Statistical Signal Processing

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Lecture - 37
Linear Models of Random Signals 2

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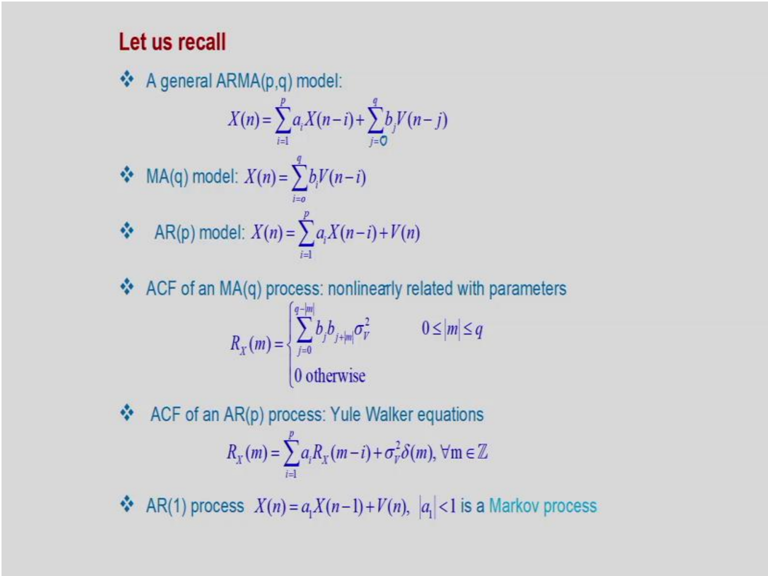

Hello students in this lecture i will discuss linear models of random signals 2. Let us recall a general ARMA p q model is given by Xn is equal to summation ai X of n - i i going from 1 to p plus summation bj a of n - j j going from 0 to q this is an ARMA p q model and we also discuss MA q model that is Xn is equal to summation bi V of n - i i going from 0 to q AR p model autoregressive of order p that is Xn is equal to summation ai X of n - i i going from 1 to p + Vn.

The ACF of an MA q process is non linearly related with parameters like this is the relationship R x of m is equal to summation j is equal to 0 to q - mod of m bj b of j plus mod of m into sigma V square where mod of m lies between 0 and q and otherwise R x of m is 0. The ACF of an ERP process is given by the Yule Walker equations that is the set of linear equation R x of m is equal to summation i is equal to 1 to p of ai into R x of m - i + Sigma V square into Delta m this is for all values of integer m.

Particularly we discussed about AR1 process Xn is equal to a 1 into X of n - 1 + Vn where mod of a 1 is less than 1 and this is required for stationarity of the signal and we have seen that this air 1 process is a Markov process.

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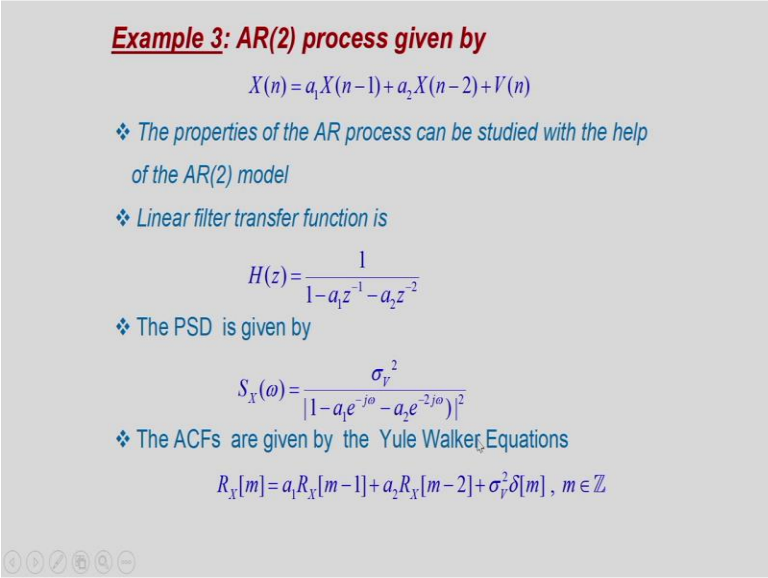

We will consider another example that is AR2 process given by Xn is equal to a1 X of n - 1 + a 2 X over n - 2 plus Vn this is a very good example the properties of AR processes can be studied with the help of this model. Let us see firstly linear filter transfer function corresponding to this process this linear filter transfer function is given by 1 by 1 - a 1 z to the power - 1 - a 2 into z to the power of – 2. So, this is an all pole model it has two poles.

The PSD power spectral density of this process is given by Sx Omega that is the power spectral density is equal to Sigma V squared divided by mod of 1 - a 1 into e to the power - j Omega - a 2 into e to the power - 2 j Omega whole square. So, this is the power spectral density. The ACF’s are given by the Yule Walker equations we can write down the Yule Walker equation that is Rx of m is equal to a 1 into R x of m - 1 plus a 2 into R x of m - 2 + Sigma v square into Delta m that is for all values of m integer.

