Statistical Signal Processing
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Lecture – 22
Linear Predictions of Signals 2

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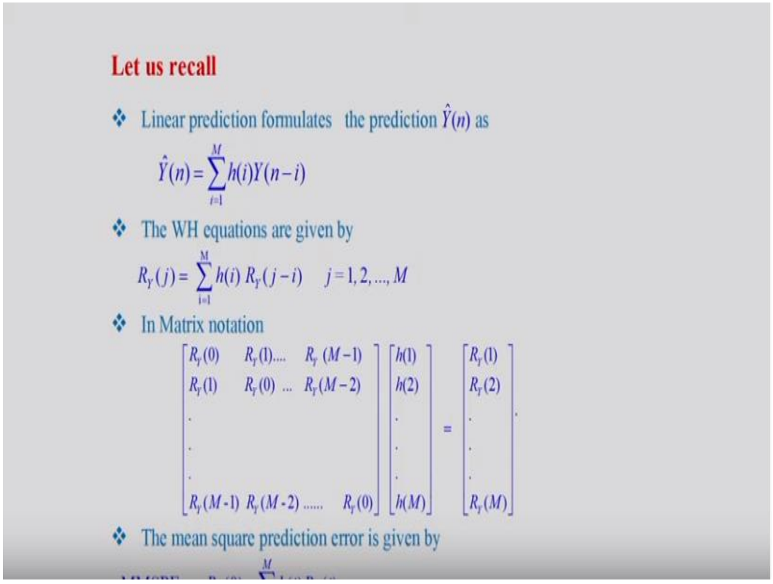

Hello students. Welcome to this lecture on Linear Predictions of Signals. Let us recall linear prediction formulates the prediction Y hat n as, this is the relationship, Y hat n is summation hiYn – i, i going from 1 to M. Corresponding Weiner Hopf equations are given by this Ryj is equal to summation of hiRyj – i, i going from 1 to M and there are M equations corresponding to j is equal to 1 to M.

In Matrix notation, the Weiner Hopf equations are given by this relationship. Here Ry, this is the autocorrelation matrix. This is the prediction coefficient factor, and this is SCF factor, autocorrelation vector. By inverting this relationship, we can find out the LPC coefficient h1, h2 upto hm, and similarly the mean square prediction error is given by MMSPE, Minimum Mean Square Prediction Error is equal to Ry0 minus summation hi into Ryi, i, i going from 1 to M.

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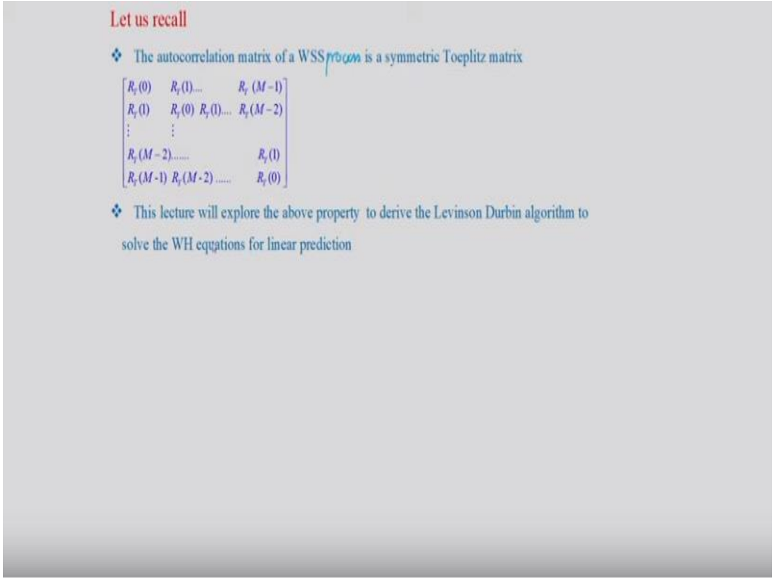

Let us also recall that the autocorrelation matrix of a WSS process is a symmetric Toeplitz matrix. If we see this RY matrix, suppose all elements in this diagonal are equal, all are Ry0, similarly if I consider this super diagonal, all elements will be Ry1. Similarly, this sub-diagonal will have all elements Ry1 like that and also it is symmetric. This element is equal to this, like this element is equal to this, like that. So, this lecture will explore the above property to derive the Levinson Durbin algorithm to solve the WH equations for linear prediction.

