Statistical Signal Processing
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Lecture – 24
Solution to Review Assignment 1

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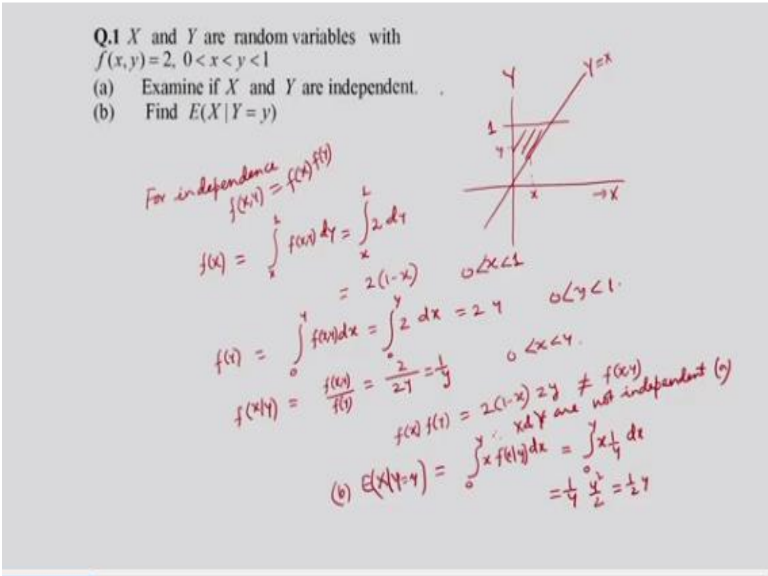

Hello students, welcome to this session on the solution to the review assignment 1, let us discuss the first question, let us solve question 1; X and Y are random variables with joint PDF, f xy is equal to 2 for 0 < x < y < 1. Examine if X and Y are independent, find the conditional expectation of X, given Y is equal to small y, this conditional expectation is very important for us.

Because this is the minimum mean square error estimator of X given Y is equal to small y, now the PDF is defined in this region, this is x, this is y and this is y is equal to x line, now 0 is < x < y, therefore y is greater than x, so it will be above y axis and it is bounded by 1 through that way this is the region, where this joint density function is defined. Now, to show independence, we have to find out this marginal density and fxy, this joint density should be equal to product of fx into fy.

This is the condition for independence now, we will find out fx, so fx is equal to; now suppose this is a point x, this is the point x, so at that point, then (()) (02:42) will be equal to; now we have to integrate joint density with respect to y and limit of integration will be from this point to this point, so that way this will be integration, this would point is equal to x and this is 1 and fxy dy.

So, this is; if we carry out the integration this is x1, this is 2 dy, so this will be 2 into; because integration of dy is y and if we put the limits, upper limit is 1, lower limit is x, therefore this will be 2 into 1 – x, now x will lie between 0 and 1. Similarly, fy we can find out; fy is equal to integration now, we have to find out at this point y, suppose this is equal to y, therefore x will be varying from 0 to y.

So, that way integration 0 to y, fxy dx that will be equal to 0 to y integration, fxy is 2 dx, so this will be simply 2y, y is lying between 0 and 1, therefore f of x given y, this is the conditional PDF will be equal to f of xy divided by fy and that will be 2 by 2y that is equal to 1 by y, this is for x lying between because once we have y; given y, x will lies between 0 and y, okay.

Now, we see that f of xy which is equal to 2 is not equal to fx into fy because fx into fy is equal to fx we have find out, 2 into 1 – x into 2 into y is not equal to f of xy, therefore X and Y are not independent, so this is the answer to part A. Now, part B we have to consider, E of X given Y is equal to small y, so part B; E of X given Y is equal to small y, so this is equal to; now we know the original PDF as a function of x we have to consider.

So, it is integration which respect to X because we are finding out the expectation of X given Y is equal to small y, so that way this will be x f of x given y dx and limit will be; x will be changing from 0 to y, so this is equal to integration 0 to y x, now this is 1 by y dx; 1 by y will come out, so this is x square by 2 and the upper limit is y, therefore it will be y square by 2 that is equal to 1/2 of y.

