Statistical Signal Processing

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Lecture 21
Linear Prediction of Signals 1

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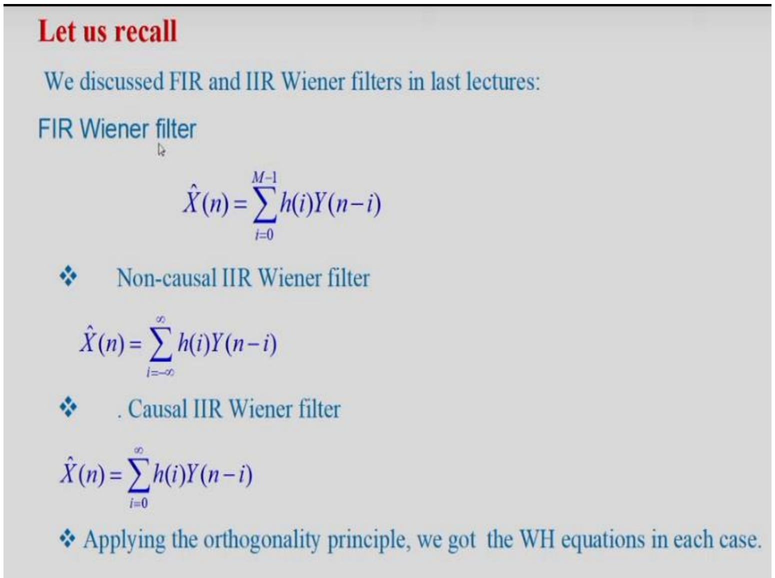

Hello students. Welcome to this lecture on linear prediction of signals. Let us recall, we discussed the FIR and IIR Wiener filters in the last lectures. A FIR Wiener filter where estimator is given by summation hi Y n – i, i going from 0 to n – 1 non-causal IIR Wiener filter where the estimator is given by summation hi Yn – i, i going from minus infinity to plus infinity and the causal IIR Wiener filter X-th n is given by summation hi into Yn – i, i going from 0 to infinity. Applying the orthogonality principle, we got the Wiener Hopf equations in each case.

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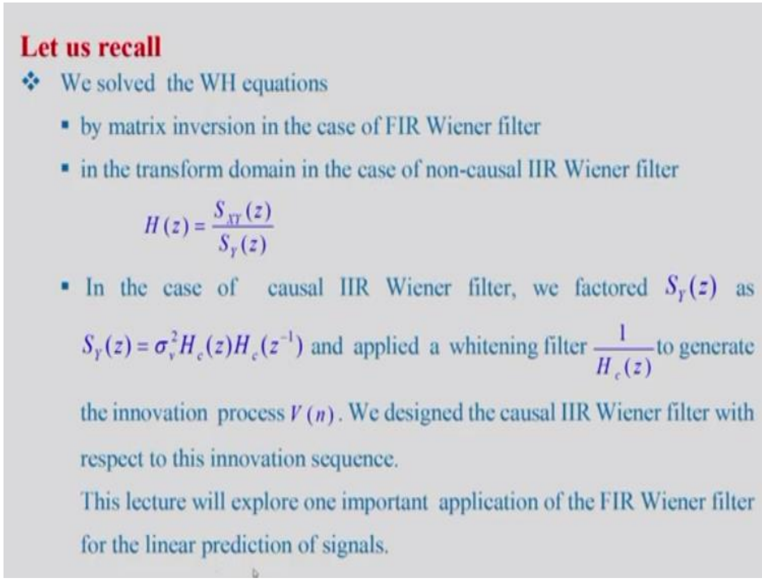

We solved the Wiener Hopf equations by matrix inversion in the case of FIR Wiener filter. In the transfer domain, in the case of non-causal IIR Wiener filter by this equation Hz is equal to Sxyz divided by Syz, whereas xyz is the cross power spectral density of the data and the unknown signal and xyz is the partial density of the data. In the case of causal IIR Wiener filter, we factored xyz that is the power spectrum of the data, zyz is equal to sigma v square into Hcz into Hcz inverse and applied a whitening filter 1 by Hz.

This Hz here 1 by Hz to generate the innovation process Vn. We design the causal IIR filter with respect to this innovation sequence. This lecture will explore one important application of FIR Wiener filter for the linear prediction of signals.

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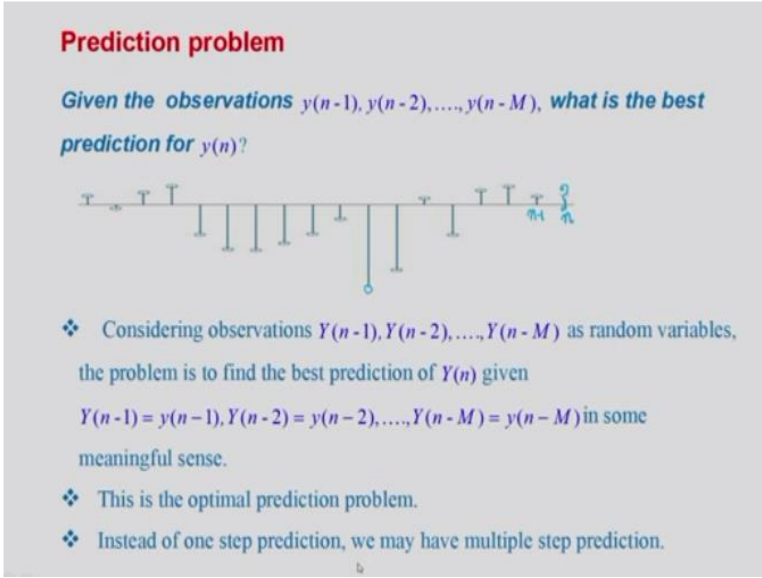

