Statistical Signal Processing
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Lecture – 20
Causal IIR Wiener Filter

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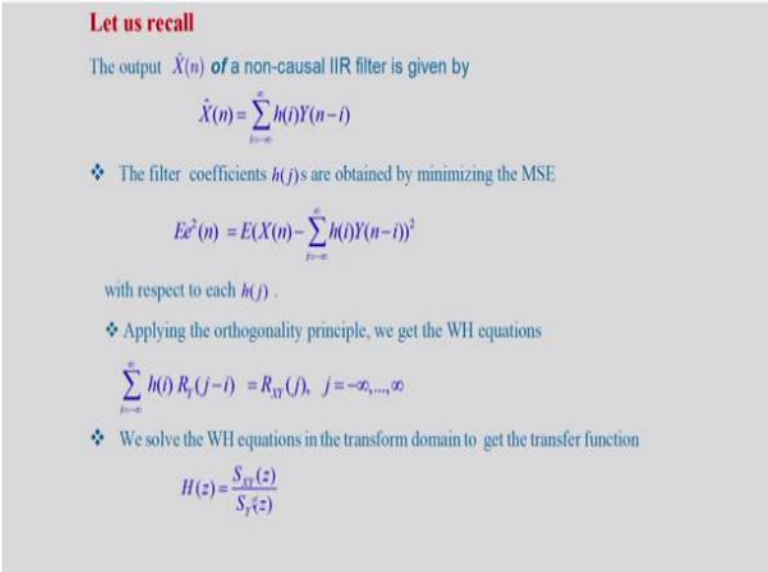

Hello students, welcome this lecture on causal IIR Wiener filter, let us recall the output X hat n of a non-causal IIR filter is given by this equation, we discussed non-causal IIR filter in the last class and the filter output is given by this equation. The filter coefficients hj's are obtained by minimizing the mean square error; mean square error is given by this expression, E of Xn minus summation hi into Y n – i; i going from minus infinity to infinity whole square.

So, we have to minimize this expression with respect to hj, now after applying the orthogonality principle we got the WH equation that is Wiener Hopf equations which are given by this relationship summation hi into Ry j – i; i going from minus infinity to infinity is equal to Rxy j for j going from minus infinity to infinity, so this is an infinite set of equations and these equations can be solved in the transform domain by applying the Z transform, we get this relationship; Hz is equal to Sxyz divided by Syz.

That is the cross power spectral density divided by power spectral density of the observed data, so this is the Wiener filter relationship in the case of non-causal IIR Wiener filter and this is quite simple.

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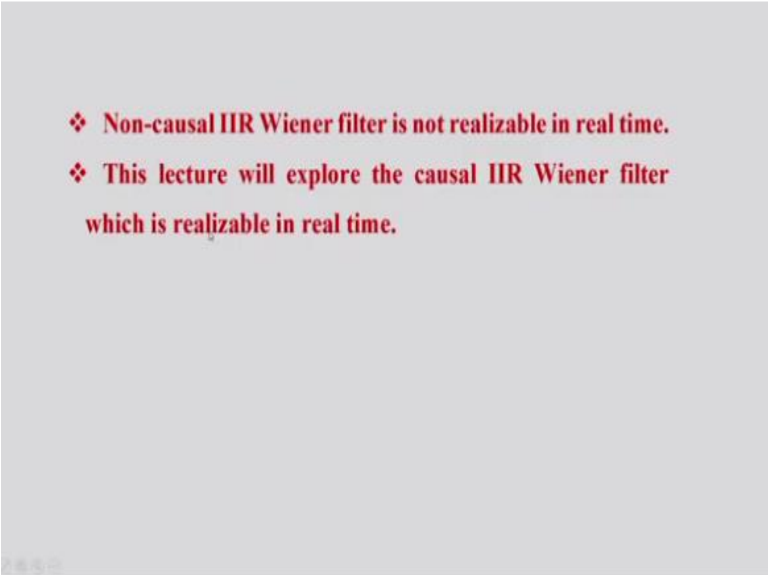

The problem with non-causal IIR Wiener filter is that it is not realizable in real time, we cannot implement it in real time; this lecture will explore the causal IIR Wiener filter which is realizable in real time.

