Statistical Signal Processing
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Lecture - 31
Solution to Review Assignment 3

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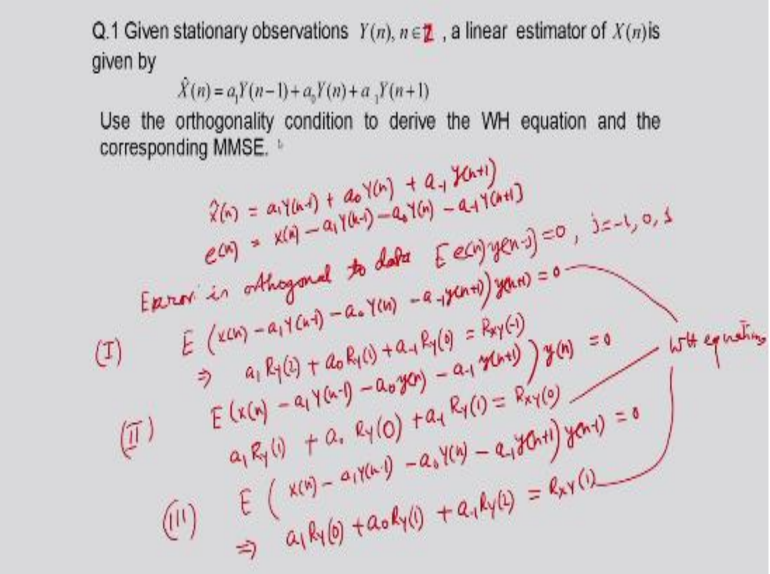

Hello students. Welcome to this session on solution to review assignment 3. Here, we will solve some problems of Wiener filter. First one is given a linear estimator of Xn is given by X hat n = a1 times Yn – 1 + a0 times Yn + a – 1 times Yn + 1. Use the orthogonality condition to derive the Wiener-Hopf equation and the corresponding minimum mean square error. Signal is WSS and X hat n is a1 Yn – 1 + a0 into Yn + a – 1 into yn + 1 filtered data is also used.

Therefore, error en will be = xn – a1 Yn -1 – a0 into Yn – a – 1 into Yn + 1. Now the orthogonality condition is error is orthogonal to data. So we will get that orthogonality condition that means of E of en into data, data is now 2 data is involved that is y of n – j that = 0 for j = - 1 0 and 1 because when j = - 1 this is yn + 1 j = 0 it will be yn and j = 1 it will be yn – 1. So that way 3 equations we will get.

Therefore, we will have E of xn – a1 Yn – 1 – a0 into Yn – a – 1 into yn + 1 whole into first one is y of n + 1 = 0 that is number one equation, equation 1 this is one equation. So that way we will get now imply that we will first compute this one so E of this one will be y of n – 1 into y of n + 1 so that way it will be Ry of 2. So that way a1 we take these values to the right hand side.

So we will consider this part first a1 into RY n – 1 – n – 1 so that way it will be Ry of 2 because of this stationarity assumption and similarly this term will be a0 into Yn into yn + 1 expected value of that, that will be RY of 1 and this term will be + a – 1 times E of yn + 1 into yn + 1 into yn + 1 that will be equal to RY of 0. So whole will be equal to now E of xn into yn + 1 so that way it will be Rxy n – n – 1 RxY of – 1.

So this is equation 1 then second equation we will get 2 equation 2 E of xn – a1 into Yn – 1 – a0 into yn – a – 1 into yn + 1 into yn = 0 that is corresponding to j = 0. So from this we will consider this first part so we will get a1 times E of Yn – 1 into yn so that way it will RY of 1 + next we will consider a0 times a0 yn into yn RY0 RY of 0. Then we will consider this one yn + 1 into yn + 1 into yn so that will be equal to + a – 1 into RY of 1.

E of yn + 1 into yn that will be equal to RY of 1 that must be = E of xn into yn that equal to RxY of 0 so this is the second equation. Third equation we will get that is E of error, error is xn – a 1 Yn – 1 – a0 into Yn – a – 1 into yn + 1 into now that equal to 1 so yn – 1 = 0. So from that what we will get we will consider this one first so that way a1 times E of yn – 1 into yn - 1 that way it will be RY of 0 + a0 times E of Yn into yn – 1 so that it will be RY of 1.