First, let us see what is a prediction problem given the observation Yn - 1, Yn - 2, etcetera, up to Yn - M, what is the base prediction for yn. That means, we have signal at n - 1, n - 2, n - 3, etc., up to some n - M, then what is the base prediction for the signal at instant n. So we want to predict Yn. Now considering the observations Yn – 1, Yn - 2 upto Yn - M as random variables.

The problem is to find the best prediction of Yn given Y n - 1 is equal to yn - 1. These are the values. Yn - 2 is equal to yn - 2 like that Yn - m equal to yn - M. These are already given values and given these values, what is the best prediction for Yn? So we have to find out the best prediction for Yn. This is the optimal prediction problem. Now here, we have used one step prediction. Instead of one step prediction, we may have multiple step prediction.

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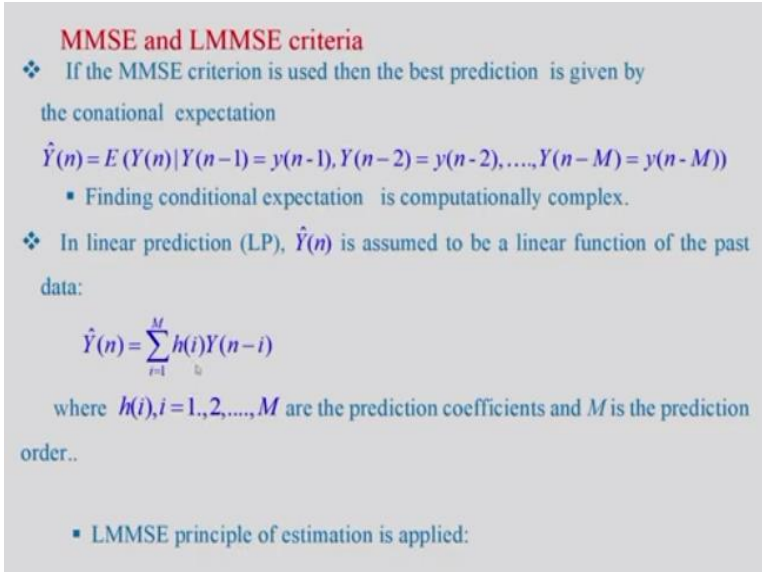

Let us see, how we can apply MMSE and LMMSE criteria. If the MMSE criterion is used, then the best prediction is given by the conditional expectation. Y hat n is equal to conditional expectation of Yn given Yn - 1 is equal to yn - 1, Yn - 2 is equal to yn - 2, etc., etc and upto Yn - M is equal to yn - M. These are the values of the corresponding random variables. So this conditional expectation is computationally complex operation.

Because we have to find out the conditional PDF for conditional PMF and after that only we can find out the conditional expectation. So we need a simpler solution. For that, we consider linear prediction. In linear prediction LP, Y hat n is assumed to be a linear function of the past data. So just like in the other cases of Wiener filter, we assume that Y hat n is a linear function of the past data. So Y hat n is equal to summation hi into Yn - i, i going from 1 to M.

Where hi, i going from 1 to M are the prediction coefficients and M is the prediction order. So this is the linear predictor. Now, we can apply the linear minimum mean square error principle to estimate the; hi parameters.

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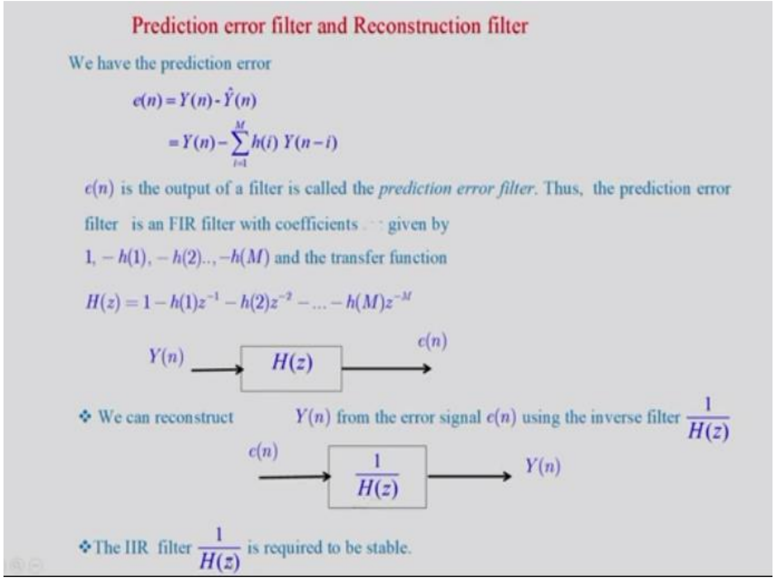

We will introduce the prediction error filter and reconstruction filter. We have the prediction error given by en is equal to Yn - Y hat n and Y hat n we can write like summation hi into Yn - i, i going from 1 to M. So that way en is equal to Yn - summation hi into Yn - i, i going from 1 to M, en is the output of a filter called the prediction error filter.

