# Statistical Signal Processing

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## Lecture - 34
## Review Assignment 4

Hello students. Welcome to this session on review assignment 4. Let us see the first problem.

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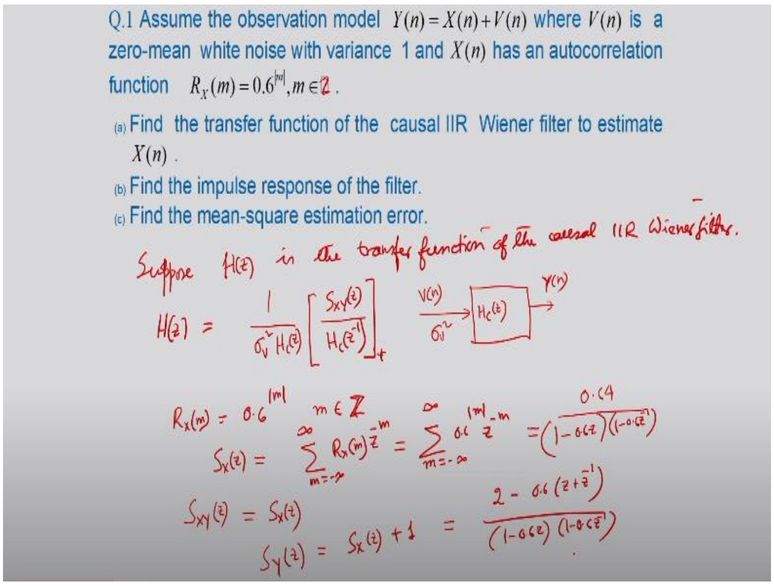


Assume the observation model Yn = Xn + Vn where Vn is a zero-mean white noise with variance 1 and Xn has an autocorrelation function Rxm is equal to 0.6 to the power of mod m, m belonging to Z. This problem we considered earlier also. Find the transfer function of the causal IIR Weiner filter to estimate Xn. So, we have to find the transfer function of the causal IIR Weiner filter.

Next, find the impulse response of the filter, and finally find the mean-square estimation error. So, in this problem we have to find out the transfer function, suppose Hz is the transfer function of the causal IIR Wiener filter. We have derived that Hz is equal to 1 by sigma V squared Hz into S xyz divided by HcZ inverse and the causal part of this. So this is the positive part, causal part. First, we have to find out HcZ.

So what is HcZ. This is the causal filter corresponding to spectral factorization of xyz. So, first we have to find out the causal filter corresponding to factorization xyz. That means, we write suppose, this is our yn, this is the observed signal and this is Vn where Vn is innovation sequence with variance sigma V squared. This one is HcZ, so that way that absorbs the stationary signal is considered as the output of a linear filter with input as the innovation sequence.

So, that we have to find out the expression for HcZ from the spectral factorization of xyz. Now, let us consider the parameters given, so Rx of m is equal to 0.6 to the power mod of m. So m belongs to set of integers. Now, Sx of that power spectral density of that domain, is given by Rx of m into Z to the power minus m, m going from minus infinity to plus infinity. So this we can carry out the algebra I already we have done it m is equal minus infinity to infinity, Rx is equal to 0.6 to the power of mod m and that to the power minus m here.

This we have shown that in earlier problem sheet, we already saw this and this is given by 0.64 divided by 1 - 0.6Z into 1 - 0.6 Z inverse. So, this is the power spectral density of Xn. Then, also we know that Sxyz because of this model signal plus noise model which are independent of each other because of that Sxyz will also be equal to Sxz that we have already established earlier and Syz = Sxz + 1, 1 is the power spectral density of the white noise because its variance is 1 and this expression, we got is 2 - 0.6 into Z + Z inverse divided by this quantity 1 - 0.6 Z into 1 - 0.6 Z inverse. So, this is the xyz.