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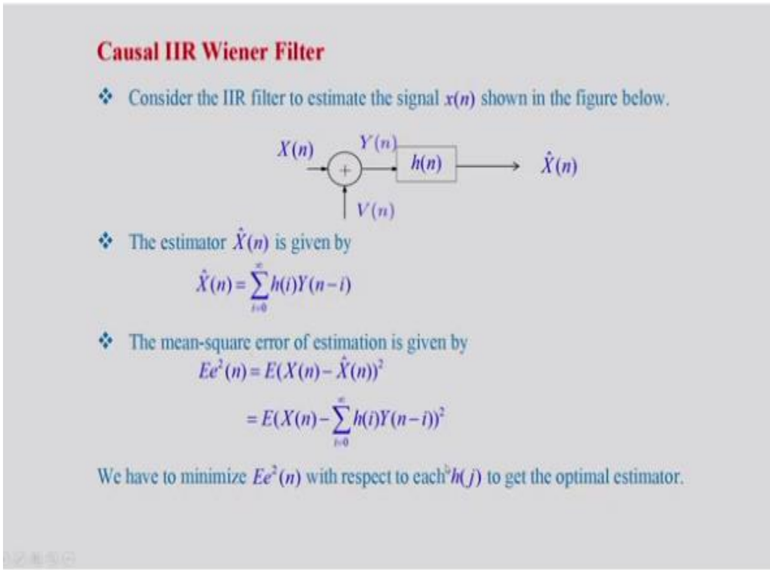

Let us consider this filtering problem; consider the IIR filter to estimate signal xn as shown in figure below, this is the figure here, Xn + Vn that is our observed signal Yn and we apply the IIR Wiener filter with filter coefficient hn to get the estimated signal X hat n, so this is the IIR Wiener filter structure, there are infinite number of coefficients from h0 to h infinity. The estimator X hat n is given by X hat n is equal to summation hi Y of n – i; i going from 0 to infinity.

Because hi’s are nonzero for i greater than equal to 0, the mean square error of estimation is given by this expression; E of e square n that is equal to E of Xn - X hat n whole square and if we substitute X hat n equals this, then we get E of e square n equal to E of Xn - summation hi Y n – i; i going from 0 to infinity whole square. So, this mean square error we have to minimize with respect to each hj.

There are infinite number of hj’s are there which respect to each hj, we have to minimize this expression to get the optimal estimator.

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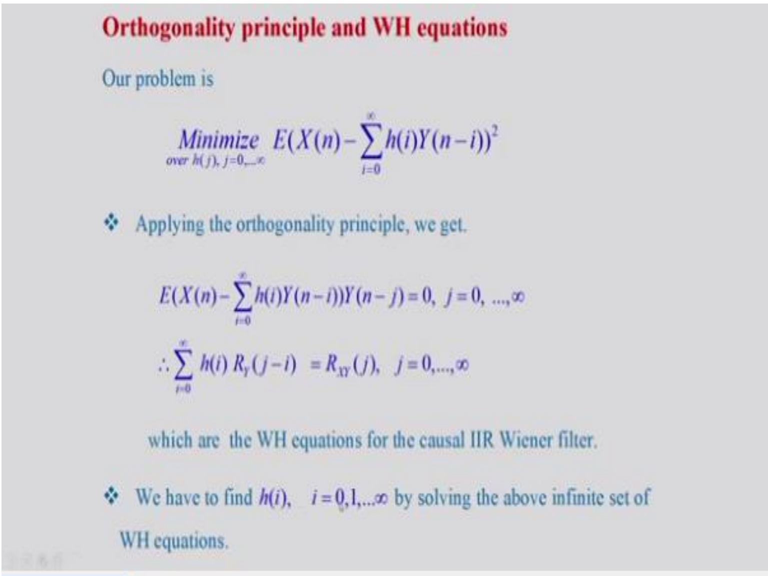

Now, we can apply the orthogonality principle and get the Wiener Hopf equations, so applying the orthogonality principle we will get that is this part is error; error is orthogonal to each data point, so that way E of Xn minus summation hi Y n – i; i going from 0 to infinity into Y n - j must be equal to 0 for j is equal to 0 to infinity, so this is the orthogonality relationship which we have studied earlier.

Now, since we are considering WSS process, so Y n - j into Y n - i that will be Ry of j - i similarly, Xn into Y n – j, if we take the expectation we will get Rxyj, therefore we will get the Wiener Hopf equation that is given by summation hi Ry into j – i; i going from 0 to infinity that is equal to Rxyj, for j equal to 0 to infinity. So, we have got the Wiener Hopf equations corresponding to causal IIR Wiener filter.