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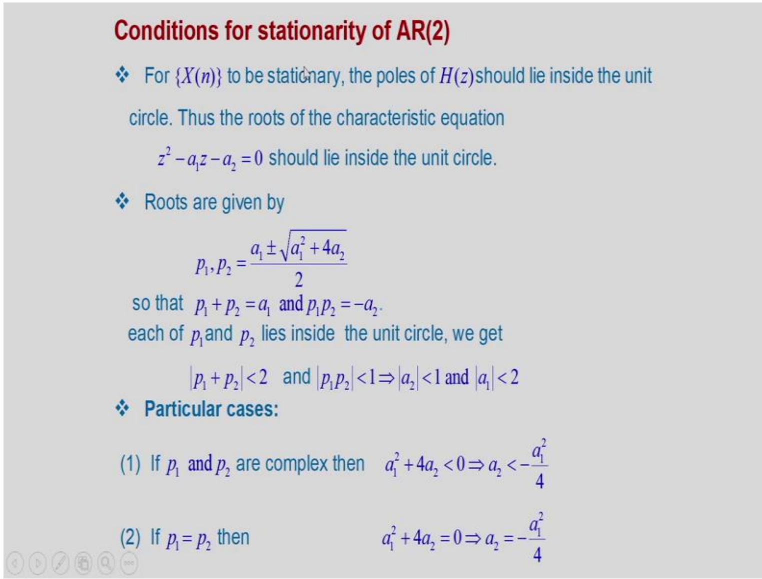

Now because AR2 process is an all pole model therefore for all values of a 1 and a 2 will not give stationarity. Let us consider the conditions for stationarity of AR2 process. For Xn to be stationary the poles of Hz to lie inside the unit circle that is the condition for stability that is the root of the characteristic equation because we can get the characteristic equation by putting the denominator is equal to 0 so that way denominator was we are Hz was Sigma V square 1 - a 1 z to the power - 1 a 2 z to the power – 2.

Now if i put this denominator is equal to 0 then we will get this equation. So, if we put this denominator is equal to 0 we will get this equation. Now the condition for stability of a Hz requires all the roots of this equation should lie inside the unit circle. Now roots are given right there will be two roots of this equation two roots p 1 and p 2 given by a 1 + - root over a 1 square + 4 a 2 divided by 2. So, that if we consider the sum of the roots p 1 + p 2 is equal to a 1 and product of the roots that is p 1 into p 2 is equal to - a 2.

So we see that that a 1 and a 2 are related to the roots by this relationship ok. So, it is a p 1 and p 2 lies inside the unit circle therefore we get this quantity because a 1 is nothing but mod of p 1 + p 2 therefore mod of p 1 + p 2 must be less than 2 and mod of p 1 into p 2 must be less than 1 and in turn these two conditions will imply that a 2 from here a 2 is less than 1 mod of a 2 is less than 1 and from this relationship we get that mod of a 1 is less than 2.

So that way a 1 will lies between - 2 n + 2 and a 2 will lies between - 1 and +1. Now let us consider the particular cases of the roots. So, p 1 and p 2 are complex then this part will be less than zero so a 1 square + 4a 2 that must be less than 0 that will in turn imply that a 2 is less than - a 1 square by 4. Now this region is a parabolic region and when roots are equal p 1 equal to p 2 then this term will be equal to 0 and in the case of a 2 equal to - a 1 square by 4 we see the parabola.

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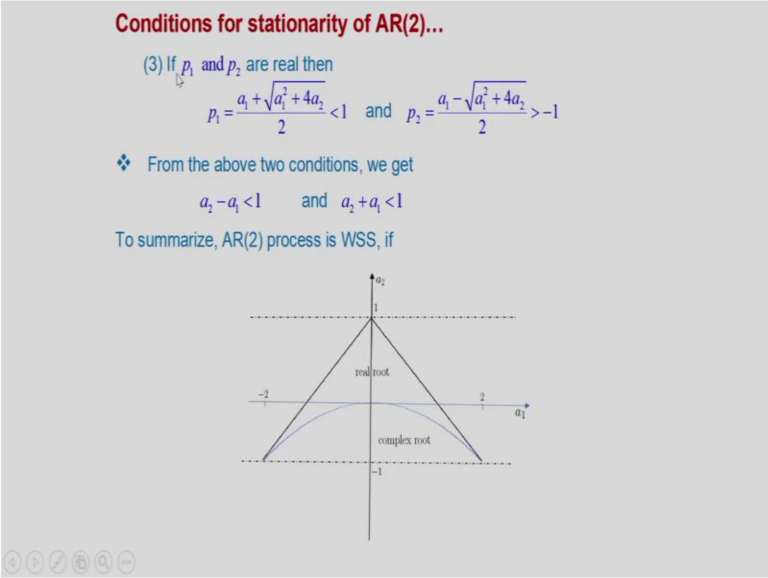

Now condition 3 if p 1 and p 2 are real then both roots p 1 and p 2 both her magnitude of both will be less than 1 now this is the larger root out of 2 this is the larger root therefore its value must be less than 1 and this is the smaller root because this minus sign, so this must be greater than -1 these are both real and from the stability conditions we get this. And from the above two conditions we will get that a 2 - a 1.

So if we simplify this condition a 2 - a 1 is less than 1 and a 1 + a 2 or a 2 + a 1 is less than 1 two conditions we get considering the real values of the poles. So, to summarize now AR2 processes WSS if suppose my a 1 and a 2 must be within this region. So, for real root it should be in this part for complex roots it should be inside this parabola. So, these are the conditions we get by analyzing the roots of the characteristic equation.

And in other words the AR 2 process is WSS a 1 and a 2 will lie within this triangular region then our roots will be stationary and if it lies inside this parabola then the roots will be complex and outside this parabola roots will be real.