So this is en, is the output of a filter called the prediction error filter with Yn as the input. Thus the prediction error filter is an FIR filter with the coefficients given by 1, this is the 1 minus h1, first coefficient will be minus h1 like that upto minus hM. So these are the filter coefficients of the prediction error filter and the transfer function Hz, we can write in the transfer domain Hz is equal to 1 - h1z to the power minus 1 minus h2 into z to the power minus 2, etc., upto minus hM into z to the power - M.

So that way yn is the input prediction error filter and prediction error en is the output. We can reconstruct Yn from the error signal en by using the inverse filter 1 by Hz. So, if I put en as the input, then this 1 by Hz will be the filter transfer function and correspondingly I will get Yn as the output. So the way en input to a filter 1 by Hz, we will get Yn. This is the reconstruction filter. The IIR filter 1 by Hz is required to be stable. So that way we have to impose condition of Hz, so that 1 by Hz is stable.

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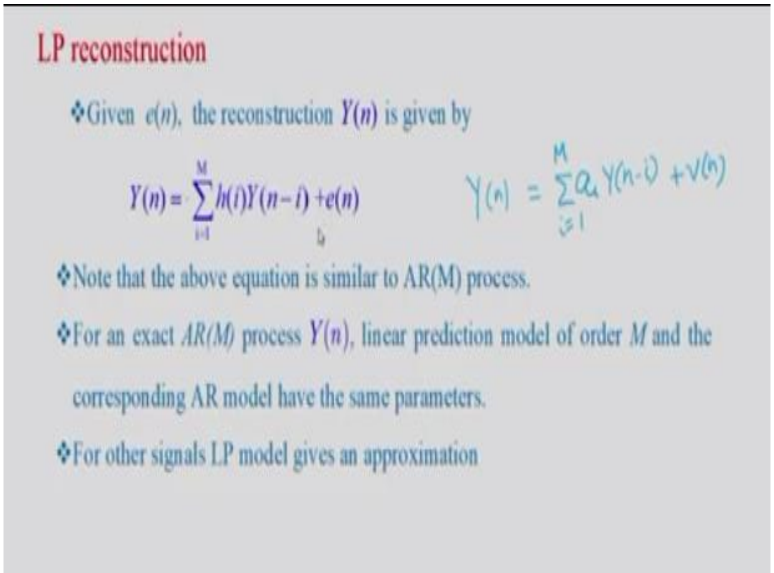

Linear prediction reconstruction, let us see, how we can reconstruct the signal given en, the reconstruction Yn is given by Yn equal to summation hi Yn minus i, i going from 1 to M plus en. This is the reconstruction filter. Note that the above equation is similar to an ARM process, because ARM process also, we got that suppose Yn is an ARM process, then Yn is equal to summation ei Yn - i, i going from 1 to M + Vn.

So that way this equation and this equation are structurally same. For an exact ARM process Yn the linear prediction model of order M and the corresponding AR model have to save parameters. That we can show. For other signal LP model gives an approximation. So LP model gives an approximation of this signal, so that we can determine the error en.

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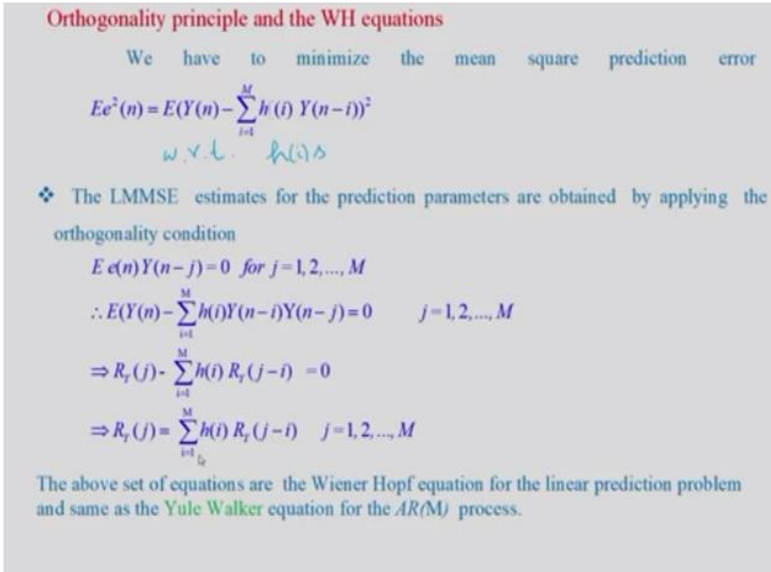

And now, we can apply the orthogonal, we have to minimize the mean square prediction error Ee square n, which is equal to E of Yn minus summation hi into Yn - i, i going from 1 to M whole square with respect to the hi parameters. The linear minimum mean square error estimates or the prediction parameters are obtained by applying the orthogonality condition. Error is orthogonal to data. Therefore, E of en into Yn - j is equal to 0 for j is equal to 1, 2 up to M.