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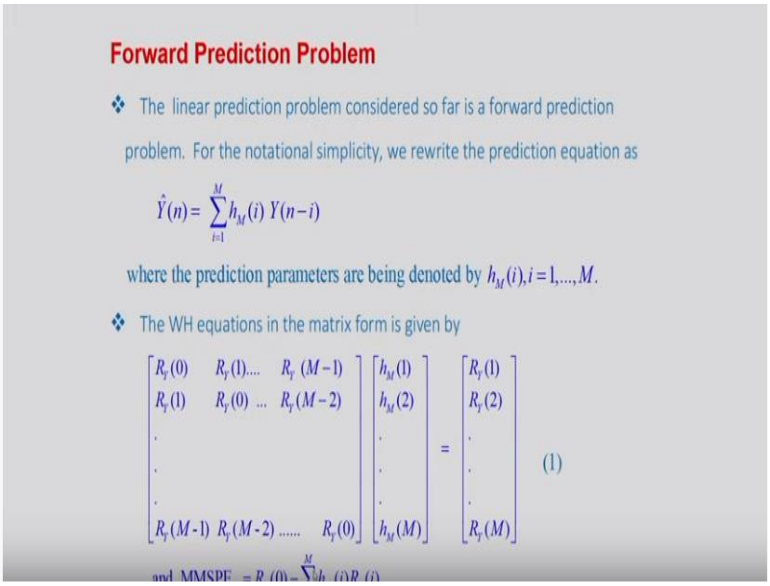

We will start with forward prediction problem. The linear prediction problem considered so far is a forward prediction problem. This is obvious from the name itself. For the notational simplicity, we rewrite the prediction equation as Y hat n equal to summation hMi into Yn-i, i going from 1 to M, where the prediction parameters are being denoted by hMi, i going from 1 to M.

Here, we have included this order parameter M as a suffix. The Wiener Hopf equation in the matrix form is given by this. This is the autocorrelation matrix, and this is the filter coefficient vector, first element hM1, hM2 like that, and right hand side is the autocorrelation vector Ry1, Ry2, etc upto RyM, and correspondingly minimum mean square prediction error is given by Ry0 minus summation hMi into Ryi, i going from 1 to M.

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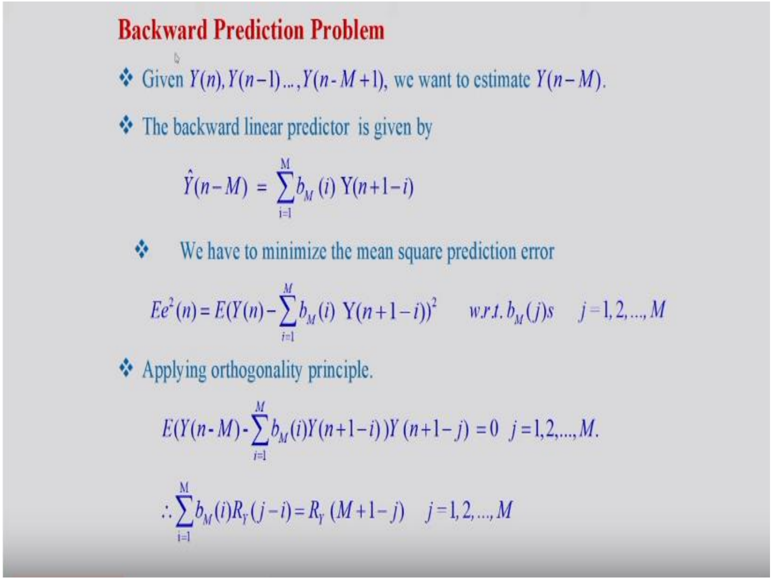

We now consider the backward prediction problem. Given Yn, Yn – 1 upto Yn – M + 1, we want to estimate Yn - M. So, what is the best estimate for Yn – M, given this data Yn, Yn – 1, upto Yn – M + 1. The backward linear prediction is given by Y hat n – M that is equal to summation bMi into Yn + 1 – i, i going from 1 to M. That is, so this will be equal to bM1 into yn + bM2 yn – 1, like that last element will be bM Myn – m + 1.

This way we predict Yn – M. We have to minimize the mean square prediction error given by E of e square n, that is equal to E of Yn – m minus summation bMi into Yn + m – i, i going from 1 to M whole square. So this is the mean square prediction error and this we have to minimize with respect to bMj, j going from 1 to M. Now we can apply the orthogonality condition. EYn – M – summation bMi into Yn + 1 – i, i going from 1 to M.

This is the error expression. Into Yn + 1 – j that must be equal to zero for j is equal to 1, 2, upto M. So taking the expectation now, we will get summation bMi into RYj – i, i going from

1 to M must be equal to RYm + 1 – j. So E of Yn – M into Yn + 1 – j that will give us this. So this set of equation is for j is equal to 1, 2, upto M.