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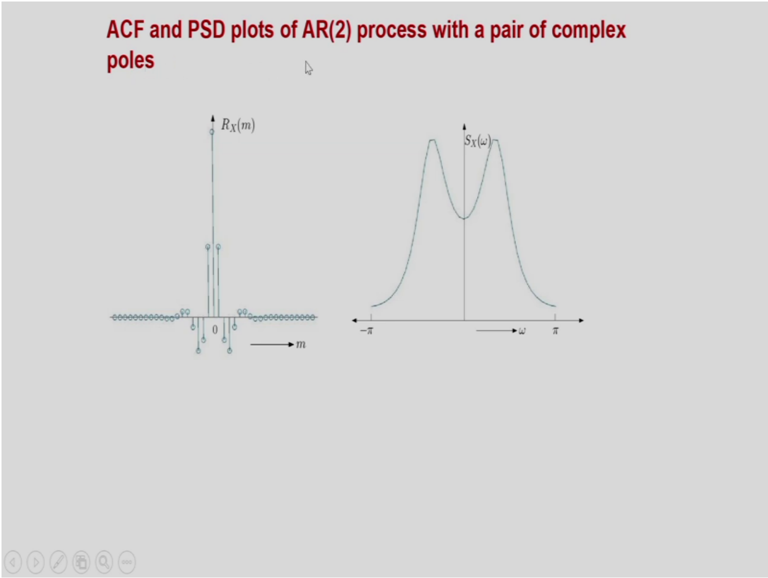

We can plot the ACF and PSD of AR2 process these are the plausibly simulated AR2 process and we see that this is a particular case for complex poles so the autocorrelation behaves like this both positive and negative values are there and power spectrum has a peak this is a negative side of the spectrum so there is a peak corresponding to this point. So, that way you suppose if we have a some dominant frequency component in a random process then we can get it by AR2 spectrum.

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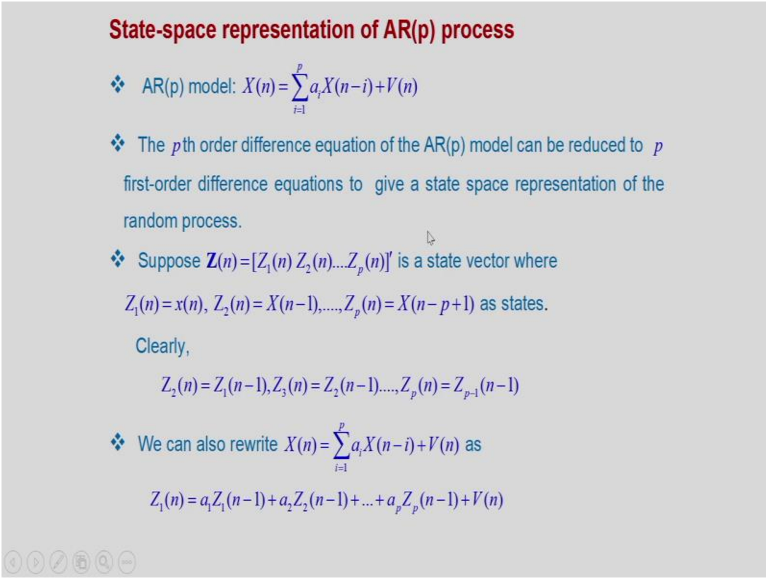

We will introduce one in important concept that is state space representation of ARP process that ARP model is Xn is equal to summation ai into X sub n - i i going from 1 to p + Vn the pth order difference equation this is the pth order differential equation of the ARP model can be reduced to p partial difference equations to give this space representation update random process. Now this is important thing because we have a pth order difference equation and corresponding to that we can get p first order difference equations.

And those will represent this state space equations. Suppose we define Zn is equal to there are pth order differentiation therefore Delta p States suppose Z n is equal to z1 and z2 n up to Zpn vector this is a vector comprising of these elements. Now each of this element will be called a state. Now let us define they Z1 end to be X n Z2n to be Xn - 1 and similarly Zpn to be X of n - p + 1. So, these p states are defined like this.

Now clearly how these sates are related Z2n is equal to z1 of n - 1 Z3n is equal to Z2 of n - 1 like that Zpn is equal to Z p - 1 of n - 1 so now we have a difference equation relationship between state variables. We can also rewrite Xn is equal to summation ai X of n - i i going from 1 to p + V n as now we will start substitute these state variables corresponding to each of the terms we get Z1n is equal to a 1 into Z 1 of n - 1 + a 2 into Z 2 of n - 1 + up to Zp into Z p of n - 1 + Vn.

So this equation now Z1n is this similarly Z2n is Z1 of n - 1 Z3n is equal to Z2 of n - 1 etcetera this can be rearranged.

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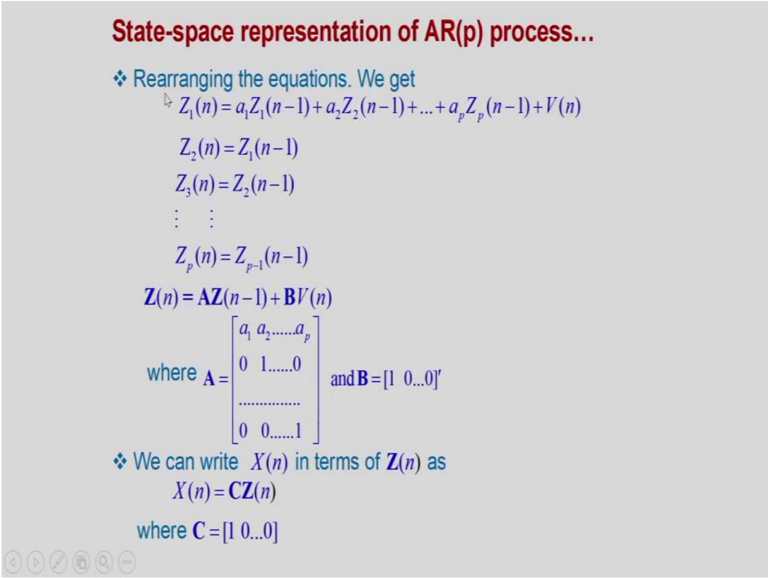