And this term will be a – 1 into E of yn + 1 into yn – 1 so that way this will be RY of 2. So that will be equal to now xn yn -1 E of xn into yn – 1 that will be RxY of 1. So that way we have 3 equations. First one is this equation second one is this equation and third one is this equation these are the Wiener-Hopf equations WH equations or normal equations. Next is use orthogonal to derive the corresponding minimum mean square error.

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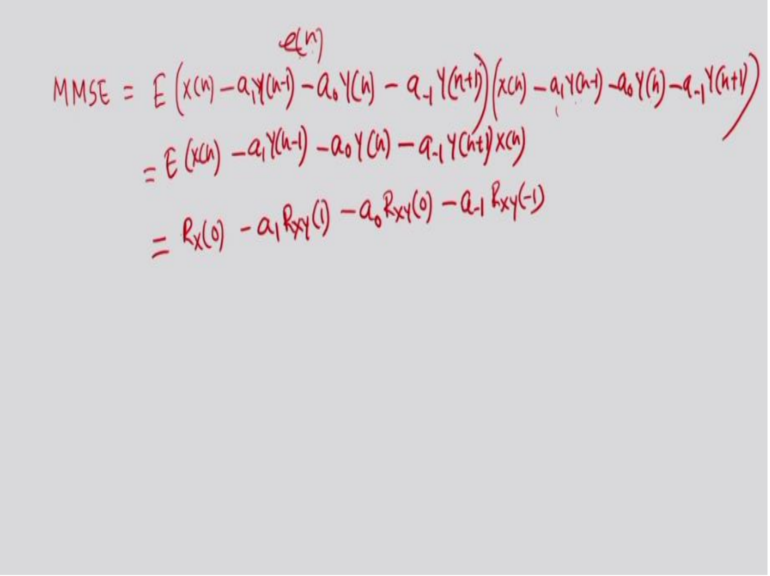

So here also we can use the orthogonality that is minimum mean square error E = E of xn – a1 Yn – 1 a0 Yn – a – 1 into Yn + 1 so this is the error into error square actually. So this we can write like this that is xn – a1 Yn – 1 – a0 into Yn – a – 1 into Yn + 1. So this is the expression now we will use the orthogonality principle this is the error, error is orthogonal to data so error is orthogonal to Yn – 1, error is orthogonal to Yn, error is orthogonal to Yn + 1 because these are the data.

So that way what we will have this contribution of this part will become 0 so that way we will have E of xn – a1 into Yn – 1 – a0 into Yn – a – 1 into Yn + 1 into xn because contribution of this part will be 0 E of en into Yn – 1 will be 0 E of en into Yn will be = 0 E of en into Yn + 1 will be equal to 0. So that way we will be left with this quantity only and this equal to xn into xn E of xn into xn will be Rx of 0 – a1 E of xn into Yn – 1 will be RxY of 1 a0 into E of Yn into xn that will be equal to RxY of 0 and similarly this one will be E of Yn + 1 into xn. So that way in terms of RxY it will be n – n – 1 RxY of – 1. So this is the MMSE minimum mean square error.

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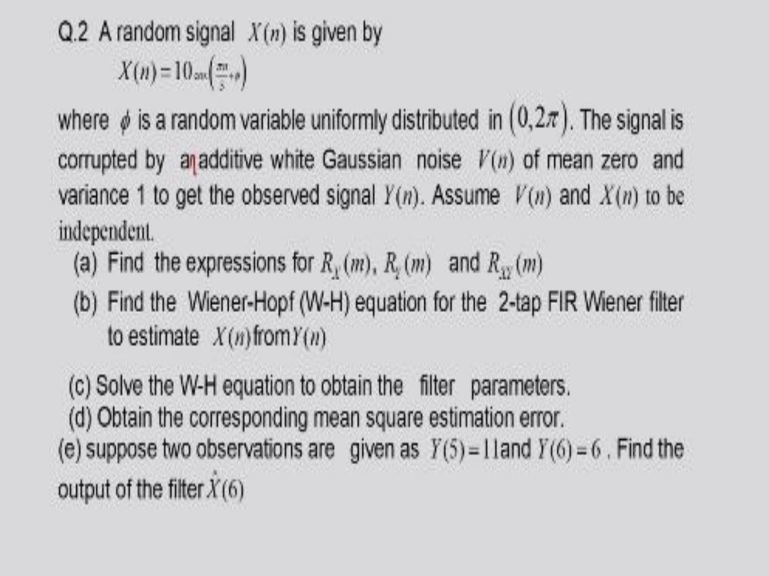

Let us consider question 2. A random signal Xn is given by Xn = 10 times cos of pi n/3 + phi where pi is a random uniformly distributed in the interval 0 to 2 pi. Pi is uniformly distributed in the interval 0 to 2 pi. This signal is corrupted by an additive by an additive white Gaussian noise Vn of mean zero and variance 1. This variance is 1 variance of Vn is 1 to get the observed signal Yn.