And now we can write en as Yn minus summation hi into Yn - i, i going from 1 to M. So you have this expression into Yn - j will be equal to 0 for j is equal to 1, 2 up to m. By applying the orthogonality condition, we get this expression and if we take the expectation, we will get Ryj minus summation hi into Ryj - i, i going from 1 to M and that must be equal to 0. So we shall give you Ry of j will be equal to summation hi into Ryj - i, i going from 1 to M and we will have M equations corresponding to j is equal to 1, 2 up to M.

The above set of equations are the Wiener Hopf equations for the linear prediction problem and same as the Yule Walker equation corresponding to ARM process, because in ARM process also we got the Yule Walker equations like this, Ry of j is equal to summation hi into Ry j - i, i going from 1 to M.

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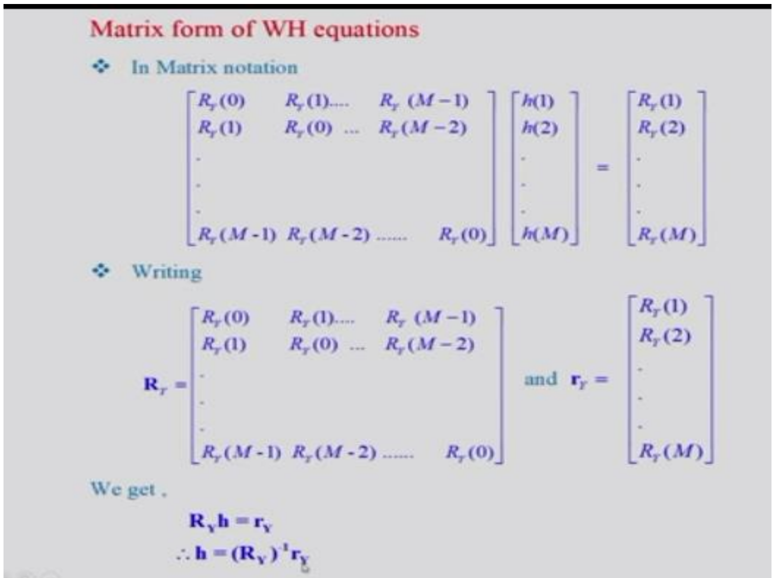

The M equations can be represented in the matrix form like this. This is the Ry matrix multiplied by this is the column vector comprising of h1, h2 up to hm and this product must be equal to the autocorrelation vector Ry1, Ry2 up to Rym. So this is the matrix form of the equation Ryj is equal to summation hi into Ryj minus i, I going from 1 to M for j is equal to 1 up to M. So this set of equation, we can write in the matrix form like this.

If we say this matrix Ry as the autocorrelation matrix and small Ry is the autocorrelation vector Ry1, Ry2, up to Rym, then we can write Ry into h vector is equal to ry. This is this ry, which means that h is equal to ry inverse into ry. So this is the prediction coefficients, we should obtained by inverting the autocorrelation matrix and then multiplying with ry vector.

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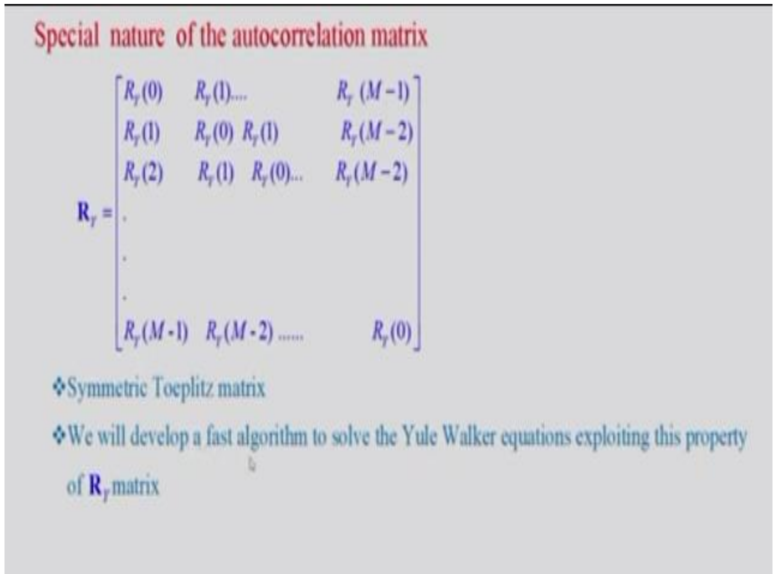

We observe that this r1 matrix has a special nature. If we consider the diagonal elements, all these diagonal element will be Ry0. Similarly if we examine the first super diagonal, then we will see that all its elements are Ry1. Similarly here also, we will get all elements along the subdiagonal will be Ry1, etc. So this type of matrix is known as Toeplitz matrix. Moreover, this matrix is symmetric, because autocorrelation function is symmetric.