Rearranging the equations we get a Z1n is equal to a 1 into Z1 n - 1 + a 2 into Z2 n - 1 plus up to Zp into Zp n - 1 + Vn so this is the representation first step. Similarly Z2n is equal to Z 1 of n - 1 Z3 n is equal to Z 2 of n - 1 etc and finally get Zpn is equal to Z p - 1 n – 1, so that we can now write Zn this is a Zn state vector Z n which comprises of element Z1n Z2n up to Zpn that is the Zn vector it is a column vector is equal to A matrix times Z n - 1 vector + B matrix time Vn, Vn is a scalar only.

So, where A is given by this matrix now it is p by p matrix first row is a 1 a 2 up to ap second row is first element is 0 then 1 and rest of the elements are 0 and similarly last row is 0 0 last element is 1 so this is the A matrix, these equations we are writing in terms of A matrix equation. And similarly B matrix is given by 1 0 0 rest of the elements are 0 transpose, transpose of this row vector so that we will get a column vector here.

We can also write Xn in terms of Zn suppose if we have to write Xn in terms of Zn then we can write Xn is equal to actually Z1n so that way if we write C is equal to 1 0 0 up to 0 all the remaining elements are 0 if we consider C to be this matrix this is a row vector actually then Xn will be C times Zn so that way we can get suppose Xn, Xn is equal to C times C is now 1 0 0 like that 0 into Zn vector and Zn vector is z1 and like that up to Z of pn.

So, if we multiply these two matrices we will get Xn is equal to Z1n that is exactly what we have shown earlier Z1n is equal to Xn that way or we can write Xn also that observed at all so in terms of state variable that is C times Zn this model is known as the state variable model and this equation Zn is equal to A into Z n - 1 + B into Vn this is known as the state equation and Xn is equal to C times Zn that is known as the observation equation.

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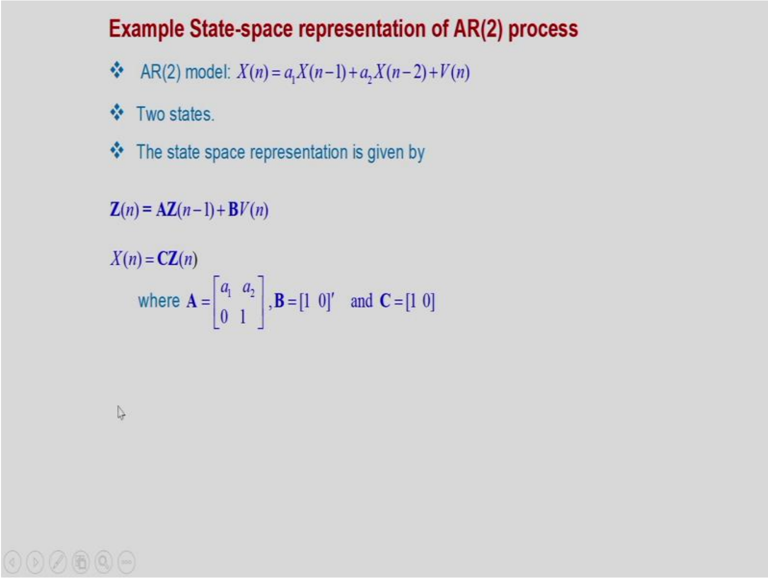

We shall consider one example that is AR2 model Xn is equal to a1X of n - 1 + a 2 into X of n - 2 + Vn this is the AR2 model how to get this state space representation. There will be two states one is Xn and another one is X of n - 1 this representation is given by now there will be two state vector Zn is equal to A into Z n - 1 + V into Vn where Xn that is the observed data is equal to C times Zn then what is A matrix where A is equal to a 1 a 2 0 1 this is a 2 by 2 matrix B is 1 0 transpose and C is 1 0 this is a row matrix.

So that way get this state space representation corresponding to this difference equation. So, this is a second order differential equation that we have converted into a matrix first order difference equation though the number of states is 2 here we can define the state variable representation in different manners there are different ways to represent state variables but this is the model we will be using.