Assume Vn and Xn to be independent so that Xn and that additive noise part are independent. Find the expressions for RX of m, RY m and RXY m. Similarly question B is find the Wiener-Hopf equation for the 2-tap FIR Wiener filter to estimate Xn from Yn. Part C is solve the Wiener-Hopf equation to obtain the filter parameters. Then part D is obtain the corresponding mean square estimation error. Suppose 2 observations are given as Y5 = 11 and Y6 = 6 find the output of the filter X hat 6. So at instance 6 what is the output of the filter.

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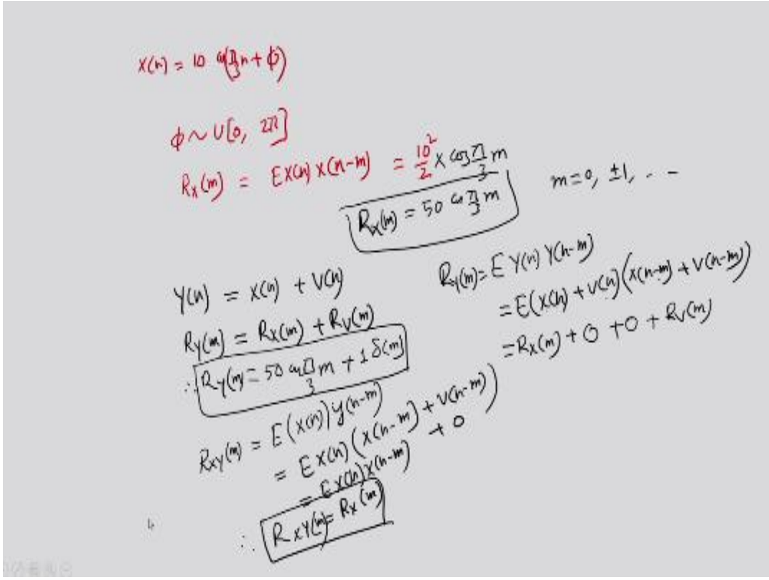

To solve this question, we are given Xn = 10 cos of pi / 3 n + phi this is the signal. Now phi is uniformly distributed phi is uniformly distributed between 0 to 2 pi. So we have to find out RX of m RX of m will be equal to E of Xn into X of n + m (()) (14:27) n – m and if we carry out this expectation operation and as we have shown in an earlier example so this will be simply 10 square / 2 into cos of pi/3 m that equal to 50 cos pi/3 m so this is for m = 0 + - 1 etcetera.

And we have to find out RX of m RY of m. Now you are given that Yn = Xn + Vn so Vn and Xn are independent and in this case we can show that RY m will be equal to Rx of m + Rv of m. How do I show this E of Yn into Yn – m that is my RY of m. So this equal to E of Xn + Vn into Yn – m will be X of n – m + V of n – m and this Xn into X of n – m if I take the expectation this one will be Rx of m.

Now this is E of vn into X of n – m because noise and signal are independent. Therefore, this will be E of vn into E of xn – m but vn is zero mean so therefore that will be equal to 0. So because of independent assumption this part will be equal to 0. Similarly, Xn into V of n – m that will be also equal to 0 because this signal and noise are independent and therefore V of xn into V of n – m will be equal to E of xn into E of vn – m, but E of vn – m = 0 therefore this cross term is also equal to 0.

And this part E of vn into V n – m that will be RV of m. Therefore, we will have RYm = RXm + Rvm and I know RX of m that is 50 cos of pi/3 m and Rv of m now noise is unity variance zero mean uncorrelated. Therefore, this will be 1 into variance is 1 into delta m where delta m = 1 for m = 0 and 0 otherwise. So this is the value of RYm then we have to find out RxYm will be = E of Xn into yn – m so this = E of Xn into X of n – m + v of n – m.

Now signal and noise are independent and also this zero mean therefore we will have this E of xn into vn – m will be simply = 0. So what we will be left with E of xn into x of n – m + 0 that = Rx of m so therefore RxY of m will be simply Rx of m. So, next part is so we have to now find out the Wiener-Hopf equation for 2-tap FIR Wiener filter. So that way we have found out Rx of m we have found out this is Rx of m Rx of m equal to this, this we have found out.