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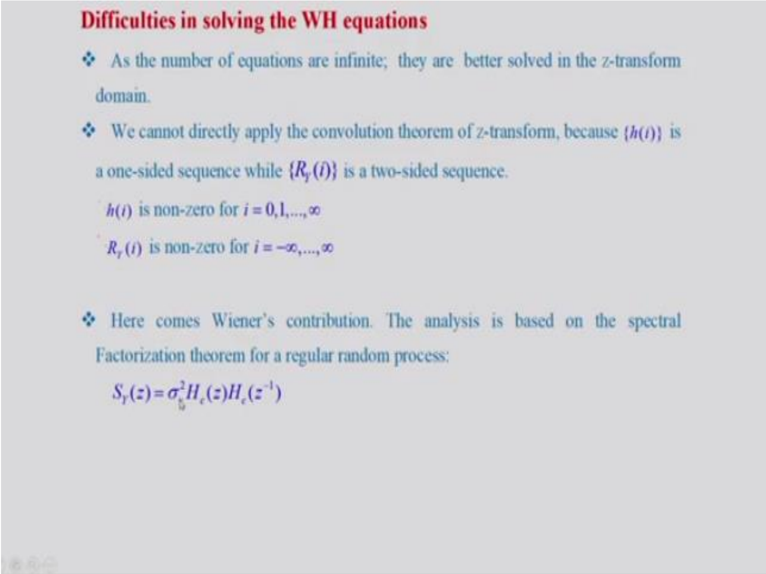

Now, we have to solve this infinite set of equations to find out hi for i is equal to 0 to infinity. Now, what are the difficulties in solving the Wiener Hopf equations for causal IIR Wiener filters, as the number of equations are infinite, they are better solved in the z transform domain, we cannot directly apply the convolution theorem of z transform because hi sequence that filter coefficient sequence is a one-sided sequence while that autocorrelation sequence is a two-sided sequence.

Thus we have hi non zero for i is equal to 0 to infinity but Ry non zero for i is equal to minus infinity to infinity, so because of this we cannot apply the convolution theorem of z transform on the Wiener Hopf equations. Here comes Wiener's contribution; the analysis is based on the spectral factorization theorem for a regular random process that is Syz is product of 2 vectors; sigma v square into Hz into Hz inverse.

And Hz is a causal minimum phase transfer function and sigma v square is the variance of the innovation process.

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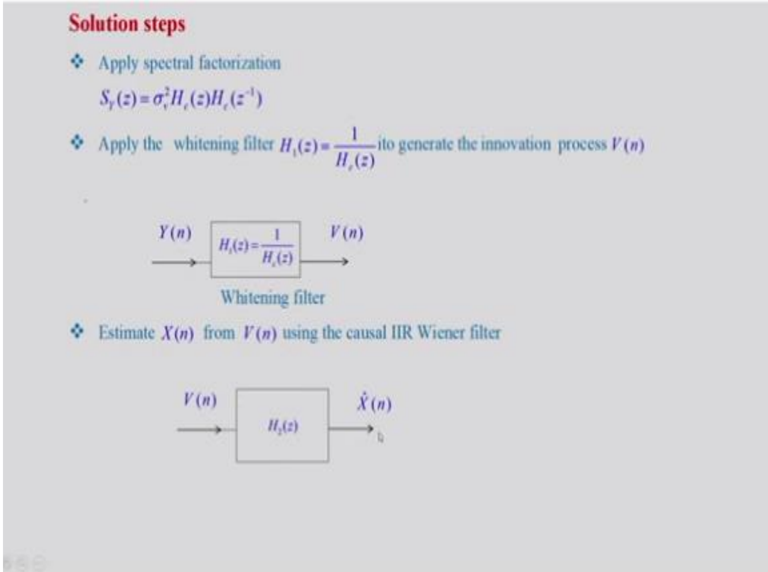

So, by applying spectral factorization, we have Syz is equal to sigma v square into Hz into He z inverse, now we can apply the whitening filter because Hz is a causal minimum phase filter therefore, we can invert it so, we can apply the whitening filter; H1z is equal to 1 by Hz to generate the innovation process Vn, this Vn is the innovation process, so this is the filtering Xn, we pass Yn through H1z is equal to 1 by Hz, we will get Vn.

Now, Vn is an uncorrelated sequence, it is easy to estimate Xn from this Vn sequence. So, we will estimate Xn from Vn by using the causal IIR Wiener filter principle, so Vn you pass it through H2z, we will get X hat n.