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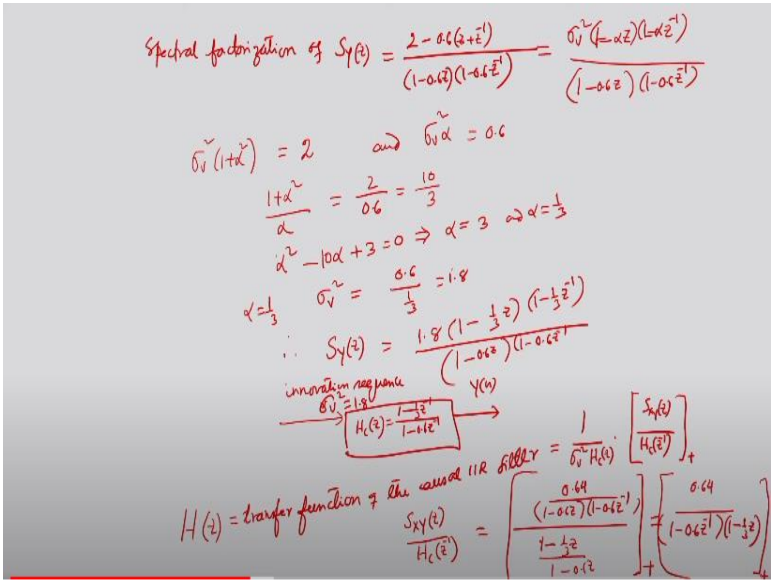


Now, we have to do spectral factorization of Syz that is equal to 2 - 0.6 into Z + Z inverse divided by 1 - 0.6Z into 1 - 0.6 Z inverse. Now, we are interested to factorize this sigma V1 into 1 - alpha Z into 1 - alpha Z inverse divided by 1 - 0.6Z into 1 - 0.6 Z inverse. Now, we have to choose this alpha and sigma V squared in such a way that corresponding filter will be minimum phase. Spectral factorization whatever this causal part will be that should be minimum phase.

So, we have this expression 2 - 0.6 into Z + Z inverse is equal to sigma V squared into 1 - alpha Z into 1 - alpha Z inverse. Now, we can equate this expression and this expression, we will get sigma V squared into 1 + alpha squared that must be equal to 2 and the other part is sigma V squared into alpha, that must be equal to 0.6. These are the equations; we have to solve to find out sigma V squared and alpha.

Now, dividing this equation by this equation, we will get 1 + alpha squared divided by alpha is equal to 2 by 0.6 that is equal to 10 by 3. So, this is the relationship we get from this alpha squared - 10 alpha + 3 = 0, which will give alpha is equal to 3 and alpha is equal to 1 by 3. For spectral factorization, the filter should be minimum phase, because of that we switch alpha is equal to 1 / 3.

So, that way alpha is equal to one-third and corresponding sigma V squared, from this expression, we can get sigma V squared is equal to 0.6 divided by one-third that is equal to 1.8. Therefore, this numerator part will be now, that is sigma V squared will be there 1.8, and then 1 - 1/3 Z into 1 - 1/3 Z inverse. So, therefore Syz is equal to 1.8 into 1 - 1/3 Z into 1 - 1/3 Z inverse divided by 1 - 0.6 Z into 1 - 0.6 Z inverse.

Therefore, this spectral factorization theorem we can apply now, that causal filter will be given by, causal part is this 1 - 1/3 Z inverse divided by 1 - 0.6 Z inverse. So, that is the causal filter so that this is my innovation sequence with variance sigma V squared is equal to 1.8 and it is passed through the system, causal system that is HcZ = 1 - 1/3 Z inverse divided by 1 - 0.6 Z inverse and output we will get Yn. So, that way we do the spectral factorization.

Now, we know that Hz that is the transfer function of the causal IIR filter and that is given by 1 by sigma V squared into HcZ into the causal part of Sxyz divided by HcZ inverse. So that way, now we know HcZ is given by this Nxyz is equal to Sxz, so Sxyz is equal to Sxz and Sxz is given by 0.64 divided by 1 - 0.6Z into 1 - 0.6Z inverse. So, we know Sxyz. So therefore Sxyz divided by Hc Z inverse that will be equal to 0.64 divided by 1 - 06Z into 1 - 0.6 Z inverse divided by Hc Z inverse, Hc Z inverse is 1 - 1/3 of Z divided by 1 - 0.6Z.