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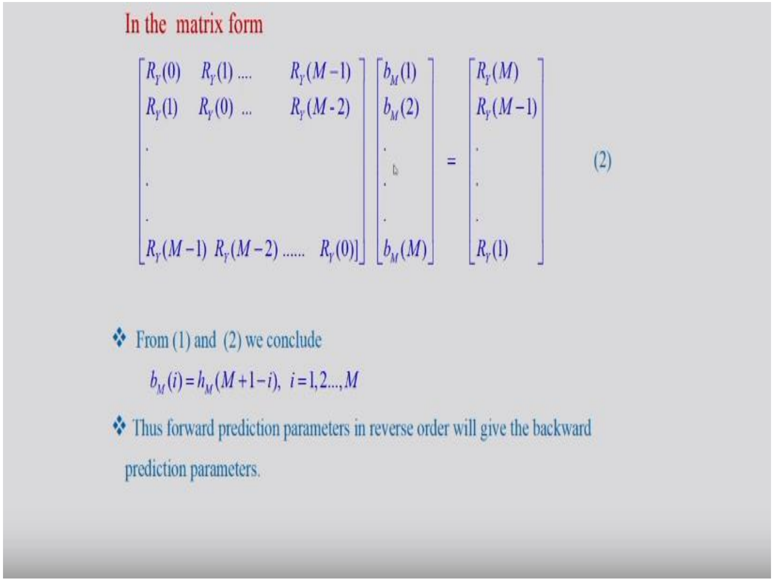

In the matrix form, we get Ry0, Ry1, etc. This is the autocorrelation matrix into that backward prediction filter coefficients bM1, bM2, upto bMM. This is the filter coefficient vector. That must be equal to RyM, and then RyM – 1, and upto Ry1. This is the autocorrelation vector, but we observe that this is in reverse order. In the forward prediction, it is in the order, first Ry1, then Ry2, etc.

But here first RyM, then RyM – 1, etc. Therefore, from the forward prediction problem 1 and 2 here, we get the relationship between the coefficients bM of i is equal to hM of M + 1 – i, where i goes from 1 to M. So, that means the forward prediction parameters in reverse order will give the backward prediction parameters. So this is an important observation, this bMi's are nothing but the forward prediction parameters in the reverse order.

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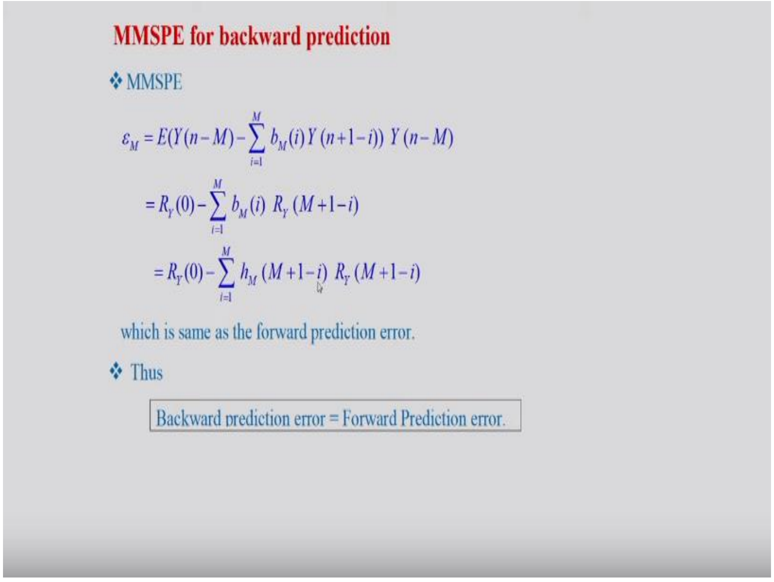

MMSPE, the minimum mean square prediction error that is given by E of Yn – M minus summation bMi into Yn + 1 – i, i going from 1 to M, into Yn – M. That is using the orthogonality condition. So, here the remaining terms will contribute zero, because of the orthogonality of the error, this is the error to the data. So, therefore we will have simply mean square prediction error, that is, epsilon M, that is the notation is equal to EYn – M minus summation bMi into Yn + 1 – i, i going from 1 to M, the whole into Yn – M.

And if we substitute, this is equal to Ry of 0 minus summation bMi RyM + 1 – i, i is equal to 1 to M, and since bM of i is equal to hM of M + 1 – i, we substitute this hM of M + 1 – i, for bMi, and we get this expression. If we carefully observe this equation, it is nothing but the forward prediction error, which is same as the forward prediction error. Thus, the mean square backward prediction error is equal to the mean square forward prediction error.

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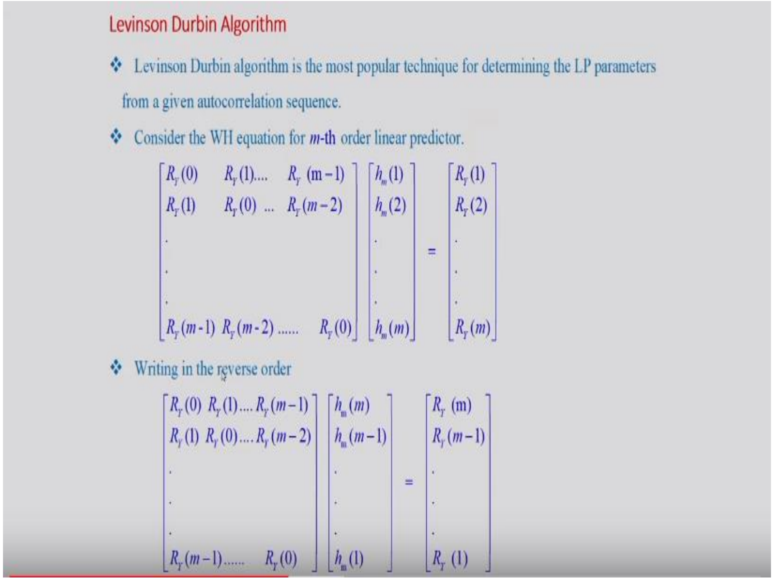