Because of that, this matrix is also symmetric and Toeplitz matrix. We will develop a fast algorithm to solve the Yule Walker equations exploiting this symmetrical Toeplitz matrix property.

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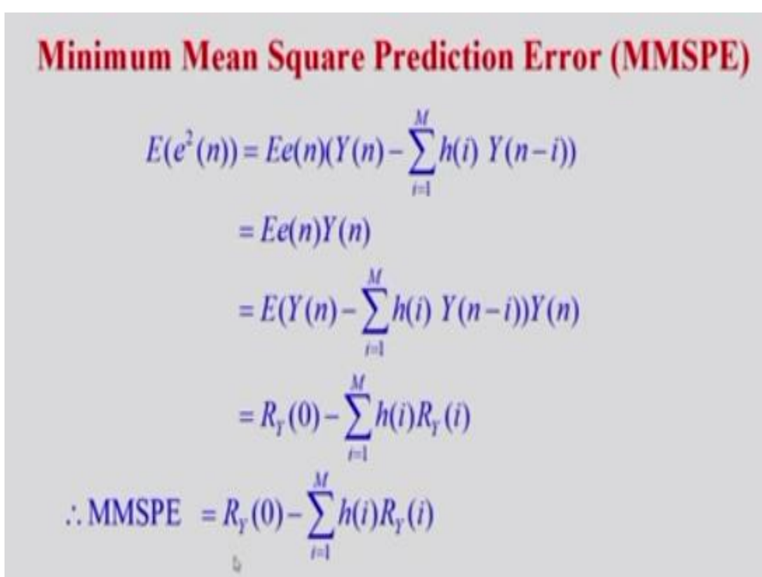

We have to find out the minimum mean square prediction error MMSPE, which will be given by E of e square n that is equal to E of en into now 1 En we are writing in terms of the prediction error. So E of en into Yn minus summation hi into Yn - i, i going from 1 to M. Now it already is orthogonal to this estimator. Therefore, we will simply have E of en into Yn and now substituting En equal to Yn minus summation hi into Yn - i, i going from 1 to M.

We will get E of e square n equal to e of Yn minus summation hi into Yn - i, i going from 1 to M into Yn and this when we simplify, we will get Ry of 0 minus summation hi into Ryi, i going from 1 to M. Therefore minimum mean square prediction error is equal to Ry0 minus summation hi into Ryi, i going from 1 to M. This is the expression for minimum mean square prediction error.

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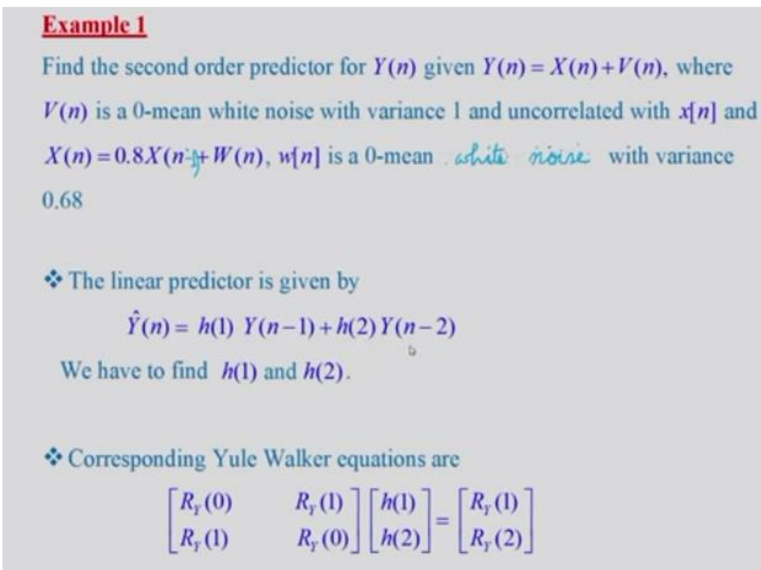

Let us consider one example. Find the second order predictor for Yn, given Yn is equal to Xn plus Vn where Vn is a 0-mean white noise with variance 1 and uncorrelated with Xn and Xn is equal to 0.8 Xn - 1 + Wn, where Wn is a 0-mean white noise with variance 0.68. So we have to find out this second-order linear predictor. The linear predictor is given by Y hat n is equal to h1 into Yn - 1 + h2 into Yn - 2. So this is the second-order predictor.

We have to find h1 and h2 and that corresponding Yule Walker equations are Ry0, Ry1, Ry1, Ry0. This is the autocorrelation matrix multiplied by h1, h2. This is the coefficient vector is equal to Ry1, Ry2. This is the autocorrelation vector. So this is the Yule Walker equation for the second order linear predictor. Now to find out Ry0, because here we have to find out Ry0, Ry1, Ry1 and this Ry0.