So, conditional expectation of X given Y is equal to small y is equal to 1/2 of y, so this is the conditional expectation, this conditional expectation we know that this is important for minimum mean square error estimation, so that way how to find out conditional expectation we have seen here.

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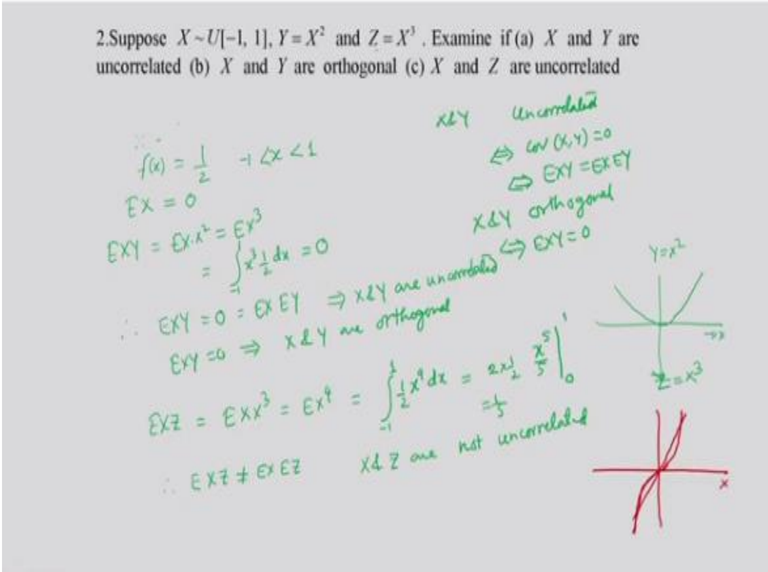

We will go to the next question; suppose, X is uniformly distributed between - 1 and 1, Y is equal to X square and Z is equal to X cube, examine if X and Y are orthogonal, if X and Z are uncorrelated, so these 3 questions we will solve. Now, we know that 2 random variables X and Y are uncorrelated, so that is covariance of uncorrelated X and Y uncorrelated imply that covariance of XY is equal to 0, which also implies that E of XY is equal to EX into EY.

And X and Y are orthogonal, so this is if and only if, this also we can write if and only if, E of XY is equal to 0. Now, coming to this question, so X is uniformly distributed between - 1 and 1, so that means fx will be equal to 1 by 2, X lies between - 1 and + 1, so this is the distribution, therefore E of X will be equal to because it is symmetrical distribution between - 1 and 1, so you have X is equal to 0.

Now, let us see what is E of XY; E of XY and that is nothing but E of X into Y is equal to X square, so into X square, so that is equal to E of X cube, now you have X cube we can compute and this is integration from - 1 to 1 X cube and then density is 1 by 2 dx and this is again identically 0, so E of XY is equal to 0 and therefore, we see if X and Y are uncorrelated now, clearly E of XY is equal to 0 is equal to EX.

Because EX is also equal to 0 into E of Y because EX is equal to 0, already this is equal to 0, so therefore E of XY is equal to EX into EY implies that X and Y are uncorrelated and also E of XY is equal to 0, this implies that X and Y are orthogonal. Now, let us consider part C; E of XZ is equal to E of X into X cube that is equal to E of X to the power 4 and this will be now equal to minus 1 to 1, 1/2 that is the PDF x to the power 4 dx.

So, this integral okay, we can carry out this integral, this will be 2 into 1/2 into x to the power 5 by 5 and limit is 1, 0 so that way this will be 1 by 5, therefore E of XZ is not equal to E of X into E of Z because E of X is equal to 0, this right hand side is 0 but this left hand side is not equal to 0, therefore E of XZ is not equal to EX into EZ, therefore X and Z are not uncorrelated, they are correlated actually.

So, here we see that Y is a function of X and Z is also function of X and Y are uncorrelated, whereas X and Z are correlated, so this is because of the nature of the function for example, Y is equal to X square, if I plot suppose, this is X, Y is equal to X square, so this will be a parabola like this, so we cannot have a linear approximation of the parabola except that what is the mean suppose, mean we can find out that way we can approximate but we cannot get a line which approximate this parabola.