Now, we go to Levinson Durbin Algorithm. Levinson Durbin algorithm is the most popular technique for determining the LP parameters from a given autocorrelation data. Consider the Weiner Hopf equations for m-th order linear prediction. Here, this is the autocorrelation matrix, Ry matrix, and this is the prediction coefficient vector hm1, hm2, up to hmm, and the right hand side is the autocorrelation vector, that is given by Ry1 up to Rym.

And if I write the reverse order, this matrix is same, but coefficient is in reverse order. If we write the coefficients in reverse order, and right hand side also accordingly, we have to inverse the order. So first Rym and then Ry1. So, that way by reversing the order of the coefficient, we get this equation. So, here we consider the m-th order predictor, m-th order prediction.

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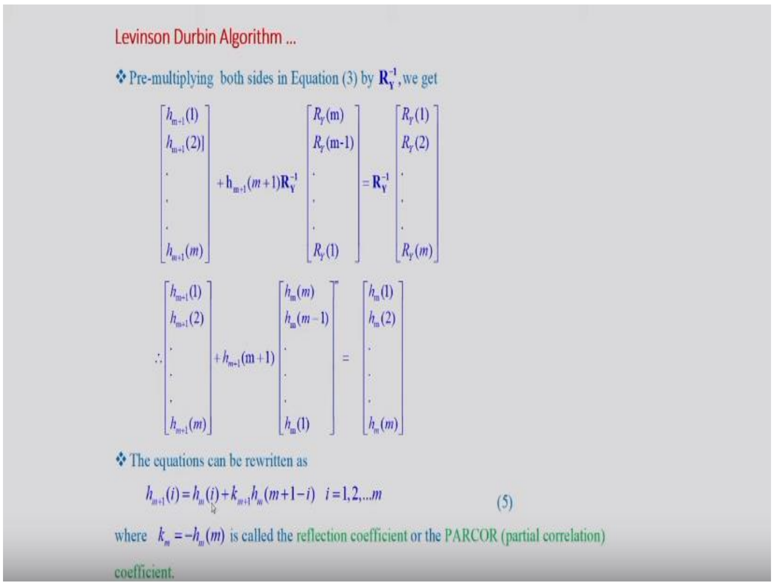

Then, the m + 1-th order predictor is given by this equation. This is the Ry matrix, this is the h vector, and this is the autocorrelation vector. Here, Ry matrix is of order m + 1 by m + 1 and this is a vector of dimension of m + 1 and similarly Ry is also a vector of dimension m + 1. So, our goal is to express this m + 1-th order predictor in terms of m-th order predictor. Let us partition equation 2, these matrices in equation 2 as shown.

So, we will partition the matrices in equation 2, this is the equation 2 like this. Let us partition the matrices in equation 2. We are partitioning by this two lines, so that we have four matrices here, this matrix, this matrix, this matrix, and this matrix. Similarly, two matrices here, and two matrices here. Now, the Weiner Hopf equation can be written in two equations. So first we will multiply this matrix, by this vector, plus this scalar multiply this vector, and then we will get the right hand side.

So, that way this Ry matrix multiplied by this h vector plus h sub m + 1 that is this element, into this vector, we will get the right hand side, that is from Ry1 to Rym, and if I consider the last row, then what we will get is that, this multiplied by this. Summation h of m + i into Ry of m + 1 – i, i going from 1 to m, then hm + 1into Ry0. So last element hm + 1 m + 1 into Ry of 0 and that must be equal to Rym + 1.

So because of this partition, we get two equations now, equation 3 and equation 4. Now, suppose I pre-multiply this equation by corresponding inverse. Then, I will get this. Similarly, I have to pre-multiply here and pre-multiply here. So, that way pre-multiplying both sides by equations 3 by Ry inverse, we get h of m+1, h of m+2, up to h of m + 1m + hm + 1 of m+1 into Ry inverse of this one, Rym, Rym – 1 up to Ry1 is equal to Ry inverse multiplied Ry1 up to Rym.

So, that we have this hm + 1 vector, plus the scalar hm + 1 of m + 1 into, now this is, if I take Ry inverse, into this vector, I will get the filter coefficient in the reverse order. So, that way it will be hmm, hmm – 1, up to hm1 and the right hand side will be hm1, hm2, up to hmm. So, what we have observed that, this is m + 1-th order coefficient, and they are related to m-th order coefficients by this matrix equation.