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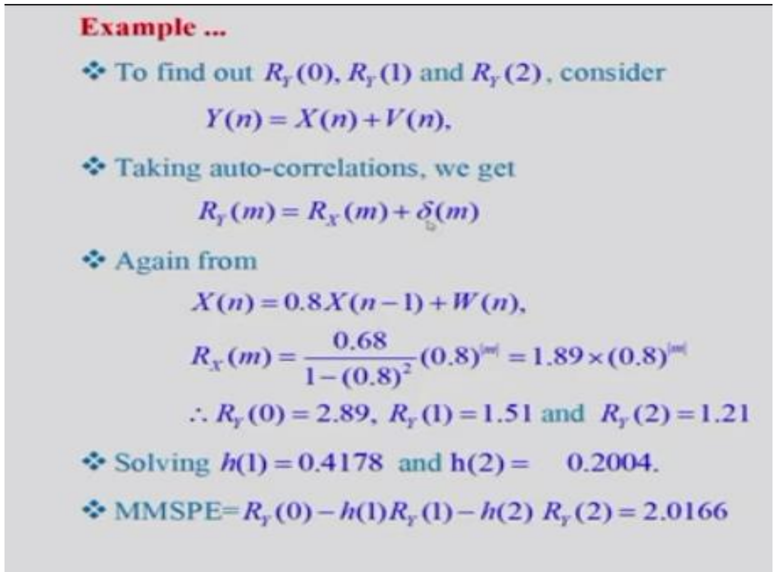

So that way to find out Ry0, Ry1 and Ry2, we consider this Yn is equal to Xn plus Vn model taking the autocorrelation, as we showed earlier Rym will be equal to Rx of m plus delta m and from Xn is equal to 0.8 Xn - 1 + Wn, we will get Rx of m is equal to 0.68 divided by 1 - 0.8 square into 0.8 to the power mod of m and this if I simplify this, it will be equal to 1.89 into 0.8 eight to the power mod of m.

Now from this relationship Rym is equal to Rxm + delta m for m is equal to 0, we will get Ry0 is equal to 2.89 and Ry1 will be equal to 1.51 and Ry2 will be equal to 1.21. This using this relationship and this expression, we can find out these values of Ry0, Ry1, Ry2. So we have found out these values Ry0, Ry1, Ry1, Ry0 and this side also Ry1, Ry2. So, all these values are known.

Therefore, we can find out h1, h2 by inverting this matrix and then multiplying with this vector. So that way, we will get h1 is equal to 0.4178 and h2 is equal to 0.2004 and the minimum mean square prediction error will be given by Ry0 - h1 into Ry1 - h2 into Ry2. So, all these values are known. If we substitute, we will get that MMSPE will be equal to 2.0166. So that way, we got the linear predictor and the corresponding mean square error of prediction.

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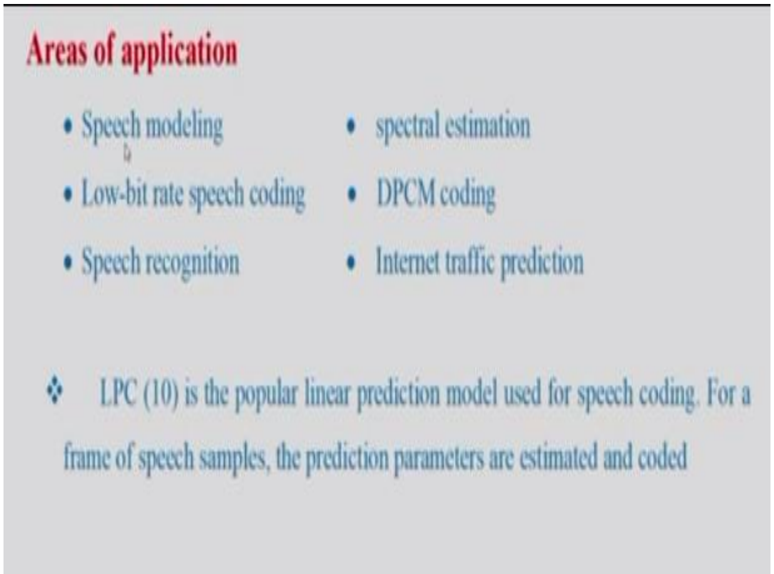

Linear prediction has wide applications. For example, it is used in speech modeling, low-bit rate speech coding, that is compression, speech recognition, spectral estimation, DPCM coding, internet traffic prediction, etc. Many prediction problem, we can apply linear prediction. LPC 10 is the popular linear prediction model used for speech coding. For a frame of speech samples, the prediction parameters are estimated and coded. Instead of coding the raw data, the prediction parameters are estimated and then coded.

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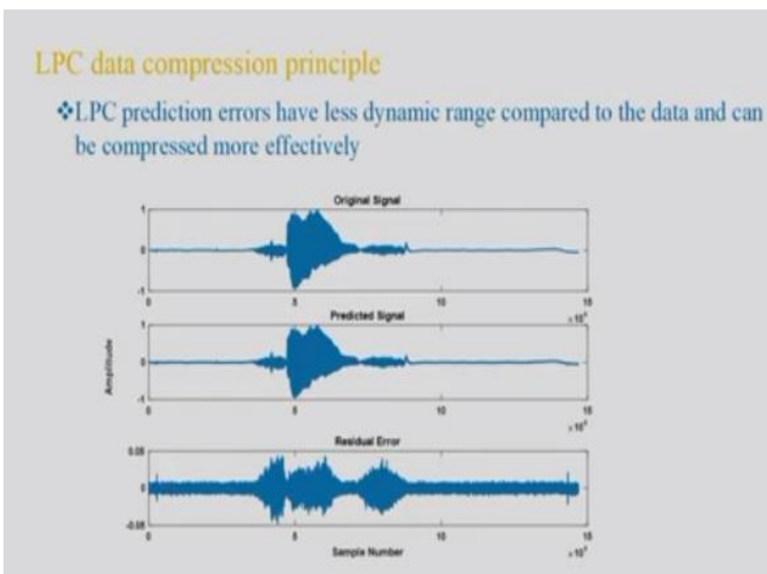