So, we have to take the causal part of this. This, we can simplify because 1 - 0.6 and 1 - 0.6 will get cancelled, so we will have this is equal to 0.64 divided by 1 - 0.6 Z inverse into 1 – 1 / 3Z. This is the expression for Sxyz divided by Hc Z inverse and we have to take the positive part or the causal part of this. So we found the causal part.

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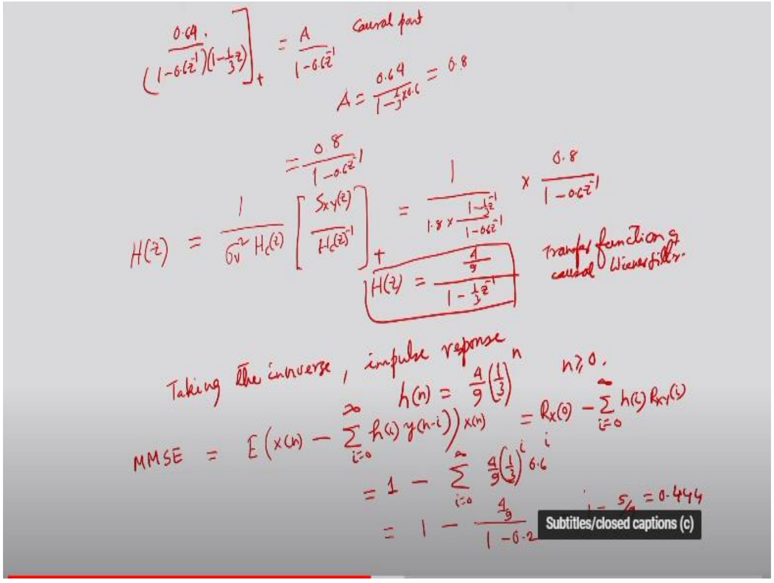


Now we have to find out the causal part of 0.64 divided by 1 - 0.6 Z inverse into 1 - 1/3 of Z. So, positive part of this we have to consider causal part of this. Now, this causal part will involve this because this is the causal transfer function, so that way this will be of this form, this will be equal to A by 1 - 0.6 Z inverse and anti-causal part will be there. This is causal part, because this part will involve the anti-causal part.

Now, to find out A, we will multiply by 1 - 0.6 Z inverse here. This and this will get cancelled and then, we will substitute Z is equal to 0.6, so that way, A will be given 0.64 divided by here we will put that is equal to 1 - 1/3 0.6. This is where we will do partial fraction, so using that we get the result and which is equal to 0.8, because it will be 1 - 0.2 will be 0.8, so this will be 0.8.

So, this part of the causal expression is equal to 0.8 divided by 1 - 0.6Z inverse. So, therefore, HZ is the transfer function of the causal IR Weiner filter is given by 1 by sigma V squared we already out, sigma 1 by sigma V squared into HcZ into Sxyz divided by Hc Z inverse that is the causal part of that. So, causal part of that we already found out. This, we have found out. Sigma V squared we have found out. So, that way this will be equal to 1 by 1.8 into 1 divided

HcZ 1 - 1/3 of Z inverse divided by 1 - 0.6 Z inverse into causal part of this is given by 0.8 divided by 1 - 0.6 Z inverse.

So, if you simplify, we will get this as 4 / 9, so this part and this part will get cancelled, only we have 1 – 1 / 3 Z inverse. This is the transfer function of the causal Weiner filter. This is the transfer function. Now taking the inverse of this, we will get the impulse response which is the causal IR Weiner filter, impulse response hn = 4 / 9, this is 1 / 3 is there into 1 / 3 to the power n, n > 0. So, this is the impulse response of the causal IIR Weiner filter.

We also have to find out the minimum mean square error MMSE, this is equal to E of Xn minus the filter hi y of n - i, i going from 0 to infinity, because error is orthogonal to data, there is only Xn will be here, and because of that it will be Rx of 0 minus summation hi into Rxy square, i going from 0 to infinity, and if we substitute the values of Rx0 and hi Rxy of i this will be equal to 1, Rx0 is 1 minus summation 4 / 9 is there, 1 / 3 to the power i, and Rxyi, that is the same as Rxi that is equal 0.6 to the power i, i going from 0 to infinity.