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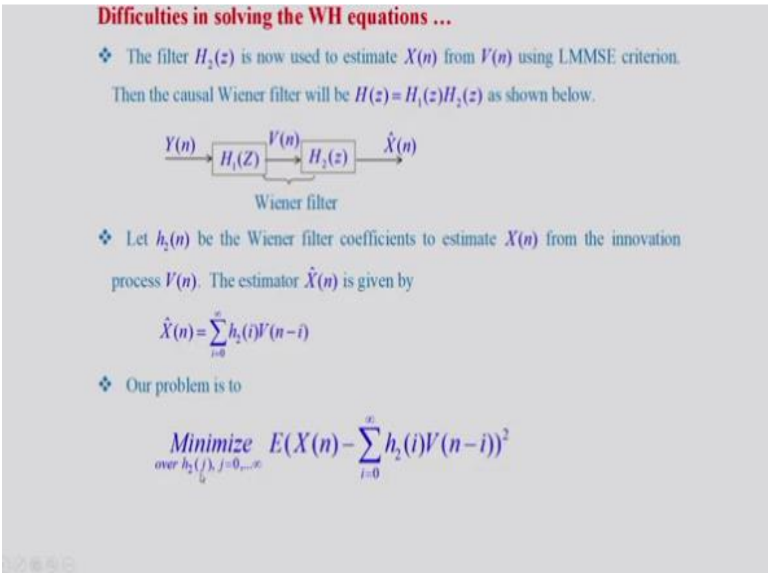

The filter H2z is now used to estimate Xn from Vn using the linear minimum mean square error criterion, then the causal Wiener filter will be Hz is equal to H1z into H2z, where H1z is equal to 1 by Hz as we have shown earlier, this is the H1z. So, we pass Yn through H1z, we get Vn and then using H2z, we get X hat n, therefore our Wiener filter will be the cascaded filters H1z and H2z.

Or in other words, our Hz will be H1z into H2z, let h2n be the Weiner filter coefficients to estimate Xn from the innovation process Vn, so here corresponding time domain parameters are h2n. The estimator X hat n is now given by X hat n is equal to summation h2i V of n – i; i going from 0 to infinity and the corresponding minimization problem is minimize E of Xn minus summation h2i into V of n – i; i going from 0 to infinity whole square with respect to h2j, j going from 0 to infinity.

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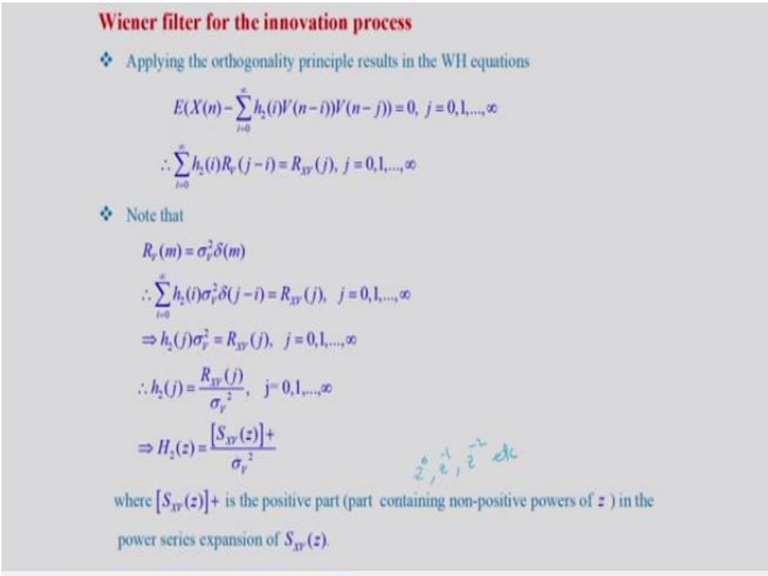

Now, we will get the Wiener filter for the innovation process, applying the orthogonality principle results in the Wiener Hopf equations like this; E of Xn minus summation h2i into V of n – i; i going from 0 to infinity, this is the error and it is orthogonal to V of n – j; j going from 0 to infinity, therefore E of Xn minus summation h2i into V of n – i; i going from 0 to infinity into V of n - j must be equal to 0 for j is equal to 0 to infinity.

Now, taking the expectation operation we will get that summation h2i into Rv j – i; i going from 0 to infinity that must be equal to Rxvj, j going from 0 to infinity. So, this set of equations are the Wiener Hopf equations for the innovation process, note that Rvm is equal to sigma v square into delta m because innovation sequence is a white noise sequence, therefore its autocorrelation function is sigma v square into delta m.