I will give one example. Suppose this is the original signal and if I apply LPC and predict the signal using linear prediction, then the predicted signal will look like this. It is almost similar and the corresponding error, we can estimate and the error will have a very low dynamic range. For example, it is between -0.05 to +0.05, whereas this signal is from minus 1 to 1. So that way, this is a very small amplitude signal, which can be quantized with less number of bits.

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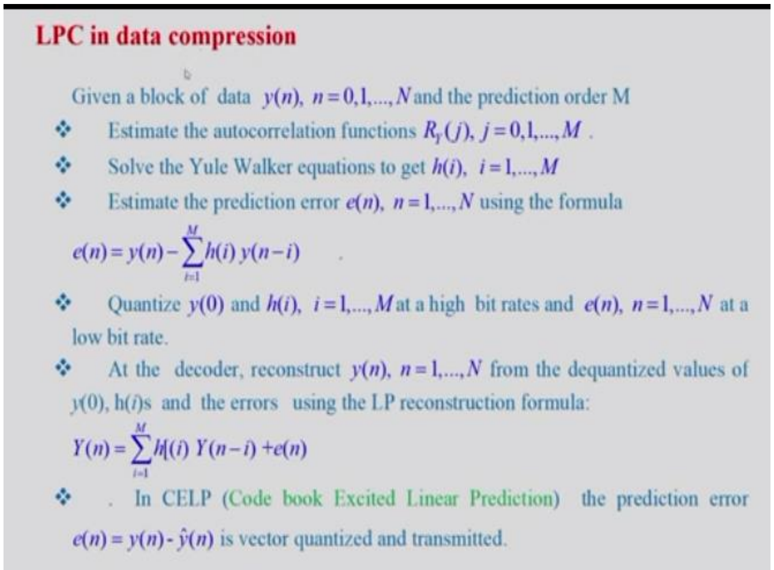

So I will give the basic procedure of LPC in data compression. Given a block of data yn, n is equal to 0 to N and the prediction order M estimate the autocorrelation function from data Ryj, j going from 0 to M. So the Yule Walker equations to get hi, we know the Yule Walker equation corresponding to M-th order linear prediction, so that we have to solve. Estimate the prediction error for n is equal to 1 up to N using the formula en is equal to yn minus summation hi into yn minus i, i going from 1 to M.

Once we have exercised, we can find out the predictor error signal. Now, because it is a linear prediction, so one sample must be there from where we can predict. So, therefore, that y0 will be as it is. We have to quantize y0 and hi for i is equal to 1, 2 up to M at a high bit rate. So y0 that is the initial sample, which will be needed to reconstruct the; signal and hi filter coefficient. These are coded with a high bit rate.

Whereas en which has a small length, en is coded with a low bit rate. At the decoder will reconstruct Yn from the dequantized values of y0, hi, and the errors using the LP reconstruction formula that we know Yn is equal to summation hi into Yn - i + en, i going from 1 to M. So if we use this formula, we can reconstruct the Yn sequence. So this is the basic outline of the LPC coding. There is an LPC codec CELP, codebook excited linear prediction.

Where the prediction error en is equal to yn minus y hat n is vector quantize and transmitted. So in addition to coding the hi and initial values of the signal, the error part we have to code it, but in the case of CELP it is coded using a vector quantizer, which can compress more than a scalar quantizer.

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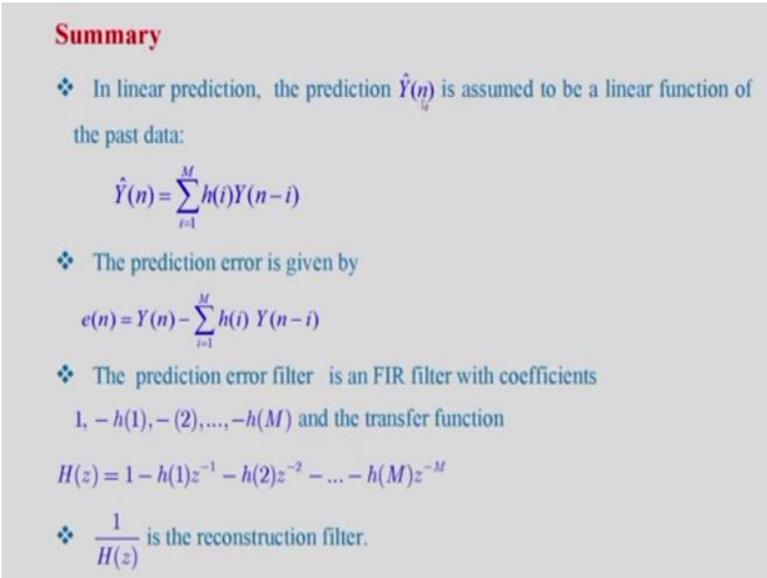