For the cubic relationship, Z is equal to X cube, this side is X we have the curve like this okay, so that we can approximate this by a line like this, so this linear approximation is possible and that is what correlation means but in this case, that type of linear approximation is not possible.

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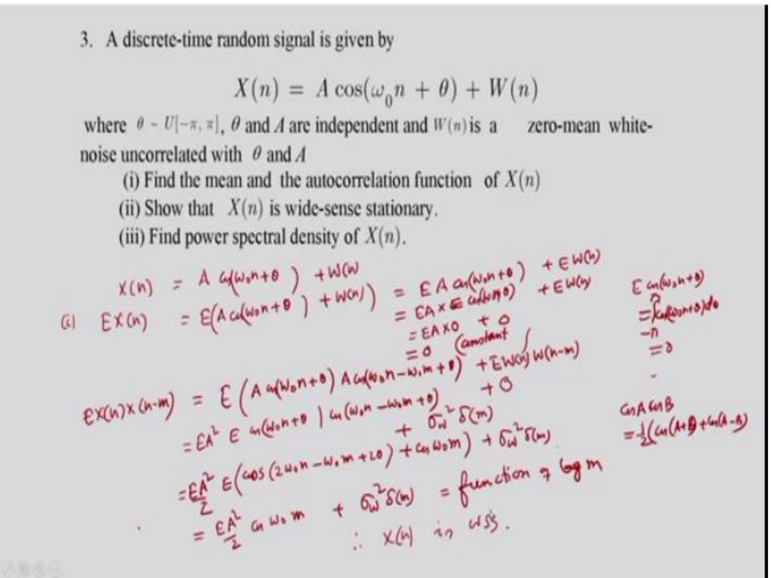

Let us go to the third question; a discrete-time random signal is given by Xn is equal to A cos omega 0n + theta + Wn, where theta is a uniform random variable distributed between minus pi to pi and theta and A are independent, this A and theta are independent and Wn is a zero mean white noise uncorrelated with theta and A, so Wn is uncorrelated with this random variable and this random variable.

Find the mean and the autocorrelation function of Xn, so that Xn is WSS, find the power spectral density of Xn, so in this case find the mean and autocorrelation of Xn, so let us solve the problem. We are given Xn is equal to A cos of omega 0n + theta + Wn and A and theta are independent and Wn is uncorrelated with theta and A and now, you have Xn, so we will consider part 1.

E of Xn will be equal to E of A cos omega 0n + theta + Wn, so expected value of the entire thing and this is some therefore expected value will be also some E of A cos of omega 0n + theta + E of Wn and A and theta are independent, so we can write E of A into cos of omega 0 + theta + E of Wn and we are given that theta is uniformly distributed between minus pi and pi.

Therefore, E of cos of omega n + theta omega 0n + theta, so this term will be equal to 0, so that way this will be EA into 0 because this term we can determine E of cos of omega 0 n + theta, this is integration from minus pi to pi cos of omega 0n + theta d theta, so this integration if we carry out this will be equal to 0, so and this part is also 0, so that way this will be 0, which is a constant.

So, mean is a constant, it does not depend on n and now, we will see the autocorrelation function E of X of n into X of n – m, so that is equal to E of A cos omega 0 n + theta into A cos of omega 0n - omega 0m + theta, so this is the at lag n – m and then E of Wn into Wn - m and plus; because this Wn and this signal are uncorrelated, therefore this joint expectation will be equal to 0 and all are zero mean that way this joint expectation will be equal to 0, so this will be 0.

Now, if I carry out this expectation, I will get this as this A and A will be A square, E of A square into E of cos of omega 0 n + theta into cos of omega 0 n - omega 0 m + theta and this one, plus this quantity is because it is the white noise within it variance 0 mean white noise, so that way will suppose, it variance is sigma W square, so that way we can write it as sigma W square into delta m, where delta m is equal to 1, for m is equal to 0 otherwise it will be 0.

Now, this part I can expand in terms of cosine formula; cos A into cos B is equal to 1/2 of cos of A + B + cos of A – B, so that way we will get this as E of A square into E of cos of 2 will be also there; cos of, if we add both twice omega 0 n - omega 0 m + 2 theta, this is the one corresponding to A + B and if we subtract we will get, cos of this and this will get cancelled, theta will get cancelled, cos of omega 0m + sigma W square delta m.