It is sigma v square, for, m is equal to 0 and 0 otherwise, therefore we will write this expression here therefore, we will get summation h2i sigma v square delta j – i; i going from 0 to infinity that must be equal to Rxv of j, for j equal to 0 to infinity. Now, this expression using the property of delta function, we get h2j into sigma v square that must be equal to Rxv of j, for j is equal to 0 to infinity, which gives h2 of j is equal to Rxv of j divided by sigma v.

And this is for j is equal to 0 to infinity, so we are able to find h2j since our input process is a whitening process and taking the z transform of h2j, we will get h2z is equal to z transform of this divided by sigma v square, so z transform of h2j must be equal to z transform of Rxv of j divided by sigma v square but here j is defined for 0 to infinity, therefore we have to consider only the positive part of the z transform that is H2 of z is equal to positive part of Sxvz divided by sigma v square, where Sxvz plus is the positive part that is the part containing the non-positive powers of z.

That is z to the power 0, z to the power – 1, z to the power - 2 etc., in the power series expansion of Sxvz, therefore H2z will be given by the positive part of Sxv z divided by sigma v square.

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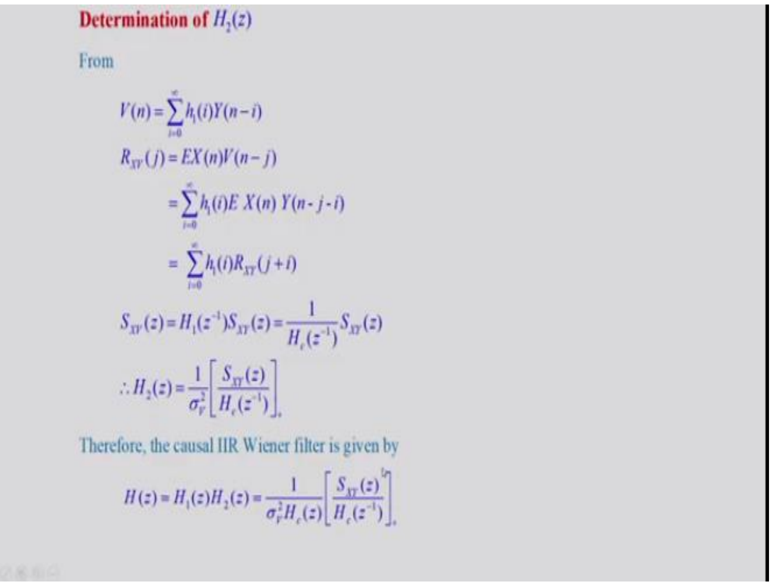

So, how to determine H2z; we know that Vn is equal to summation i is equal to 0 to infinity h1i into Y n - i, so this is from the innovation representation of Yn, so we get this relationship and taking the autocorrelation function, we will get Rxv of j is equal to E of Xn into Vn – j. So, therefore the Rxv of j will be equal to summation h1i into E of Xn into Y n – j – i; i going from 0 to infinity that will be equal to summation h1i into Rxy j + i because E of Xn into Y n – j - i will give me Rxy of j + i.

And now if we take the z transform; Sxv z will be equal to H1 z inverse into Sxyz, here j + i because of that this is H1 of z inverse into Sxy of z and I know H1 of z inverse is equal to 1 by He z inverse, therefore Sxv of z will be 1 by He z inverse into Sxyz, so we have H2 of z is equal to Sxv of z plus divided by sigma v square, therefore we will substitute this expression, so we will get H2z is equal to 1 by sigma v square into Sxy of z divided by He of z inverse plus; plus means the positive part.

Therefore, the causal IIR Wiener filter is given by Hz which is equal to H1z into H2z and H1z is given by 1 by He z and H2z is given by this, therefore the combined expression will be 1 by sigma v square into Hez into positive part of Sxyz divided by He z inverse, so this is the causal IIR Wiener filter expression.