Let us summarize the lecture. In linear prediction, the prediction y hat n is assumed to be a linear function of the past data Y hat n is equal to summation hi into Yn - i, i going from 1 to M. The prediction error is given by en that is equal to Yn minus summation hi into Yn - i, i going from 1 to M. The prediction error filter is an FIR filter with the coefficients 1 corresponding to this 1 - h1 corresponding to the first term - h1, - h2, etcetera up to - hm.

And the corresponding transfer function will be given by 1 - h1 into z to the power -1 - h2 into z to the power -2, etc., upto hM into z to the power -M. This is the transfer function of the prediction error filter. Hence, once we have Hz, we can use one by Hz as the reconstruction filter. This inverse filter is used to reconstruct Yn from the error sequence en.

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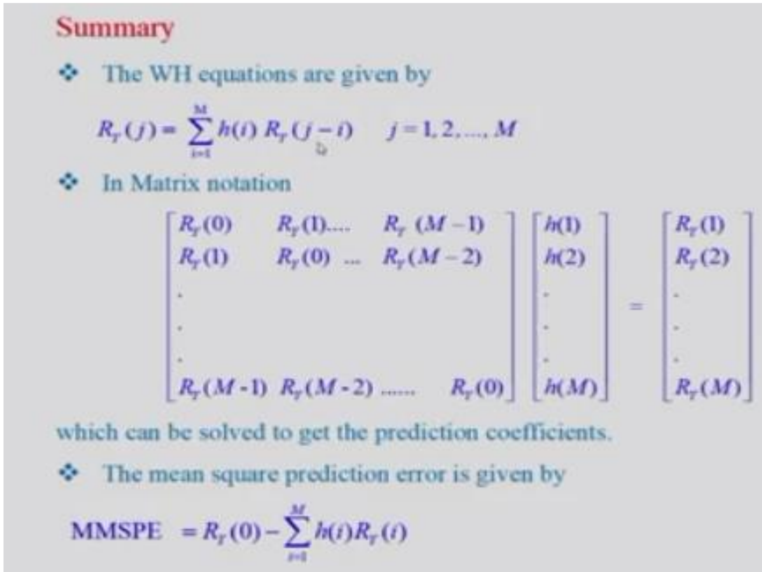

For the linear prediction problem, the Wiener Hopf equations are given by this Ry of j is equal to summation hi into Ryj - i, i going from 1 to n. This is for j equal to 1 up to M. So in the matrix notation, I can write like this. So this Ry of z that brought to the right-hand side. So that way, this column will be Ryz, Ry1, Ry2 up to RyM and this is the M by M autocorrelation matrix.

So therefore, in the matrix, notation we will have this matrix into the coefficient vector is equal to the autocorrelation vector and we can solve this problem by applying the inverse transfer. So Ry inverse into this Ry vector will give us h1, h2 up to hM. The mean square prediction error is given by MMSPE that is equal to Ry of 0 minus summation hi into Ryi, i going from 1 to M. This is the mean square prediction error.

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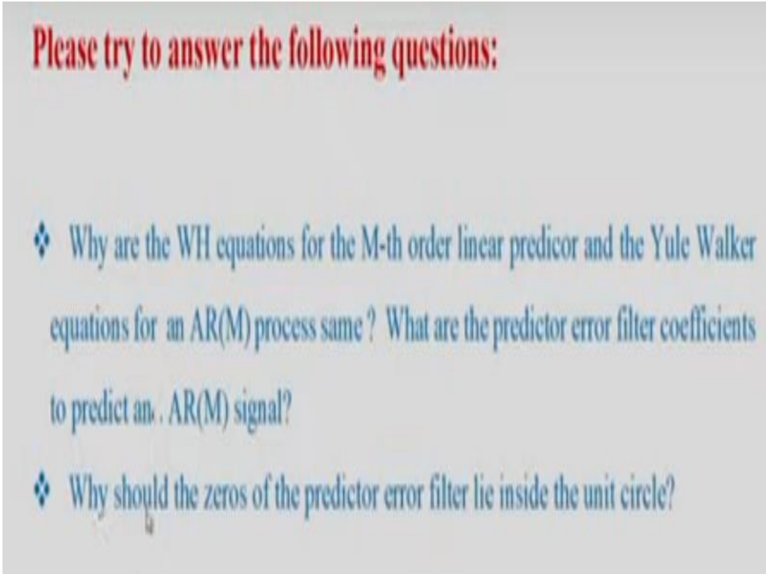

Please try to answer the following questions. Why are the Wiener Hopf equations for the M-th order linear predictor and the Yule Walker equations for an ARM process same? What are the predictor error filter coefficients to predict an ARM signal? Why should the zeroes of the predictor error filter lie inside the unit circle? So this question also you try to answer. Thank you.