And this part again we can show that expected value of this part will be equal to 0 because if I integrate this function over minus pi to pi, then I will get exactly 0, so that way we will be left with E of A square by 2 into cos of omega 0m + sigma W square delta m, where sigma W square is the variance of the white noise. Now, we see that this is a function of m only; this is a function of lag m.

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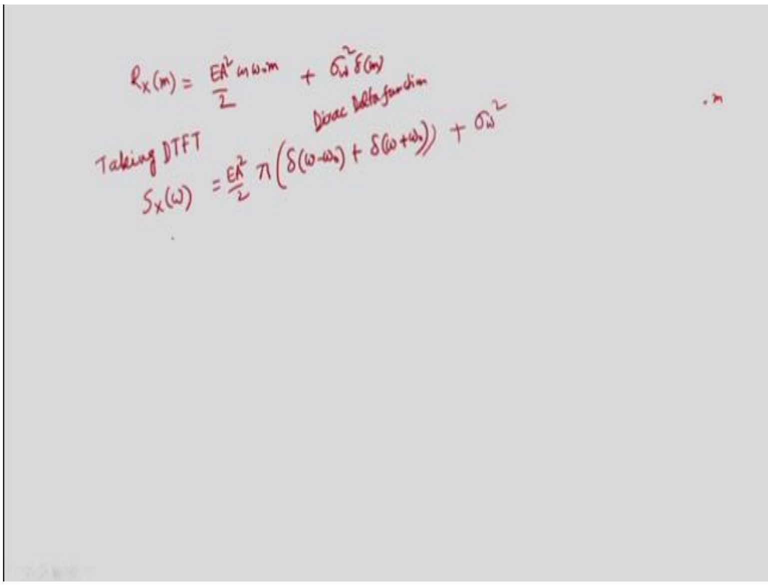

And here we have seen that E of Xn is equal to 0 which is a constant, therefore Xn is WSS, now we have to find out the power spectral density we have Rx of m is equal to E of A square into cos of omega 0m divided by 2 + sigma W square into delta m, we have the autocorrelation function like this, so by taking the DTFT now, we will get the PSD Sx omega that will be DTFT of this E of A square by 2 and this DTFT is given by pi into delta omega - omega 0 + delta omega + omega 0, where delta is the dirac delta function.

This is Dirac delta plus and DTFT of this term will be equal to simply sigma W square, so this is the power spectrum of the signal Xn.

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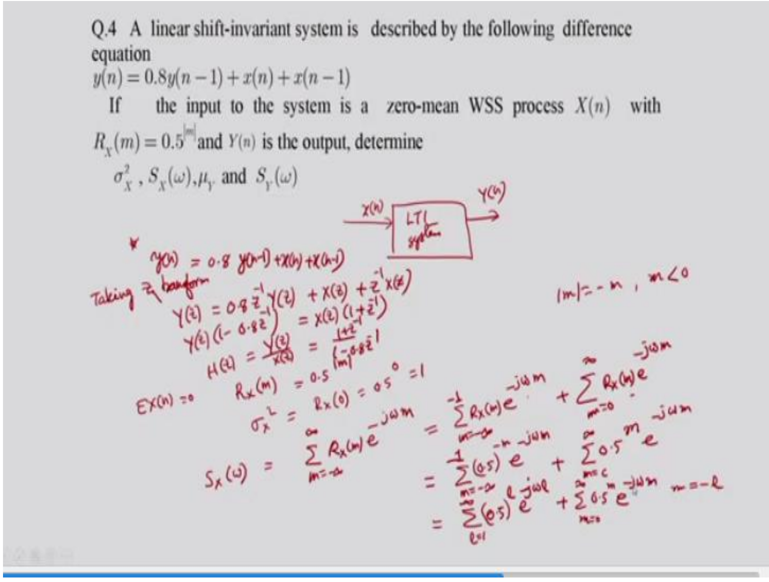

Let us go to question number 4; a linear shift invariant system is described by the following difference equation; yn is equal to 0.8 yn - 1 + xn + xn – 1, if the input to the system is a zero mean WSS process Xn with Rx of m is equal to 0.5 to the power m and Yn is the output, determine sigma x square SX omega mu Y and SY omega, so this is this signal xn suppose and this is the LTI system, linear shaped invariant system.