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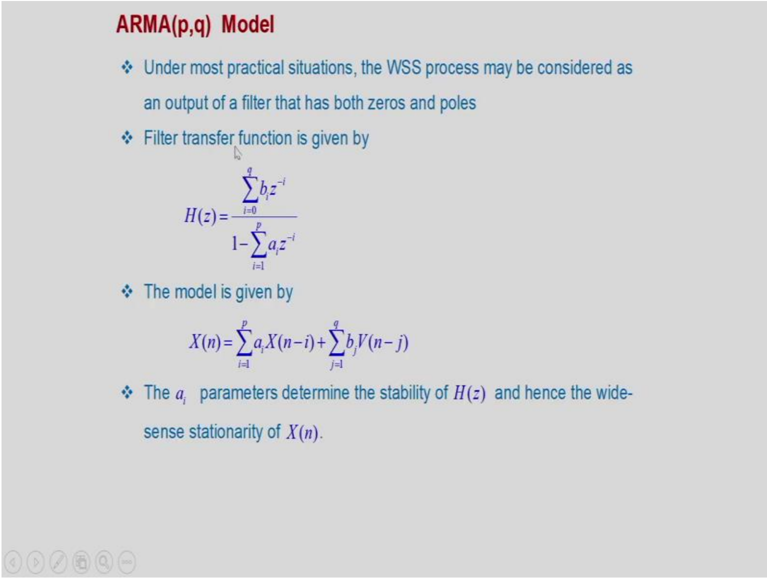

Let us consider the ARMA p q model I am not in most practical situations the WSS process may be considered as an output of a filter that has both zeros and poles. So, ma model and AR model are particular cases in ma model we have only zeros in AR models we have poles but most generally we will have both poles and zeros. The filter transfer function corresponding to ARMA p q model is given by Hz is equal to this is the relationship ratio of summation bi is e to the power - i i going from 0 to keep q divided by 1 - summation I going from 1 to p of ai into z to the power – i.

This is the transfer function corresponding to a ARMA p q model. The model is given by Xn is equal to summation i going from 1 to p of ai X of n - i + summation j going from 0 to q of b j V of n - j this is the difference equation model for the ARMA p q model that filter Hz has both poles and zeros and stability is determined by poles only therefore the ai parameters determine the stability of a Hz and hence the wide sense stationarity of Xn. So, Xn will be WSS under certain conditions on ai's.

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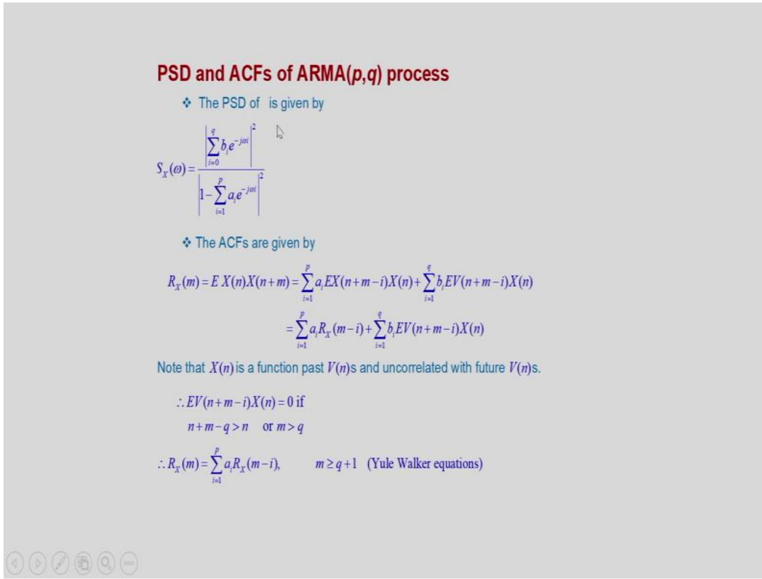

PSD and ACF's of ARMA p q process the PSD is given by Sx Omega Sx Omega is equal to this is the mod of the numerator term squared divided by mod of the denominator term square Sx Omega is given by this ratio that is equal to summation i going from 0 to q bi e to the power - j Omega whole mod square divided by 1 - summation i going from 1 to p ai e to the power - j Omega whole mod square so this is the value for Sx Omega.

This is the expression for Sx Omega and here in harm applicable model we consider from j is equal to 0 to q therefore there is a b 0 V of n and this b 0 absorbs the variance of Vn so that way Vn will be considered as unit variance so that it is variance will be taken care off by b 0 so this b 0 here there will be an expression suppose there will be b 0 V 0 will be there and this V 0, b 0 V0 n will be there and this V 0 n is now unit variance.

Because variance of the white noise process really taken care of by b 0 so that is why we do not have any Sigma b squared term here. The ACF of the ARMA p q process is given by this expression x, Rx of m by definition it is equal to e of X n into X of n + m now we will be writing X of n + m in terms of the difference equation model. So, that way we will have this summation i going from 1 to p X of n + m - i that way X of n + m will be written + bi will be there.

The ACF’s of Xn are given by this expression Rx of m equal to E of X n into X of n + m now we can write X of n + m this we can write as X of n + m is equal to summation ai X of n + m – i, i going from 1 to p + summation b i n + m – i, i going from 0 to q we are expanding X of n + m then this autocorrelation function will be given by summation i going from 1 to p ai into E of X of n + m - i into Xn + summation i going from 0 to q of bi into your V of n + m - i into X n.

So, this is the autocorrelation function of Xn at lag m - i so that way we can write summation i is equal to 1 to p ai into Rx of m - i this Xn and X of n + m - i so there are differences m - i here and plus summation bi into E of V of n + m - i into Xn i going from 0 to q. So, this is the expression for autocorrelation function of Xn Rx of m is equal to this and we see that there are these bi terms are contributing to this expression.

But we know that Xn is a function of past Vn’s and uncorrelated so that where action is uncorrelated with future Vn’s. So, suppose Xn and any future Vn’s for example Vn + 1 Vn + 2 etcetera uncorrelated because Xn is generated due to the earlier Vn’s so that way we can write E of V of n + m - i into Xn is equal to 0 if suppose this quantity is greater than n and n + m - q is greater than n okay so this term is greater than this so in the case we get m is greater if m is greater than q then this term will be equal to 0.