So, this is the autocorrelation function cross correlation function is same as the autocorrelation function. So, this is an infinite series and carry out this summation, this is an infinite series and common ratio is less than 1, so this will be simply first term, that is i is equal to 0, 4 / 9 divided by 1 minus common ratio, common ratio is 1 / 3 into 0.6. So that way, this will be 1 - 0.2, and this is equal to 1 – 5 / 9, that is equal to 0.444. So, this is the minimum mean square error.

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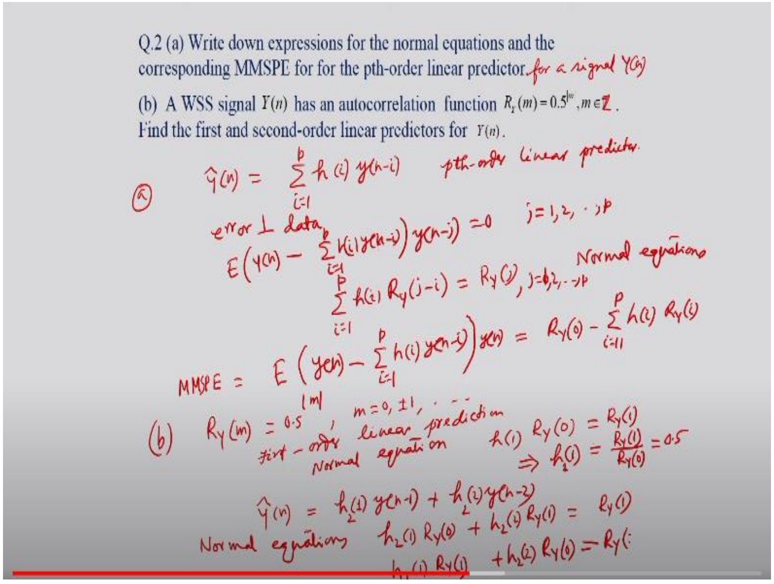

So this is the first problem. So, we will go to the next problem. One is theoretical, write down the expression for the normal equations and the corresponding minimum mean square prediction error for the pth order linear predictor. Suppose; this is for a signal yn. So, yn is this signal, where y head n is equal to summation hi into y of n - i, i going from 1 to p. This is pth order linear predictor.

Now normal equation is obtained from the condition, error is orthogonal to data. So, that way E of yn - summation hi yn - i, i going from 1 to p into yn - j that will be equal to 0, j = 1, 2 up to p, pth order linear prediction. So, from this we will get, we can write the expression first, summation hi into Ry of j – i, i going from 1 to p that must be equal to Ry of j. So this normal, j = 1, 2 up to p. So, this is the normal equation.

Expression for mean square prediction error, minimum mean square prediction error, MMSPE is equal to E of yn, that is the data minus prediction hi y of n - i, i going from 1 to p and there will be y of n here, this error is orthogonal to the remaining part of the error. So, that way this will be equal to Ry0 minus summation hi Ryi, i going from 1 to p. This is the expression for minimum mean square prediction error.

Part b is Rym is equal to 0.5 to the power mod of m, so that is m is equal to 0, plus minus 1 etc. Now, for first order linear predictor. For that, what we will have, so here is p is 1, so there will be one term, so normal equation will be, there will be only term h1 into Ry, i going from 1 to p, p is 1, one term only, Ry of 0 is equal to Ry of 1, imply that h1 is equal to Ry of 1 divided Ry of 0, so that will be equal to 0.5. We know Ry of 1 is equal to 0.5 and Ry of 0 is equal to 1. So that way it is 0.5, which we have found out.