Now, we can write its element like this, hmm + 1i is equal to hmi, this is hmi from this vector, plus km + 1 into hm of m + 1 – i, because it is in reverse order. This is true for i is equal to 1, 2, upto m. Since this is hm + 1 of m + 1, so therefore, this we will write as km + 1 hm of m + 1 – 1, because this is m + 1, here also it will be m + 1. So, that way we are now able to find m + 1-th order LP coefficients in terms of the m-th order LP coefficients and this is true for only i is equal to 1, 2, up to m. This km coefficient is called the reflection coefficient or the PARCOR, partial correlation coefficient.

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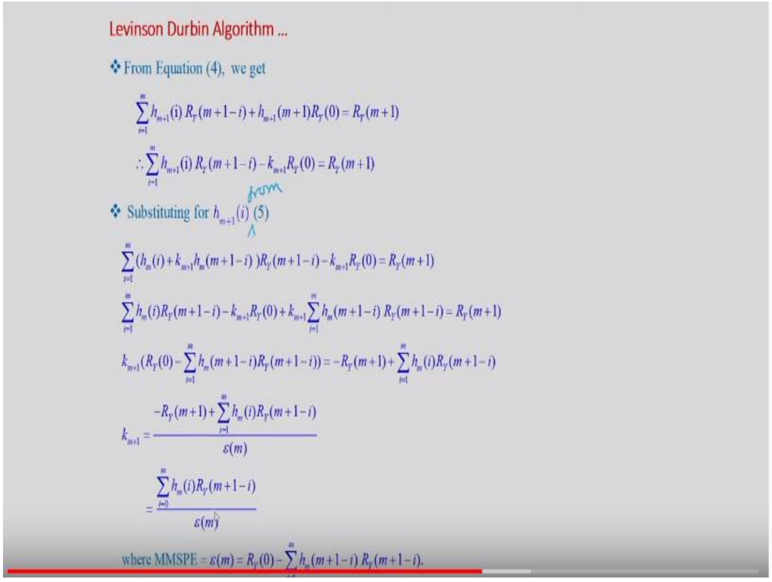

Now from equation 4, we get summation h of m+1i into Rym + 1 – i, i going from 1 to m, plus hm + 1 at point m + 1 into Ry0 is equal to Ty of m + 1. This we obtain from this equation 4. So, substituting h of m + 1 is equal to minus km + 1, we get this relationship. Substituting for h of m + 1i from 5, we have already equation 5, that is the recursive relationship. So, we can write summation hmi + km + 1 into hm at m + 1 – i into Rym + 1 – i, i going from 1 to m minus km + 1 into Ry of 0.

That must be equal to Ry of m + 1 and after simplifying, we will get km + 1 into Ry0 minus summation hm at point m + 1 – i into Rym + 1 – i, i going from 1 to m, that must be equal to minus Ry of m + 1 plus summation hmi into Rym + 1 – i, i going from 1 to m. So, from this expression, we get k of m + 1 equals to minus Ry of m + 1. This is minus Ry of m + 1 plus summation hmi into Ry m + 1 – i, i going from 1 to m, divided by epsilon m, where this epsilon m is the MMSPE, mean square error given by this expression.

So that way, Ry0 minus summation hmm + 1 – i into Rym + 1 – i, i going from 1 to m. Therefore, km + 1 is related with the mean square prediction error by this expression. This is one important relationship and the reflexion coefficient of order m+1 is given by this expression divided by mean square prediction error.

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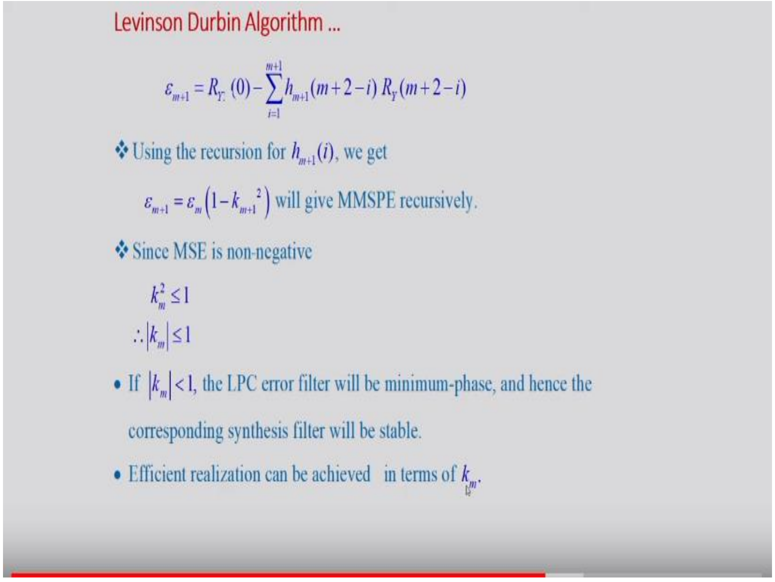

Now, the mean square prediction error at order m+1 is given by Ry of 0 minus summation hm + 1 at point m + 2 – i into Rym + 2 – i, i going from 1 to m + 1. Now using the recursion for h of m + 1i, we get this relationship. This mean square prediction also we will be getting recursively. Epsilon m+1 is equal to epsilon m multiplied by 1 – km + 1 squared, reflexion coefficient square. This will be the MMSPE at order m + 1.