In discrete case, we will call it linear shaped invariant system, so that way linear shift invariant or time invariant system and this is yn, so we have yn is equal to 0.8 y n - 1 + xn + x of n – 1, now we want to find out the transfer function of this system for that we will take Z transform, taking Z transform and assuming that initial conditions are 0, so we will get Yz is equal to 0.8 into z to the power - 1 into Yz + Xz + z to the power - 1 into Xz.

So, we can write by rearranging the term we can write, Yz into 1 - 0.8 z inverse is equal to Xz into 1 + z inverse, so that Xz is equal to by definition it is Yz by Xz is equal to 1 + z inverse divided by 1 - 0.8 z inverse, so this is the transfer function and now given that E of Xn is equal to 0 and Rx of m is equal to 0.5 to the power mod of m, so we have to find out sigma x square, so sigma x square will be simply because it is 0 mean, sigma x is equal to Rx of 0 that is equal to 0.5 to the power 0 is equal to 1.

Then we will have to find out Sx of omega, Rx of m is given that is equal to summation Rx of m, e to the power - j omega m; m going from minus infinity to infinity. Now, my autocorrelation function is in terms of mod of m, therefore I will write separate out the summation, so first part will be summation Rx of m, e to the power – j omega m, m going from - infinity to - 1 + summation from m is equal to 0 to infinity Rx of m, e to the power - j omega m like this.

So, this now I know that this is equal to Rx of m is equal to 0.5 to the power mod of m and when m is negative, mod of m is equal to – m for m less than 0, so therefore this sum will be equal to 0.5 to the power - m e to the power – j omega m; m going from minus infinity to minus 1 this is 0.5 to the power m into e to the power – j omega m; m going from 0 to infinity okay.

Now this I can write suppose, if I write m is equal to suppose, minus l, then this will be summation from l is equal to 1 to infinity 0.5 to the power l e to the power + j omega l, so this we are writing l is equal to minus m, so that way this limit will be now interchange like this and this will remain same, summation m is equal to 0 to infinity 0.5 to the power m, e to the power – j omega m, okay.

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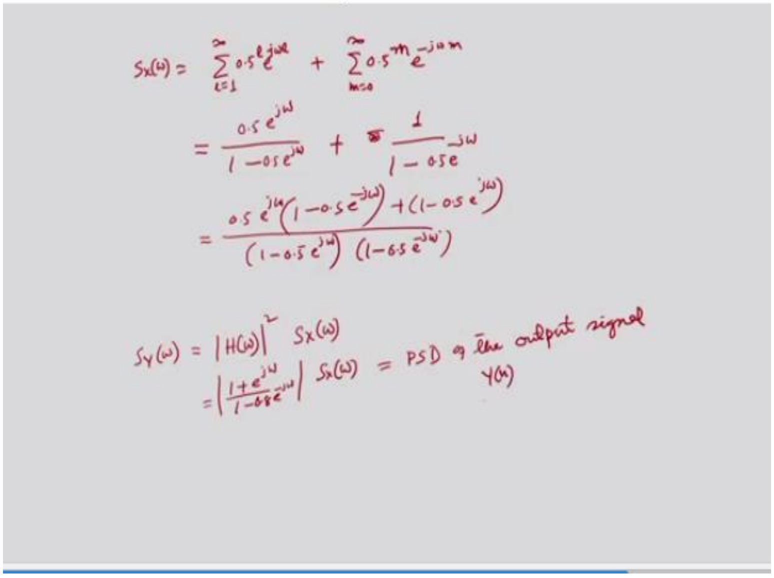

So, we can find out Sx omega in this manner, so Sx omega is equal to summation 0.5 to the power l e to the power j omega l; l going from 1 to infinity plus summation 0.5 to the power m, e to the power – j omega m; m going from 0 to infinity, now this is a geometric series with common ratio here 0.5 into e to the power j omega, here also common ratio 0.5 into e to the power – j omega.