Similarly, RY of m equal to; therefore RY of m equal to this, this also we have found out and we have found out RxY of m is m also we have found out. So these are the expression for Rx of m, RxY m and RY of m.

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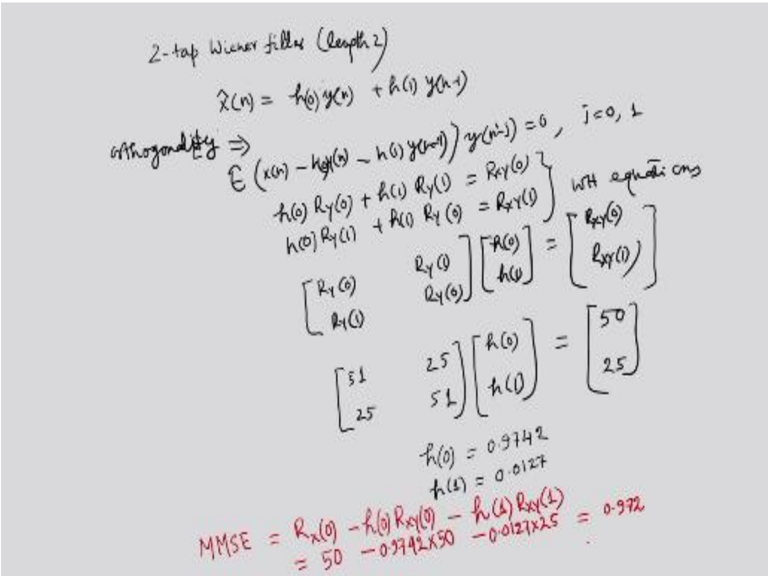

Next we have to find out the Wiener-Hopf equation so it is a 2-tap Wiener filter 2-tap Wiener filter this is length-2 Wiener filter where this is length 2. So that therefore estimated signal x hat n will be = h of 0 into yn + h of 1 into yn – 1 this is the filter equation and filter output is the estimated signal and using orthogonality condition E of xn – h 0 Yn h0 yn – h1 yn – 1 into yn – j = 0 for j = 0 and 1.

So this is the condition so from this we will get h0 into RY of this one will give RY of for j = 0 I will h0 into RY of 0 + h1 into RY of 1 will be = xn into yn that is RxY of 0 this is the first equation. Second equation when we put j = 1 so that way this one will be h0 into yn into yn –

1 expected value so that way it will be RY of 1 and similarly this part h1 into RY of 1 and here we will have RxY of 1.

So this two equations we have these are the Wiener-Hopf equations. So we can substitute the values now. So in matrix form also we can write it as RY0 here RY1 here RY1 here RY0 here and then into h0 h1 here must be equal to RxY 0 RxY 1 and we can substitute the value so RY0 is from this expression RY0 will be 50 because cos of 0 will be equal to 1 this is 1 so that way it will be 51 so that way this will be 51.

RY1 will be RY1 will be equal to 50 into cos pi/3 that will be 25 cos of pi / 3 is half and this is 0 so that way this is 25 here also 25 here 51 and into h0 h1 and this side similarly we will have Rx of 0 that is 50 and this is 25. So this is the Wiener-Hopf equation we can directly solve this either two linear equations or by matrix inversion and we can get h0 = 0.9742 and h1 = 0.0127 so these are the filter coefficients.

So we have found out h0 h1 so MMSE that is minimum mean square error that is equal to Rx0 – h0 into RxY0 – h1 into RxY 1. So all these data are available Rx0 is 50 and h0 we have found out 0.9742 into RxY0 RxY0 was 50 only – h1 h1 was 0.0127 and RxY of 1 is 0 that is 25 so if we carry out this calculation this will be 0.972. So we have found out the minimum mean square error. Next we are given Y5 = 11 Y6 = 6 to find the filter output X hat 6 so we have to estimate the signal at instant 6.