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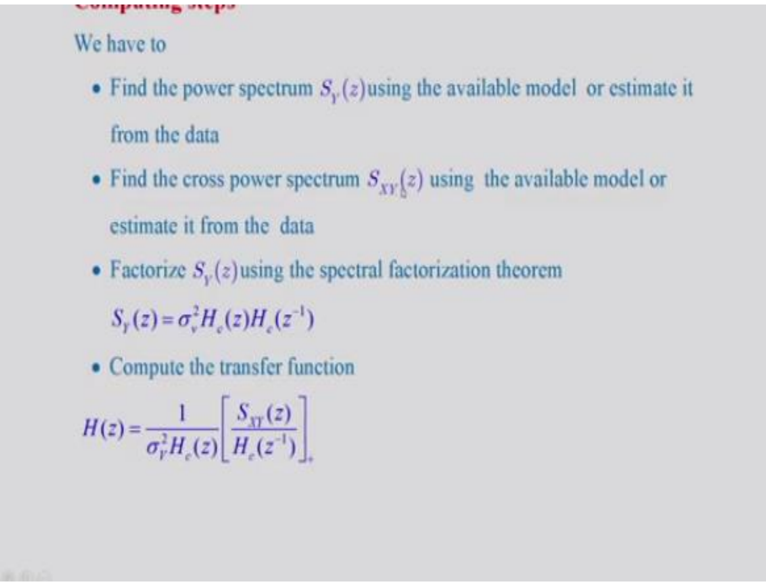

Let us see the computing steps; we have to find the power spectral density Syz using the available model or estimate it from the data, we have to find the cross power spectrum Sxy z using the available model or estimated from the data, then we have to factorize Sy z using the spectral factorization theorem; Syz is equal to sigma v square into He z into He z inverse. If these things are available, then the transfer function of the IIR Wiener filter is given by Hz that is equal to 1 by sigma v square into He z into the positive part of Sxy z divided by He z inverse.

So, He z inverse we get here, He z also we get here, sigma v square also we will get here and Sxy z we estimate or find from the model here.

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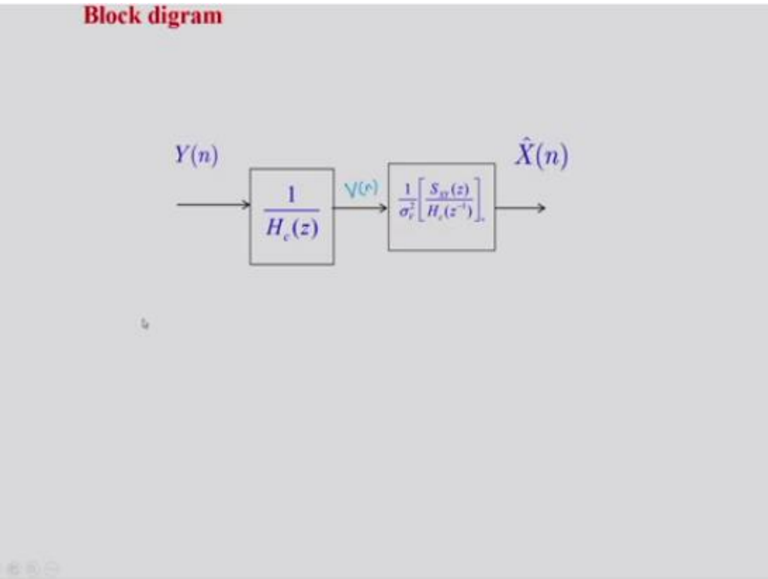

We can represent the procedure in a block diagram; Yn is passed through this filter of transform function 1 by He z to get the innovation process Vn, which is now passed through this filter 1 by sigma v square into the positive part of Sxyz divided by Hz inverse and we will get the estimator X hat n. This is the principle of causal IIR Wiener filter.

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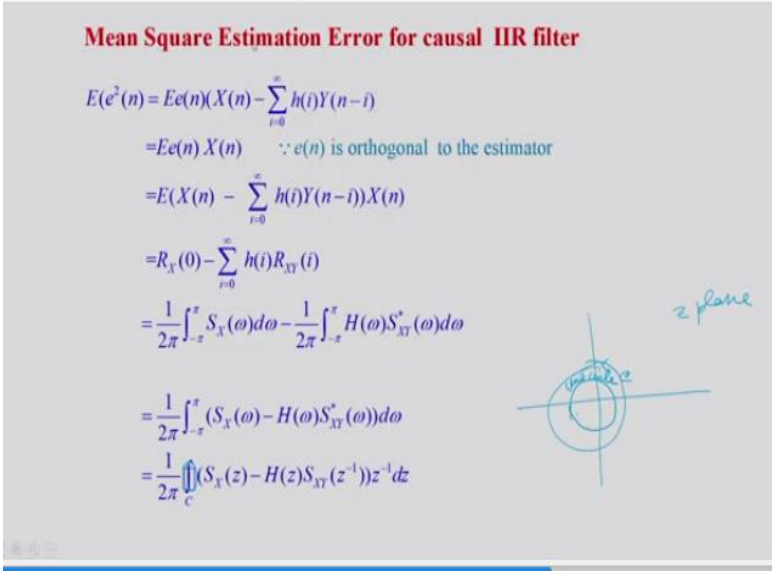