Therefore, we will be able to determine minimum mean square prediction error also recursively. Now, this quantity is always not negative and since MSE is always non-negative, so from this expression because this is non-negative, this is non-negative, we will get km square, that is the reflection coefficient square is always less than equal to 1, implying that reflection coefficient km, its magnitude will be less than equal to 1.

If magnitude of km less than 1, the LPC error filter will be minimum-phase and hence the corresponding synthesis filter will be also stable. Because here we can ascertain that this quantity is less than 1, so that corresponding LPC error filter will be minimum-phase, because we are elementing in terms of km, and the LPC error filter will be minimum-phase and hence corresponding synthesis filter will be also stable.

So, that is one advantage of denoting the modal parameters by the reflection coefficients and the corresponding estimation error. Efficient realization can be achieved in terms of km, instead of using hi's, we can use this km, the reflection coefficient to get some efficient implementation of the prediction error filter, that we will discuss later on.

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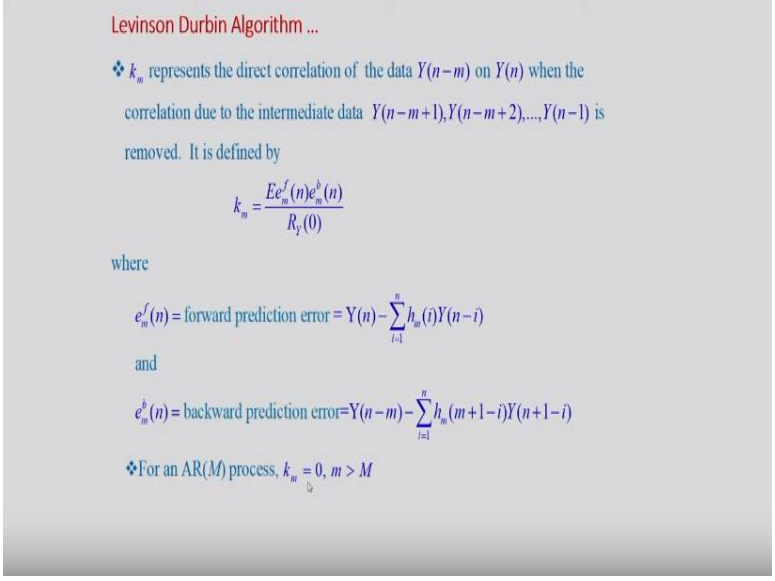

Km is an very important parameter. It represents the direct correlation of the data Yn-m on Yn, when the correlation due to the intermediate data Yn – m + 1 upto Yn-1 is removed. If we remove the correlation because of this data, then whatever correlation is present, that is measured by E of forward error into backward error divided by Ry0. That is the partial correlation coefficient.

So, therefore km is equal to E emfn into embn, where emfn is the forward prediction error is equal to Yn minus summation hmi into Yn – i, i going from 1 to n and embn is equal to backward prediction error. That is equal to Y of n – m minus summation hmm + 1 – i into Yn + 1 – i, i going from 1 to m. Now, it can be shown that for an ARM process, this km will become zero for m greater than M

So that way it is a test for this partial relation coefficient can be used as a test for the order of ARM process. So that we discussed in the earlier class, that how to determine the order of moving average process, similarly order of AR process can be determined by examining the km sequence.

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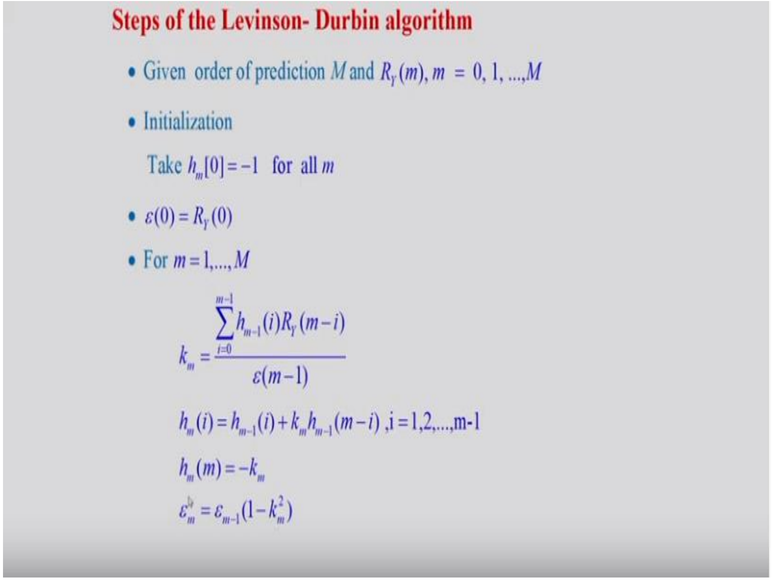