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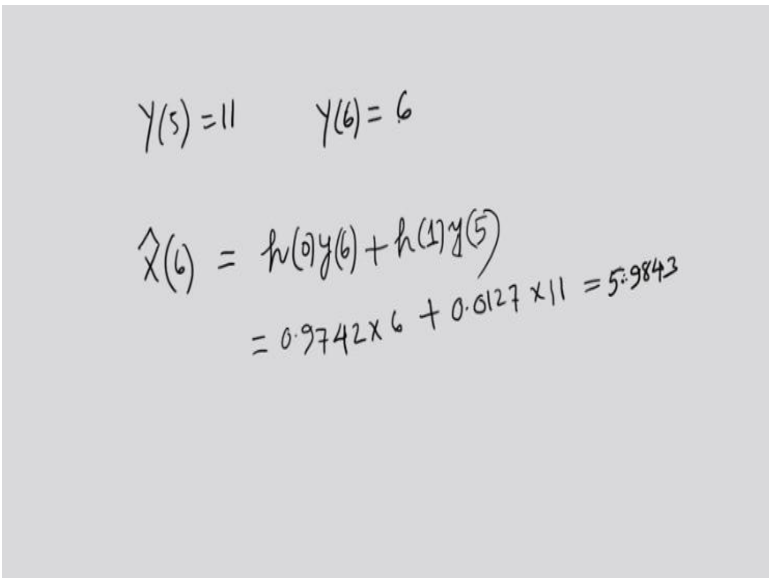

So we are given Y5 Y5 = 11 and Y6 = 6 so we have to estimate X hat 6. So X hat 6 that equal to h0 into y of 6 + h1 into y of 5. So we can substitute the values so this equal to h0 is 0.9742 multiplied by y6 that equal to 6 + h1 is 0.0127 multiplied by y5 that is 11. So this we will get as 5.9843 so this is the estimated value of signal at 0.6 so X hat 6 is 5.9843 observed value was 6 estimated value is 5.9843.

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Q.3 Assume the observation model $Y(n)=X(n)+V(n)$ where $V(n)$ is a zero-mean white noise with variance 1 ~~noise with variance~~ 1 and $X(n)$ has an autocorrelation function $R_X(m)=0.6^{|m|}, m\in\mathbb{Z}$.

(a) Find the expressions for $S_X(z)$, $S_Y(z)$ and $S_{XY}(z)$

(b) Find the transfer function of the noncausal IIR Wiener filter to estimate $X(n)$.

(c) What is the transfer function of the optimal non-causal Wiener filter if $V(n)=0$ in the model?

Let us go to the next question assume the observation model Yn = Xn + Vn where Vn is a zero mean white noise with variance 1 and Xn has an autocorrelation function RXm = 0.6 to the power mod of m where m belongs to z integer. Find the expression for SXz SYz and SXYz. Part B is find the transfer function the non-causal IIR Wiener filter to estimate Xn. Part C is what is the transfer function of the optimal non-causal Wiener filter if Vn = 0 in the model.

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$$R_x(m) = 0.6^{|m|}$$

$$S_x(z) = \sum_{m=-\infty}^{\infty} R_x(m) z^{-m} = \sum_{m=-\infty}^{\infty} 0.6^{|m|} z^{-m} = \sum_{m=-\infty}^{-1} 0.6^{-m} z^{-m} + \sum_{m=0}^{\infty} 0.6^{m} z^{-m}$$

$$= \sum_{l=1}^{\infty} 0.6^{l} z^{l} + \sum_{m=0}^{\infty} 0.6^{m} z^{-m}$$

$$= \frac{0.6z}{1-0.6z} + \frac{1}{1-0.6z^{-1}}$$

$$= \frac{0.6z(1-0.6z^{-1}) + (1-0.6z)}{(1-0.6z)(1-0.6z^{-1})} = \frac{0.64}{(1-0.6z)(1-0.6z^{-1})}$$

$$R_y(m) = R_x(m) + \overset{1\,\delta(m)}{R_v(m)}$$

$$\therefore S_y(z) = S_x(z) + S_v(z) = \frac{0.64}{(1-0.6z)(1-0.6z^{-1})} + 1 = \frac{2 - 0.6(z+z^{-1})}{(1-0.6z)(1-0.6z^{-1})}$$

$$R_{xy}(m) = R_x(m)$$

$$\therefore S_{xy}(z) = S_x(z) = \frac{0.64}{(1-0.6z)(1-0.6z^{-1})}$$

So we will solve this we are given Rx of m = 0.6 to the power m Rx of m = 0.6 to the power mod of m. So to find out the non-causal Wiener filter transfer function we have to find the Generalized power spectral density. So we will find out SXz. SXz that equal to by definition RX of m into z to the power – m m going from minus infinity to infinity so that way this will be summation m going from minus infinity to plus infinity 0.6 to the power mod of m into z to the power – m.