Suppose, this is first order prediction that is we will denote here 1. Now, we will go to the second order predictor. Second order prediction is Y hat n is equal to let me call it h2 of 1 y into n - 1 + h2 of 2 into y n - 2. Now, we will have the normal equations, h2 of 1 into Ry of 0 + h2 of 2 into Ry of 1 that must be equal to Ry of 1. This is one equation. Next equation will be h2 of 1 = Ry of 1 + h2 of 2 Ry of 0, that must be equal to Ry of 2. So, this is the normal equations for second order case. So, here we can substitute the values.

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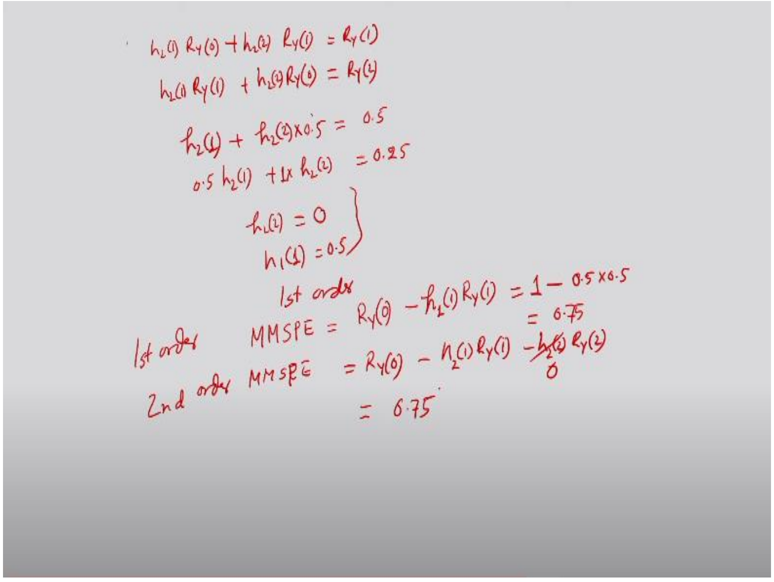


So, here we have h2 of 1 into Ry of 0 + h2 of 2 into Ry of 1 = Ry of 1 that is first equation. Second equation will be h2 of 1 Ry of 1 + h2 of 2 into Ry of 0 that is equal to Ry of 2. So, from this h2 of 1, Ry of 0 + h2 of 2 into Ry of 1 = 0.5, so that will be equal to Ry of 1 is 0.5 and from second equation, 0.5 h2 of 1 x h2 of 2 that must be equal to Ry of 2, that is equal to 0.25. So, these two equations if we solve, we get h2 of 2 = 0 and h1 of 1 = 0.5.

So, this is the linear predictor filter coefficients for order 2 and we observe here that h2 of 2 is equal to zero, because from the data, this is the autocorrelation function of the first order ER process, therefore, if we predict second order terms will be equal to 0. So that is expected. We have to find out the MMSPE also, minimum mean square prediction error, that is for first order case.

Suppose, that will be equal to Ry0 minus h1 of 1 into Ry of 1 and this is equal to Ry of 0 and this is 0.5 into 0.5, so that way it will be 0.75 and in the second case, this is first order and this is second order MMSPE is equal to again Ry0 - h21, Ry1 - h2 of 2 Ry2. Now, this is 0 and this is same as 0.5 and this also equal to 0.75 only.

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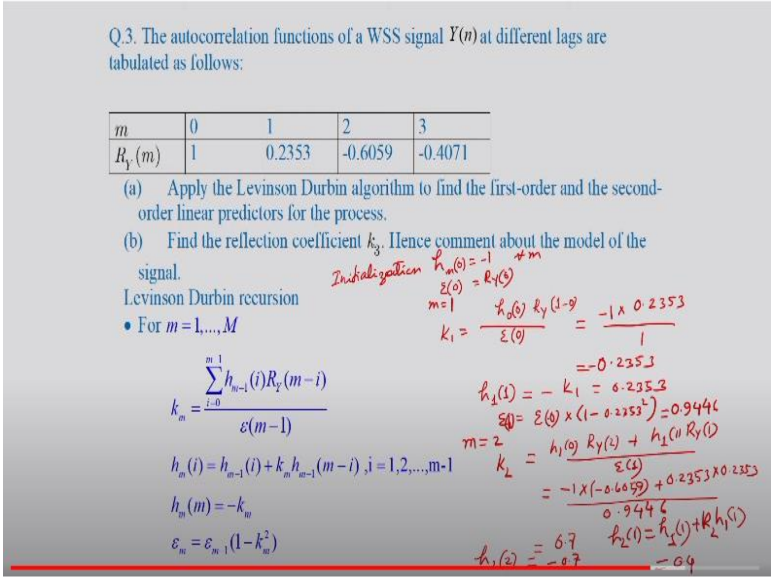