Therefore for m greater than equal to q + 1 we will get this equation R x of m is equal to summation ai into Rx of m – i, i going from 1 to p. So, this is part values of lag greater than equal to q + 1. So, this is the set of Yule Walker equation.

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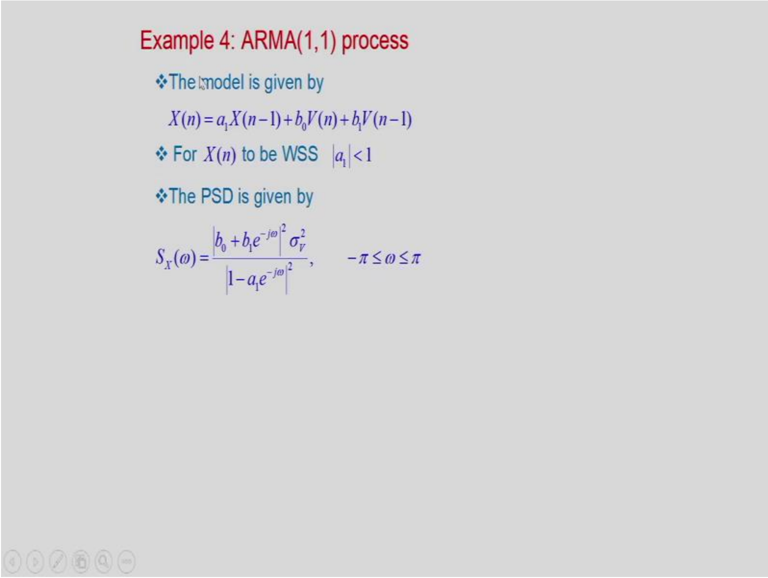

We can consider one example suppose ARMA 1, 1 process the model is given by Xn into a 1 X of n - 1 that is the AR part + b 0 into Vn + b 1 into Vn – 1. Now here the condition for what is in stationarity will be determined by the a1 parameters like AR1 process. So, mod of a1 should be less than 1 for a White sense stationarity of Xn and the PSD will be again given by and this expression but here Sigma V square we have already taken Sigma square is equal to 1.

So, that way this Sigma b Square will be equal to 1, so that way we have Sx of Omega is equal to b 0 + b 1 into e to the power - j Omega whole mod square divided by 1 - a 1 into e to the power - j Omega whole mod Square and for Omega lying between - PI and PI, this is the expression for power spectral density.

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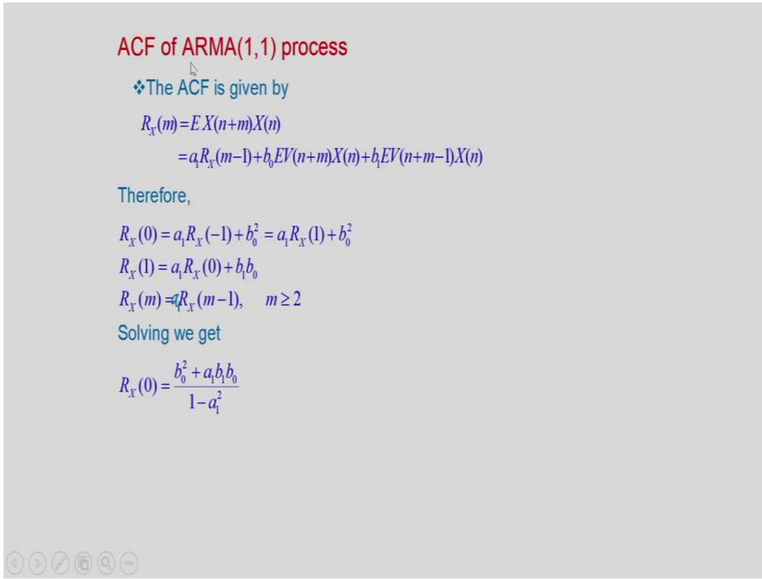

ACF of ARMA 1, 1 process the ACF autocorrelation function is given by Rx of m that is e of X of n + m into Xn and using the previous relation we can write this is equal to a 1 into Rx of m - 1 + b 0 into now EV of n + m into Xn + b 1 into EV of n + m - 1 into Xn. Now Rx of 0 from this equation this part we can write Rx of 0 is equal to a 1 into Rx of - 1 plus b 0 square so this part b 0 again when m is equal to 0 suppose when m is equal to zero then this will be Vn into Xn and this will contribute b 0 into Vn squares.

So that way we will get here b0 square similarly Rx of 1 will be equal to a 1 times Rx of 0 + b1 b0 that way we will get Rx of 1 and these two equations Rx of 0 is given by this Rx of 1 is given by this they can be solved to find out Rx of 0 and Rx of 1. For m greater than equal to 2 now we will have the Yule Walker equation that is Rx of m will be equal to a 1 times Rx of m – 1. So, that way we can get the remaining autocorrelation functions like Rx of 2 we can get Rx of 3 etc we can obtain.