And this we can write as for negative m this will be 0.6 to the power – m so that way 0.6 to the power – m z to the power – m m going from minus infinity to – 1 + 0.6 to the power m for positive m mod of m means equal to m into z to the power – m m going from 0 to infinity. Now this we can further simplify by putting minus m = l so that this summation can be rewritten as 0.6 to the power l z to the power l l is going from 1 to infinity.

And this is one is summation m going from 0 to infinity 0.6 to the power m z to the power – m. These are geometric series provided the common ratio less than 1 we can find out the infinite sum and in the region of convergence we will have the summation like this for this first term will be 6 z divided by common ratio is 0.6 z 1 – 0.6 z and this part is first term is 1 so 1 – common ratio is 0.6 into 6 to the power – 1.

So this is the SXz and this we can simplify as 1 – 0.6 z into 1 – 0.6 z inverse and here if we multiply by 0.6 z into 1 – 0.6 z inverse + 1 – 0.6 z. So if simplify then we will get because this 0.6 z and – 0.6 z will get cancel 0.6 z inverse will be equal to - 0.36 so that way the result will be 1 – 0.36 that is 0.64 divided by 1 – 0.6 z into 1 – 0.6 z inverse. So we have found out SX of z now signal and noise are uncorrelated.

Therefore, RYm autocorrelation of the output will be equal to Rxm + Rvm v is the white noise. So taking the z transform we will get SYz = Sxz + Svz. Now this we know this quantity is 1 into delta m because variance 1. Therefore, how we can find out the z transform of this that will be simply 1. So will be we have found out 0.64 / 1 – 0.6 z into 1 – 0.6 z inverse plus this one is 1. So if I carry out the algebra we will create this as denominator is 1 – 0.6 z into 1 – 0.6 z inverse.

And numerator will be 2 – 0.6 into z + inverse also RXYm in this case will be equal to Rx of m only. Therefore, SxYz will be equal to Sx of z that is equal to we have already know this that is equal to 0.64 divided by 1 – 06 z into 1 – 0.6 z inverse. So this we have found out.

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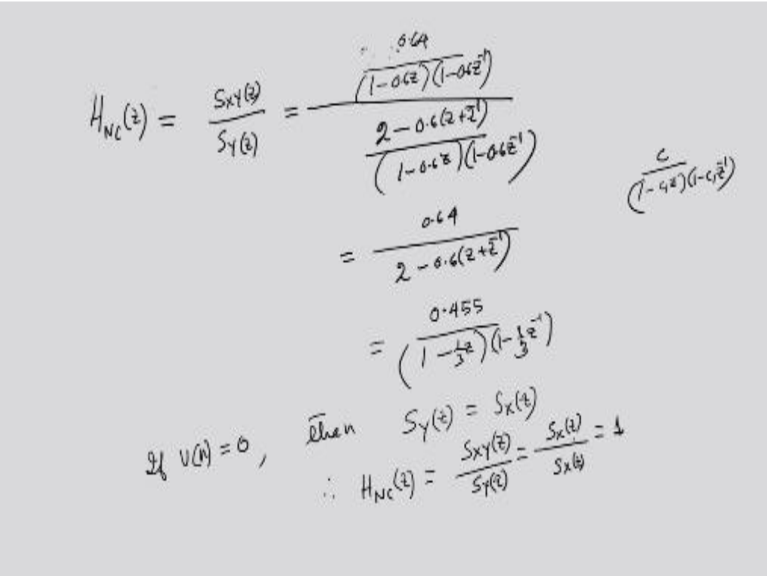

We can find out now the transfer function of the non causal Wiener filter that is HNCz that equal to SxYz / Syz and this we have already found out 0.64 / 1 – 0.6 z into 1 – 0.6 z inverse / 2 – 0.6 z + z inverse / 1 – 0.6 z into 1 – 0.6 z inverse. So if we simplify we will get as 0.64 / 2 – 0.6 into z + z inverse. This we can further write in this standard form in the form like this suppose maybe some constant / 1 – c1z into 1 – c1 z inverse.

So in this standard form we can write so z form it will come out we will see later on how to get this type of expression, but essentially this has to be factorized. So we will get this as 0.455 / 1 – 1 / 3 of z into 1 – 1 / 3 of z inverse. So this is the transfer function of the non-causal Wiener filter HNCz. Next part is what happened when vn = 0. So if vn = 0 then SYz will be simply equal to Sxz. Therefore, HNCz will be that equal to SxYz divided by SYz and this we have shown that this equal to Sxz divided by Sxz that equal to 1 because SYz = Sxz therefore it will be 1 that means the signal will be passed as it is. Thank you.