So, let us see the third equation, the autocorrelation function of a WSS signal Y of n at different lags are tabulated as follows. That lag 0, it is 1, lag 1, it is 0.2353, lag 2, - 0.6059, lag 3, - 0. 4071. So, apply Levinson Durbin algorithm to find the first-order and the second-order linear predictors for the process. Second part is, find the reflection coefficient k3, for third order predictor what is the reflection coefficient. Hence comment about the model of the signal.

We know the Levinson-Durbin recursion. This is the recursion. We have to first determine the reflection coefficient that is given by km given by summation h m - 1 i into Ry m - i, i going from 0 to m1, divided by minimum mean square prediction error of the m - 1 order. These are the parameters of the m - 1 th order, so from that, we will determine the km, that is the reflection coefficient of order m.

Then for order m, we will update the filter parameters by this equation, h of mi is equal to h m - 1 i + km h m - 1 m - i and finally hm is equal to –k, hm that is mth order last parameter will be – km and mean square prediction error will be updated by this equation. We also have to initialize the algorithm. So, for that initialization that hm of 0 = - 1, for all m that is one initialization and epsilon 0 is equal to Ry of 0.

Because there is no prediction for 0th order predictor and therefore mean square prediction will be simply will be variance that is Ry0. Now, we have to apply and find out the Levinson-Durbin algorithm to find out the first order and the second order linear predictors. So, suppose m is equal to 1 and then K1 will be equal to from this expression, a sub it is up to m - 1, m is equal to 1.

And it will be 1 term will be there and is equal to 0, h0 of 0 into Ry, m = 1 - 0 divided by mean-square error, that is divided by epsilon 0. So that way, h0 of 0 = - 1 into Ry of 1 = 0.2353 divided by mean-square error, that is epsilon 0, so h0 of 0 = - 1 into Ry of 1 = 0.2353 divided by epsilon 0, that is same Ry of 0 that is equal to 1, so that is equal to minus 0.2353. So, here only one parameter is there, replacement parameter and that will give us the corresponding linear predictor parameter that is h1 of 1 will be equal to negative of this minus of K1 that is equal to 0.2353.

Next, we will go to m is equal to 2 and here we have to update the mean-square error also, before going to m is equal to 2, so, that way epsilon 1, that is the mean-square prediction error. Minimum mean-square prediction error of the first-order predictor, epsilon 1 is equal to epsilon 0 into 1 - km, we have already found out 1 - 0.2353 squared, and that will be equal to 0.9446 and now we will go to m is equal to 2, so k2 we have to determine.

And if I put here, m is equal to 2, so it will be summation from i is equal 0 to 1. Then this term will be h1 of 0 and h1 of 1. So, that way first term will be h1 of 0 into Ry of 2 + h1 of 1 into Ry of 1 divided by mean-square error epsilon 1 and if we substitute the values, h1 of 0 that is equal to -1, Ry2 is given, that is equal to -06059 + h1 of 1, that is equal to 0.2353 into Ry of 1, also is equal to 0. 2353 divided by V1; also we found out 0.9446, this will give us k2 is equal to 0.7.

Therefore, we are interested in second-order parameters. So h2 of 2 will be equal to minus 0.7. We have to find out h1 of 1, h2 of 1 will be equal to, from this that is h1 of 1 k2 into h1, that is 2 – 1, that is h1 of 1. So, this we can again substitute the values and we will get this as 0.4. We know h1 of 1, k2 we know, so therefore it will be 0.4. So, we have found out the first order and second order linear predictor.