So that way this is a; this is also infinite geometric series, this is also infinite geometric series, so that way we can find out this is equal to first term will be 0.5 into e to the power j omega and common ratio will be 1 minus, common ratio is 0.5; 0.5 into e to the power j omega. Similarly, this part will be also equal to summation, so that way a first term here will be m is equal to 0, this is 1 divided by 1 - 0.5 into e to the power – j omega.

So, this now we can simplify this expression and we will get the final result, so we can carry out 1 - 0.5 into e to the power j omega into 1 - 0.5 into e to the power – j omega and here 0.5 into e to the power j omega multiplied by 1 - 0.5 into e to power –j omega + 1 into 1 - 0.5 into e to the power j omega, we can simplify this expression and we can get the Sx omega, so that way we can find out the Sx omega.

And then we have to find out Sy omega, so Sy omega will be simply because we know Sz; Sy omega will be simply mod of H omega whole square into Sx of omega, so mod of H omega we have; we can find out because here Hz is given, so if we put z is equal to e to the power j omega, then we can find out H of omega, so that way we can find out mod of H omega square into Sx omega.

And hence find out the Sy omega, so this will be 1 + e to the power j omega divided by 1 minus, this one will be equal to 1 - 0.8 e to the power – j omega and this we have to take the mod into Sx omega, so this will give me the PSD of the output signal. So, we can further simplify, you can carry out the simplification and this will be the PSD of the output signal Yn.

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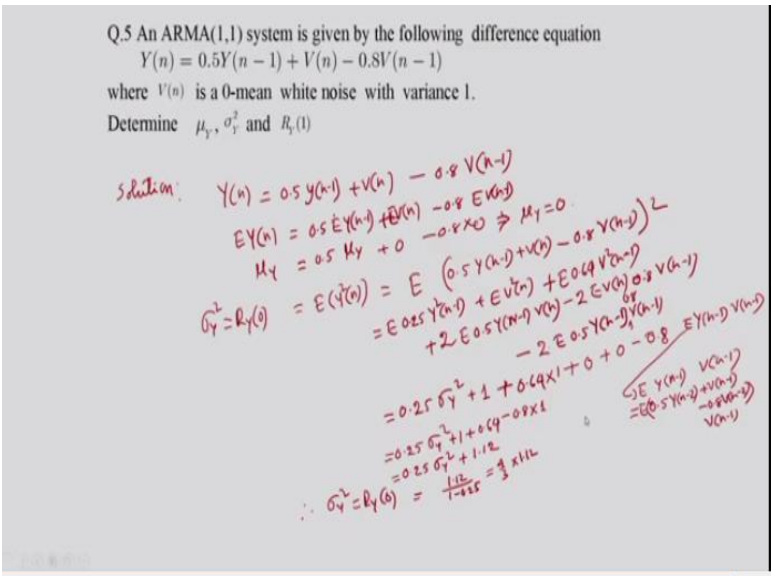

We will go to the next question; An ARMA 1, 1 system is given by the following difference equation, this is a linear difference equation; Yn is equal to 0.5 Y n - 1 + Vn - 0.8 Vn - 1, where Vn is a 0 mean white noise with variance 1, determine mu y sigma y square and Ry of 1. So, here solution Yn is equal to 0.5 Y n - 1 + Vn - 0.8 Vn – 1, therefore E of Yn will be equal to 0.5 into E of Y n - 1 + E of Vn - 0.8 E of Vn - 1.

Now, Yn is WSS, so this will be mu y is equal to 0.5 mu y, this is also mu y because of stationarity and E of Vn is equal to 0 - 0.8 into this is also equal to 0, so that way we will have mu y is equal to 0.5 mu y, implies that mu y is equal to 0, then we have to find out sigma y square; sigma y square this is same as Ry of 0 because it is 0 mean, mu y is 0, so therefore sigma y square is simply Ry of 0.

That is equal to E of y square n, so this will be equal to E of 0.5 Y n - 1 + Vn - 0.8 Vn - 1 whole square, so that is equal to E of 0.25 into y square n - 1 + E of V square n + then E of 0.8 square is 0.64 V square n - 1 and then cross terms will come, so plus twice, now first one will be E of 0.5 Yn - 1 into Vn + next term will be - 2 into E of Vn into 0.8 Vn - 1 and last one will be - 2 into E of 0.5 Y n - 1 into V n – 1, 0.8 Vn – 1.