Let us see the steps of the Levinson-Durbin algorithm. Given the order of prediction M and the autocorrelation data for m is equal to 0 to M. We initialize hm of 0 is equal to minus 1, for all m, then we put Epsilon 0, that is the mean square error of the zero-th order predictor E is equal to Ry0 and for m equal to 1 upto M, we do these steps. First, we determine km by this relationship. So this km, reflection coefficient of order M.

These are determined from the m – 1-th order filter coefficient and m - 1 order estimation error and then we will update hmi is equal to h of m – 1i + km into h of m – 1 m – i. This we will do for i is equal to 1, 2, upto m - 1. Then, we will put hm of m is equal to minus km, because here we have determined upto m - 1 by this relationship, and since we have determined km, hm of m will be equal to minus km.

So m-th order of all the parameters are determined. Then, we will update the mean square prediction error, that is epsilon m will be equal to epsilon m - 1 into 1 – km squared. So, that way we will iterate up to M and we will get the prediction coefficients.

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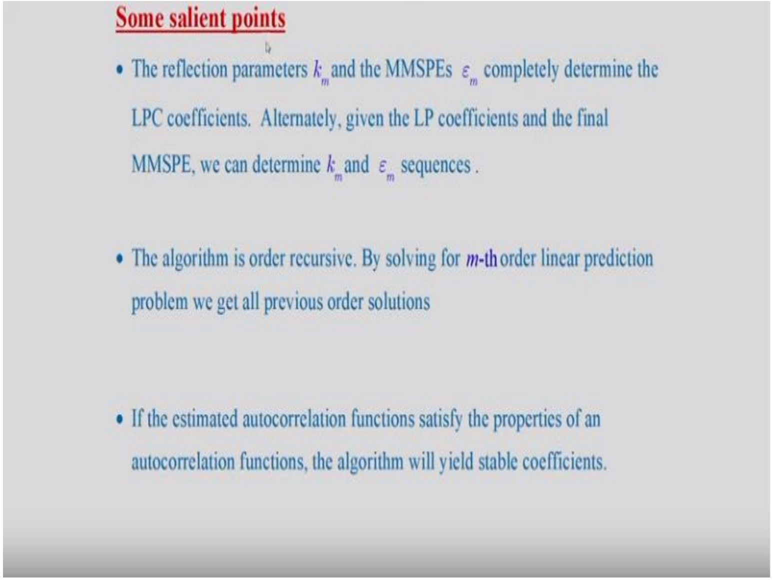

Some salient points we have to remember, the reflection parameters km and the minimum mean square prediction errors, epsilon m, completely determine the LPC coefficients. Alternatively, given the LP coefficients and the final minimum mean square prediction error values, we can determine km and epsilon m for the sequence. This may be required for efficient implementation of the LP error filter.

The algorithm is order recursive, because we first determine, suppose order 1 LP parameters, then determine order 2 LP parameters, like that, therefore the algorithm is order recursive. From the second order LP parameters, we determine the third order LP parameters. By solving the m-th order linear prediction problem, we get all the previous order solutions. So, if we try to solve the m-th order linear prediction problem, we will get the earlier order solutions also, m-1 order solutions like that.

If the estimated autocorrelation functions, because all these algorithm we have to use the autocorrelation function, which is actually estimated from data. If the estimated autocorrelation functions satisfy the properties of an autocorrelation functions, that is even symmetry, positive semi-definiteness, etc, the algorithm will yield stable coefficients. That means corresponding construction filter will be also stable. That is one advantage of Levinson Durbin algorithm.

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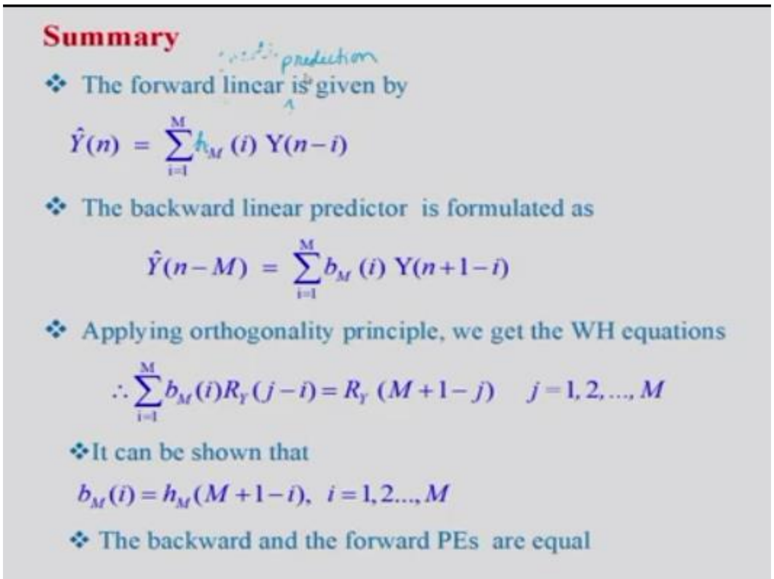

