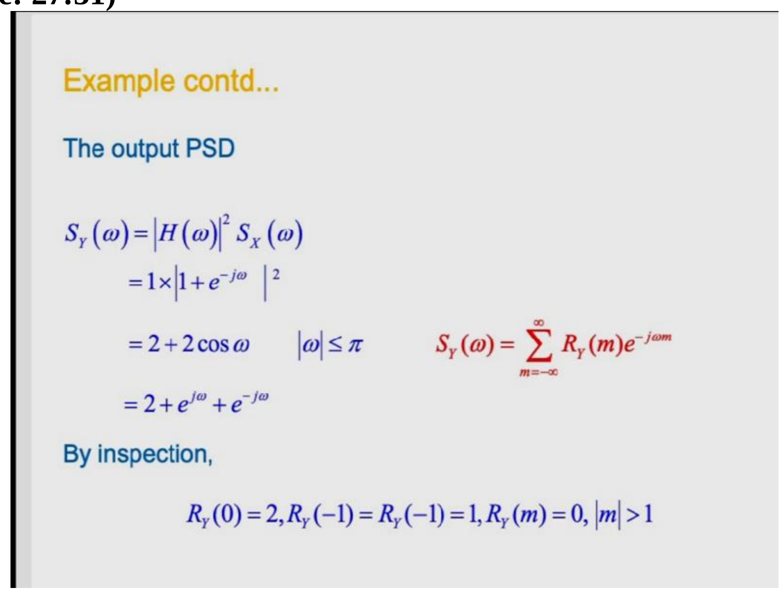

Now, we can find out the output PSD. So, the output PSD is given by S y Omega, that is equal to mod of x omega square into S x omega and we know S x omega is equal to 1 therefore, 1 into 1 + h omega is 1 + e to the power - j omega mod of 1 + e to the power - j omega square and this we can write as if we expand it, we will get it as 2 + 2 cos omega or mod of omega is less than equal to pi, because DTFT is uniquely defined in a period from - pi to, pi.

And this cos omega again we write in terms of e to the power j omega + e to the power - j omega. So, that we can now compare with this expression S y omega is equal to summation R x m e to the power of - j omega m, m going from minus infinity to plus infinity. Here we see that only 2 times are there corresponding to omega is equal to 0, omega is equal to -1 and omega is equal to + 1.

So, therefore, the corresponding coefficient will be the autocorrelation function, therefore, by inspection R y of 0 = 2 R y of - 1 will be equal to R y of 1 will be equal to 1 and for rest of the values of m R y m will be equal to 0.

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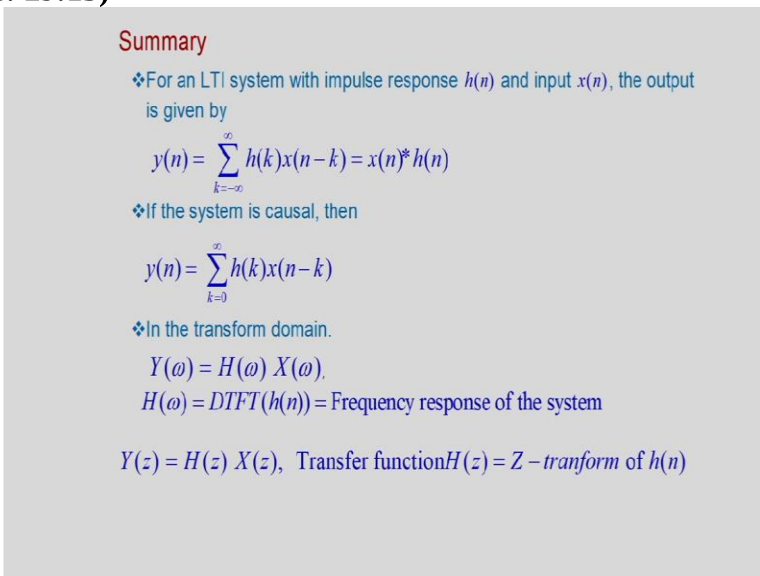

Let us summaries what we have learned today for an LTI system with impulse response hn and input xn the output is given by this convolution relation xn convolve with hn and if we expand that will be equal to k summation hk into x of n - k, k going from minus infinity to plus infinity this is the output of a linear time invariant system due to an input action. Now, if the system is causal, in that case, hk is equals 0 for k less than 0 therefore, a yn will be equal to summation hk x of n - k, k going from 0 to infinity.

In the transform domain we have Y omega is equal to H omega into X omega. This is by taking the DTFT we get this relationship and this H omega DTFT of hn is known as the frequency response of the system. And similarly we can write z transform domain Y z = H z into X z and where transfer function H z is equal to z transform of hn.

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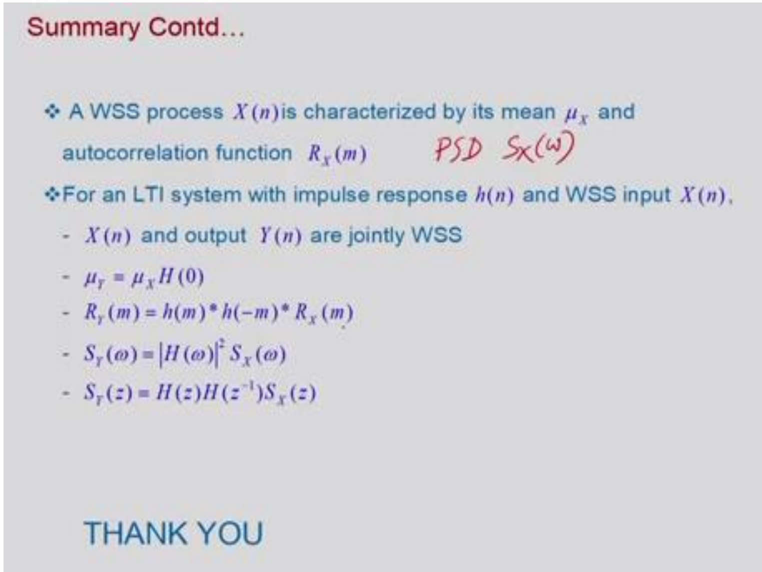

Coming to the random process a WSS process Xn is characterized by its mean mu x and the autocorrelation function R x of m. So, WSS process Xn is represented by its mean mu x and the autocorrelation function R x of m also deeper PSD S x of omega for an LTI system with impulse response hn and WSS input Xn x input Xn and output Yn are jointly WSS that we established. Then mu y that is mean of the output process is equal to mean of the input process multiplied by H0 where H0 is the this gain of this system or 0 frequency gain of the system.

The autocorrelation function of the output process R y m is given by hm convolved with h of - m convolved with R x of m it is the convolution of hm h - m and R x of m and in the frequency domain we get. S y omega is equal to mod of H omega square into S x of omega also in terms of z transform S y z that is generalized PSD is equal to Hz, Hz inverse into S x of z. Thank you.