We have to find the mean square estimation error for the causal IIR Wiener filter and it is given by E of e square n which can be written as E of en into Xn minus summation hi into Y n – i; i going from 0 to infinity and using the orthogonality principle we will get this expression is equal to E of en into Xn again, we are expanding en, so we will get E of Xn minus summation hi into Y n –i; i going from 0 to infinity into Xn.

And after taking the expectation operation, we will get E of Xn into Xn that is Rx of 0, similarly this part will be summation hi into Rxyi; i going from 0 to infinity, so this is the mean square error of estimation and in terms of the transfer function we can get it in the frequency domain, Rx of 0 is equal to 1 by 2 pi integration of Sx omega d omega from minus pi to pi minus 1 by 2 pi integration from minus pi to pi of H omega into Sxy star omega d omega.

So, this is the expression for the mean square estimation error which we get by applying the inverse DTFT. Now, this same integral we can get in the z transform domain also where we have to perform the contour, integral over a contour that include the unit circle, so we will consider one contour suppose in this z plane, this is the z plane and this is the contour c which include the unit circle maybe somewhere a unit circle is here, this is unit circle.

So, if we perform this integral over this unit circle that is contour integral of Sxz - Sz into Sxyz inverse into z inverse dz, we will get the mean square error, so this way we can find out the mean square estimation error.

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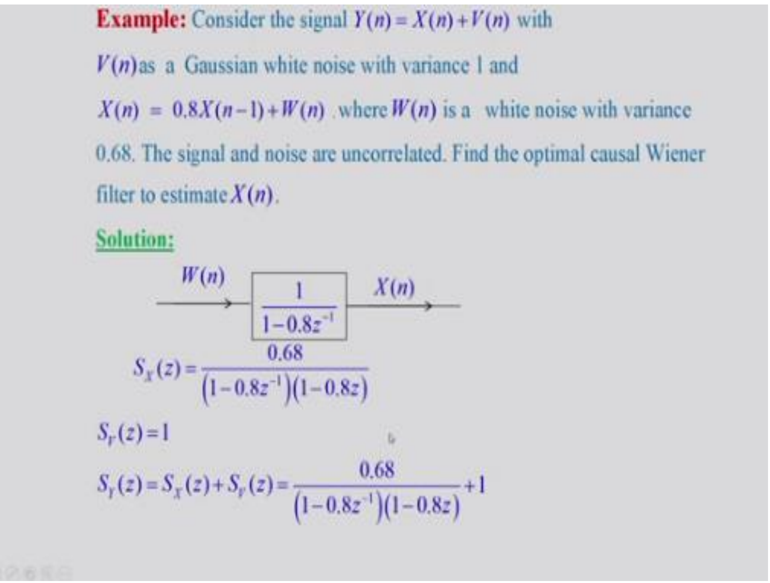

Let us consider one example; consider the signal Yn is equal to Xn + Vn with Vn as a Gaussian white noise of variance 1 and Xn given by this model; Xn equal to 0.8 X n - 1 + Wn, where Wn is a white noise with variance 0.68, this signal Xn and the noise Vn are uncorrelated. Find the optimal causal Wiener filter to estimate Xn? So, first we will see what is the power spectral density of Xn.

So that way this model, this is here 1 model, you are passing WN that is the white noise through the filter 1 by 1 - 0.8 z inverse and we will get Xn that is the WSS output, so we can find out the power spectral density of Xn that is given by Sxz, which is equal to 0.68 that is the variance of Wn divided by 1 - 0.8 z inverse that is 8z into 1 - 0.8 z, which is 8z inverse, so that way we can find out the power spectral density of Xn.

Sxz is given by this and since Vn is a white noise of variance 1, its power spectral density is equal to 1, so we have to find out the power spectral density of Yn, so this is my Yn and I know that autocorrelation function of Yn is equal to autocorrelation function of Xn + autocorrelation function of Vn, that we have analysed earlier, therefore we will get Syz that is power spectral density of Y is equal to Sxz + Svz.