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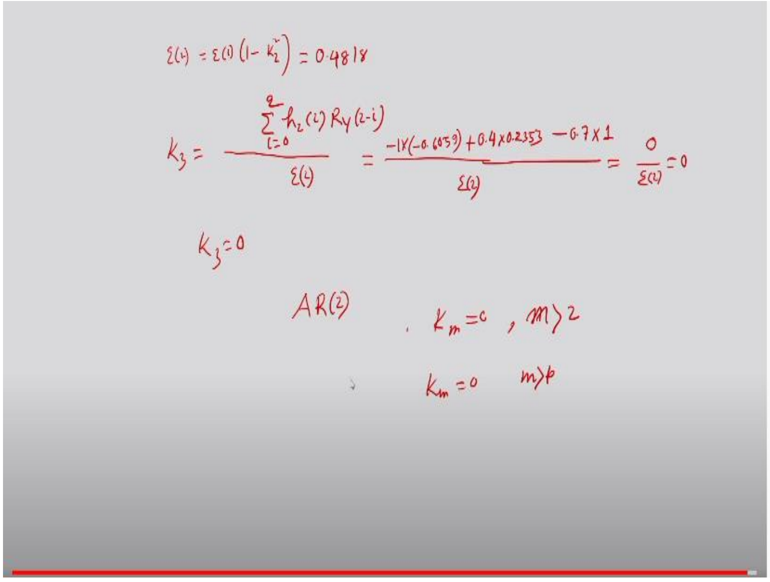


We have to find out k3. Once, we have these values h2 of 1, h2 of 2, we can find out h2 of 2 is equal to - 0.7 and h2 of 1 is equal to 0.4, and now we can update the mean-square prediction error, that is the mean-square prediction error, Epsilon 2 is equal to Epsilon 1 minus K2 squared. So, that will be Epsilon 1, mean square prediction, for a first order predictor we know and K2 we know. So substitute, this will be equal to 0.4818.

Complete the two recursion in Levinson-Durbin algorithm to find out the first order and second order model. Now, let us go to find out K3. K3 is equal to summation h2 of i Ry of 2 - i, i going from 0 to 2, divided by epsilon 2, minimum mean-square error at iteration 2. Now, we have already values of this, so this will be equal to first one is i is equal to 0, h2 of 0 is - 1 and Ry of 2, that is equal to minus 0.6059, then second term will be h2 of 1, that is 0.4 into Ry of 1, that is 0.2353 - h2 of 2 that is - 0.7 into Ry of 2 - 2 is 0, Ry of 0 is 1.

So, this is divided by Epsilon 2, that is the mean-square prediction error at the order 2 and this part, if we carry out the algebra, this will be equal to 0 by epsilon 2, that is equal to 0. So, K3 is equal to 0. That is 3rd order replacement coefficient is equal to zero. Since K3 is equal to 0, that optimal linear predictor will be of order 2 only. So, that is our observation that optimal linear predictor will be of order 2 only.

We will not get any benefit by increasing the order and this is why because the corresponding model is AR2 model. So, that is the advantage of this replacement coefficient. Since it is an AR2 model, so after m is equal to 2, Km will become zero, so this is the observation that since it is an AR2 model, here after m is equal to 2, that is K3 is equal to 0, therefore it will be an AR2 model, after m is equal to 2, this replacement coefficient Km = 0. Thus, K3 = 0. So, from this we can say that, we can get the optimal linear predictor of order 2 only, by increasing the order we do not improve the prediction.

Secondly, this K3 = 0 happens when the corresponding model is AR2, so for AR2 model, this replacement parameters will become 0. So, that is a test for AR2 model. So, if it is an AR2 process, it will be equal to 0. Similarly, if it is an ARP process, then Kp + 1 will be equal to 0. So, K3 = 0. It implies that, we do not get any improvement by increasing the linear-prediction model order beyond 2 and this happens because corresponding signal model is AR2 signal, because for AR2 signal, that Km will be equal to 0 for m > 2. Similarly, for any ARP signal, Km will be equal to 0 for m greater than p. So this is the importance of the replacement coefficients. Thank you.