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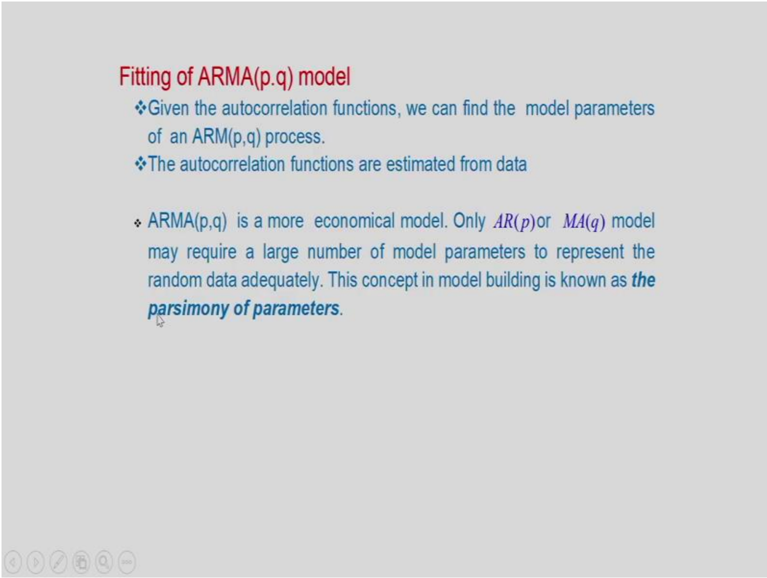

How to fit an ARMA p, q model, so given the autocorrelation functions we can find the model parameters of n ARMA p q process so because of the relationship between the autocorrelation functions and the model parameters we can find out the model parameters. The autocorrelation functions are estimated from data generally autocorrelation functions will be estimated from the data and those autocorrelation functions are used to estimate the model parameters.

ARMA p, q is more economical model economical means it is less number of parameters only AR p or MA q model may require a large number of model parameters. Suppose it were given data if i try to model by MA q model then this merely require very large value of q. so, only AR p or MA q model may require a large number of model parameters to represent the random data adequately that was given random data if we try to fit on the AP p and MA q model they may require large values of p and q.

Therefore if we use a ARMA p q model then a p and q need not be very large. This concept of model building is known as the parsimony of parameters. We should be parsimonious in selecting the parameters so that p and q should be as small as possible.

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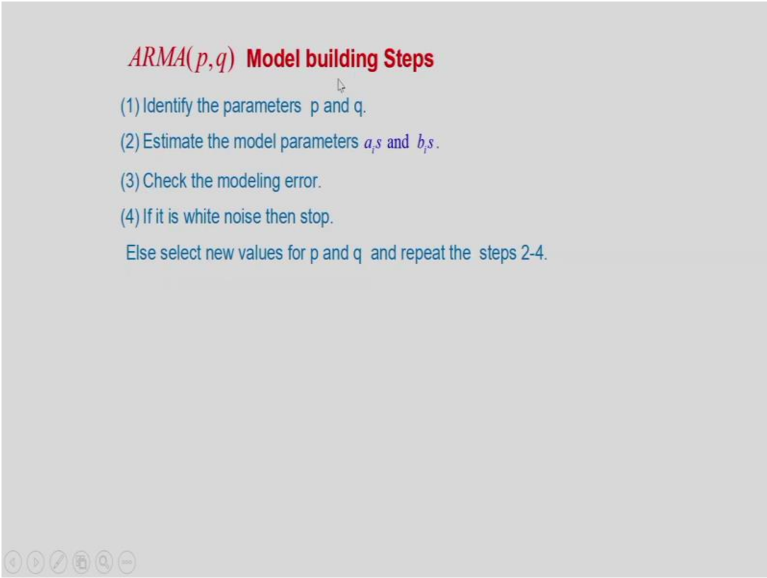

We will broadly describe ARMA p q model building steps, first step is identify the model order parameters p and q. After p and q are identified estimate the model parameters ai's and bi's check for the modeling error, if suppose we will find out the difference between actual sample value and the value predicted by the model. If it is white noise then stop so model building is over else select a new values of p and q and repeat these steps 2 to 4.

So that way we can build AMMA p q model though this is a quite involving process in a nutshell these steps can be described as this.

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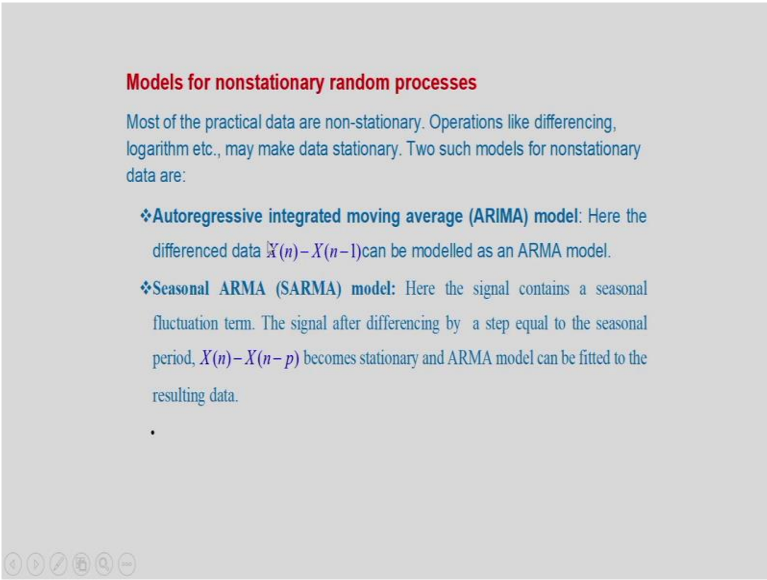

So, far we have discussed how to model WSS random process but most of the practical that there are non stationary operations like differencing logarithm etcetera may be are found to make data stationary. There are different ways to make data stationary to such models for non stationary data. First one is ARIMA model autoregressive integrated moving average model. Here the difference data Xn - Xn - 1 can be modeled as a ARMA model after differencing this data will be fitted into an ARMA model.