Let us summarize the lecture. The forward linear prediction is given by Y hat n is equal to summation hMi into Yn – i, i going from 1 to M. Similarly, the backward linear predictor is formulated as Y hat n – M that is equal to summation bMi into Yn + 1 – i, i going from 1 to M. So, these are two prediction problems. Applying orthogonality principle to the backward linear predictor, we get the Weiner Hopf equation as this summation bM of i into Ryj – i, i going from 1 to M is equal to Ry of M + 1 – j, j going from 1 to M.

So from this relationship and earlier our forward prediction relationship, it can be shown that bM of i equals hM m + 1 – i, where i is equal to 1, 2 upto M and also we showed that the backward and the forward prediction errors are equal.

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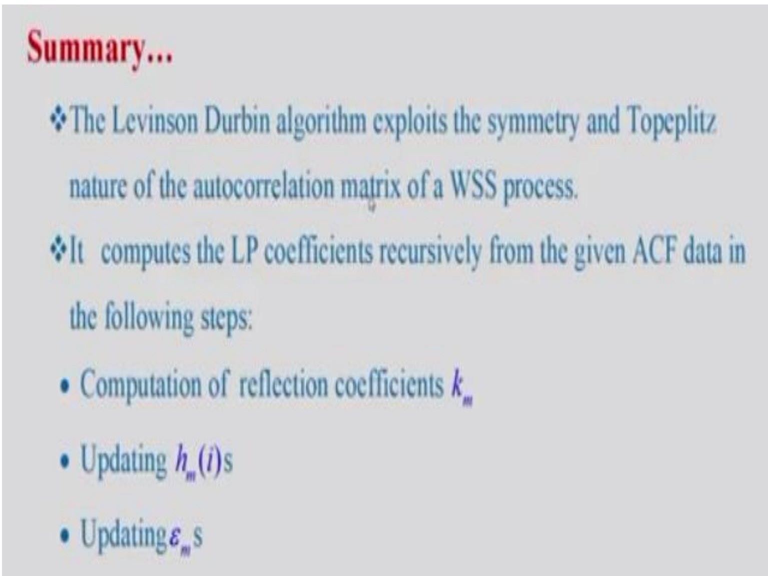

The Levinson Durbin algorithm exploits the symmetry and Toeplitz nature of the autocorrelation matrix of a WSS process. It computes the LP coefficients recursively from the given ACF data in the following steps. First step is the computation of the reflection coefficients km, then updating the predictor parameters hmi's, and then updating the mean square prediction errors, epsilon m. In this way, we can iteratively we can find out the LP coefficients.

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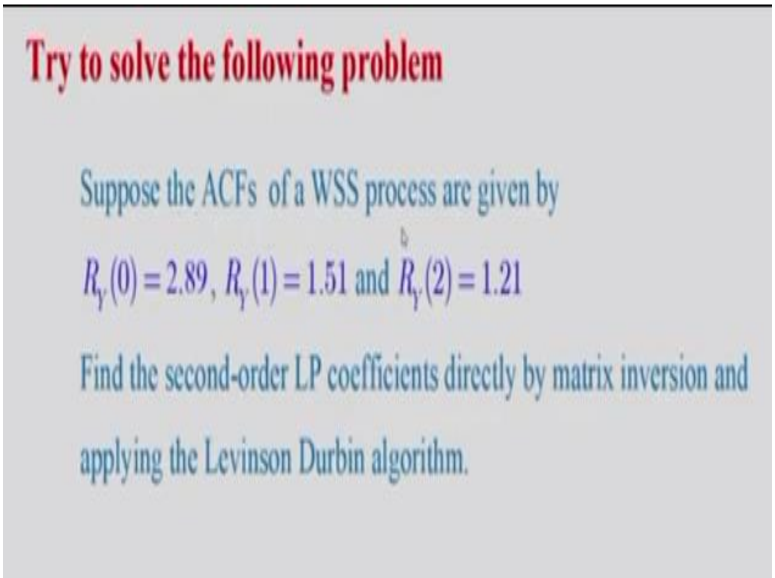

Try to solve the following problem: Suppose the ACFs of a WSS process is given by Ry of 0 is equal to 2.89, Ry of 1 is equal to 1.51, and Ry of 2 is equal to 1.21. Find the second order LP coefficients directly by matrix inversion and then applying the Levinson Durbin algorithm. Thank you.