So, first term will be, so 0.25 E of Y square n - 1 is sigma y square and this one will be sigma V square that is equal to 1, this one will be 0.64, again E of V square n - 1 is also equal to 1, then this is Y n – 1, signal at instant n - 1 and noise at Vn, instant n, they are uncorrelated, so this term will become 0; plus 0 and similarly, Vn and Vn – 1, these 2 noises are uncorrelated, so this is also equal to 0, only we are left with this term; 0.5 into 2 E of Y n - 1 into 0.8 will also come, 2 into 0.5 is 1.

And then 0.8, so this will be 0.8 into E of Y n - 1 into V n – 1, now this term we can write as E of Y n - 1 into V n - 1 will be equal to E of; now Y n - 1 is equal to 0.5 Y n - 2 + Vn – 1 – 0.8 V n – 2 into V n - 1 because this Vn - 1 now, this and this are uncorrelated, similarly this and this are uncorrelated, we are left with only V of n - 1 into V of n - 1 that is V square n – 1, so that way this quantity will be simply E of V square n – 1.

So, therefore what we will have is 0.25 sigma y square + 1 + 0.64 - 0.8 into E of V square n - 1 that is equal to 1, so that way we will get this, so this term will be 0.25 sigma y square and if I carry out this, this will be 1.12, okay. So, therefore we can find out sigma y square that is equal to Ry of 0 that is equal to; I can take this to the left hand side and then carry out the algebra, so that we will get this is equal to 1.12 divided by 1 - 0.25.

So, this will be equal to 4 by 3 into 1.12 okay, we can find out the value similarly, now we want to find out Ry of 1.

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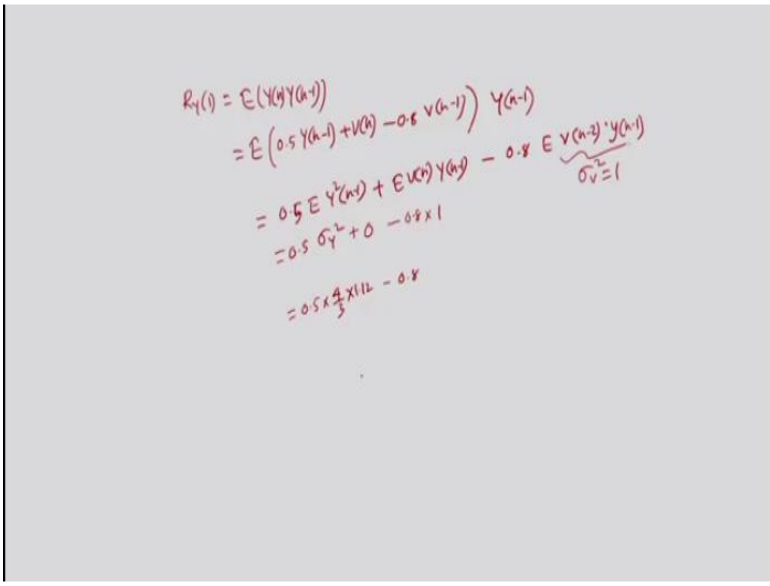

So, Ry of 1 is equal to E of Yn into Y n – 1, okay so, this is equal to we can write in the same manner, Yn is equal to 0.5 Y n - 1 + Vn - 0.8 V n - 1 into Y n - 1, so that way this will be 0.5 into E of Y square n - 1 + E of Vn into Y n - 1 - 0.8 into E of V of n - 1 into Y n – 1. Now, this is equal to 0.5 into sigma y square that we know well, E of Vn into Y n - 1 this is the white noise path signal, they are uncorrelated so, this is equal to 0.

And this is minus 0.8 into; now E of Vn - 1 into Y n – 1, so in the same way like the previous problem we can show that this is equal to; this will be equal to sigma V square that is equal to 1, this we can show, so that way minus 0.8 into 1. So, we have (()) (42:36) sigma y square, so we can put the value of sigma y square here that is sigma y square was 4 by 3 into 1.12 - 0.8 and we can compute this to find out Ry1, thank you.