That model includes integration therefore after differencing it is becoming ARMA model. Seasonal ARMA that is SARMS, SARMA model here signal contains a seasonal fluctuation terms for example weather data the signal after differencing by step equal to the seasonal period that is p becomes stationary and ARMA model can be fitted to the resulting data. So, Xn – Xn - p so this difference data can be fitted to an ARMA model that is it SARMA, seasonal ARMA model. So, these are some of the ways to deal with non stationarity but we will not discuss this in details.

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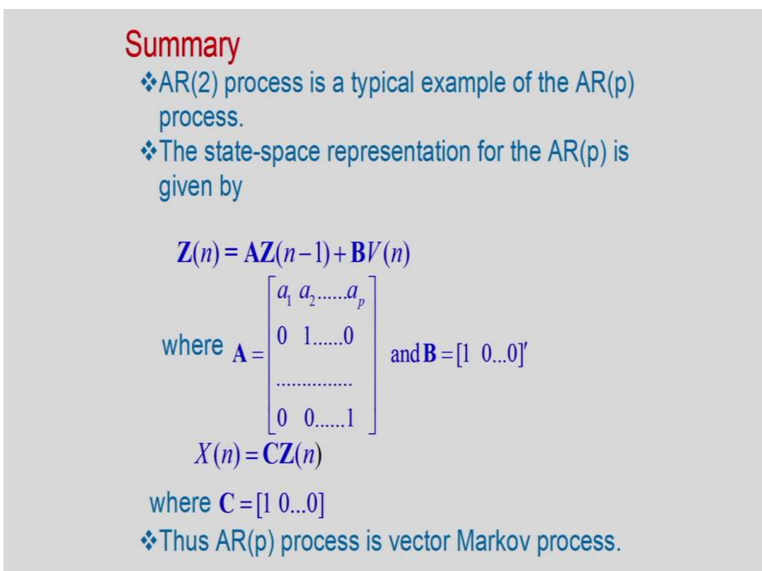

Let us summarize an AR 2 process is a typical example of ARp process so we discussed how to analyze this AR2 process stationarity of the process. AR2 process is a typical example of the ARp process we discussed how to analyze the AR2 process in terms of stationarity autocorrelation function power spectral density etc. This state space representation for an ARp process is given by the Zn is equal to AZ n - 1 + B Vn where a is p by p matrix given by this, this is the AR parameters a 1, a 2 up to ap and rest of the rows will contain 0 except 1, 1.

So there will be one in one place and even in place it will be 0 so this is the A matrix similarly b matrix we can find out B is equal to 1 0 0 up to 0 transpose so this way we can write Zn is equal to AZ n - 1 + B Vn this is the state equation. Similarly we have the observation equation so this data Xn can be written as C times Zn where C is the row matrix thus ARp process is a vector Markov process. So, because this is a first order in terms of the state variable vectors so this is a first order in terms of vector it is a first order AR process and therefore it is a vector Markov process.

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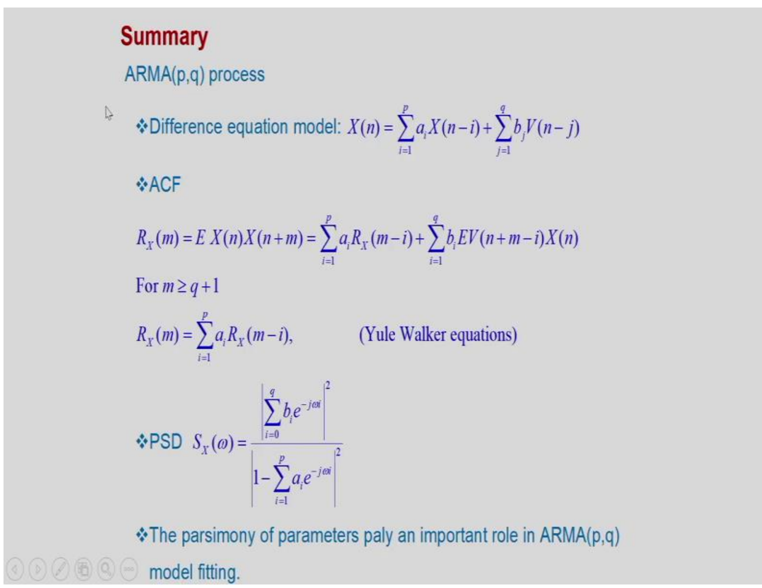

We discussed about ARMA p, q process the difference equation model is given by Xn is equal to summation ai X n – i, i going from 1 to p plus summation bj V n – j, j going from 0 to q this is the ARMA p, q model difference equation. Autocorrelation function is given by Rx of m is equal to this is the expression summation ai Rx of m – i, i going from 1 to p + summation bi EV of n + m - i into Xn i going from 0 to q and for m greater than equal to q + 1 this autocorrelation expression will result into Yule Walker equations given by Rx of m is equal to summation ai R x of m – i, i going from 1 to p.

And power spectral density is simply given by this expression Sx Omega is equal to summation bi e to the power - j Omega i whole mod square divided by 1 - summation ai e to the power - j Omega i, i going from 1 to p whole mod square this is the expression for power spectral density for ARMA p q process and we also discussed very briefly about model building and the parsimony of parameters play an important role in ARMA p, q model fitting, thank you.