And this can be written like this 0.68 divided by 1 – 0.8 z inverse into 1 - 0.8 z plus 1, so this is the power spectral density for Yn.

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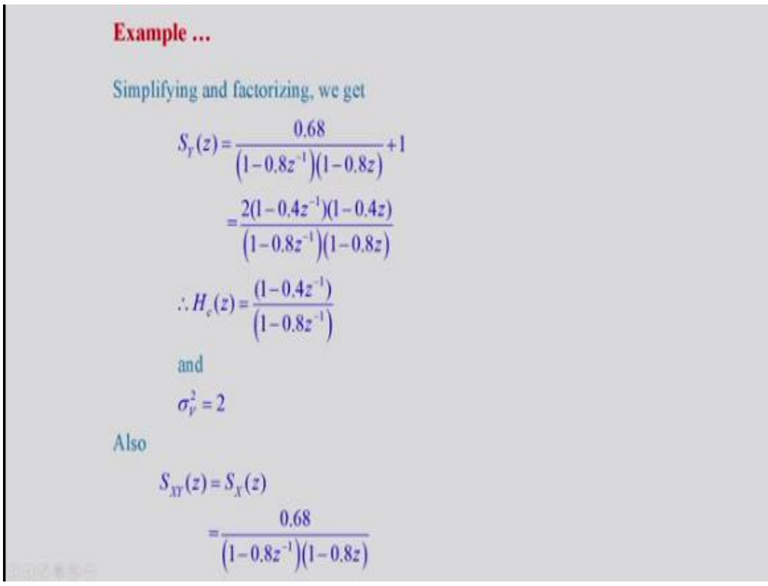

Simplifying and factorizing we will get this Syz can be written as this expression; 2 into 1 - 0.4 z inverse into 1 - 0.4 z divided by 1 - 0.8 z inverse into 1 - 0.8 z, so from this we will get sigma v square is equal to 2 and causal part is 1 - 0.4 z inverse divided by 1 - 0.8 z inverse, this is the causal transfer function, therefore Hez will be equal to 1 – 0.4z inverse divided by 1 - 0.8 z inverse.

Also because it is signal and noise are uncorrelated, therefore we will get from the same model, Sxy z is equal to simply Sxz and which is given by 0.68 divided by 1 - 0.8 z inverse into 1 - 0.8 z, so we have found out Sxy z, we have found out Hez and we have the He z inverse also, this part is the He z inverse.

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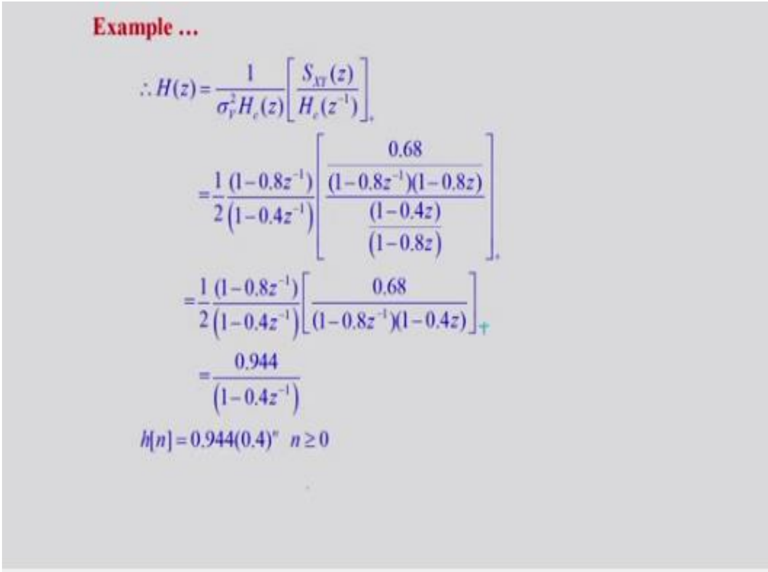

Now, Hz is equal to 1 by sigma v square into He z into positive part of Sxy z divided by He z inverse, so we will substitute each term sigma v square is equal to 2, He z is equal to this part and this part also, Sxy z is equal to this part and similarly, He z inverse is this part, so ultimately we will get an expression like this, we have to consider the positive part of this and we can get the positive part of this expression, this step transformed by partial expansion of this quantity 0.68 divided by 1 - 0.8 z inverse into 1 - 0.4 z.

So, if we have the partial expansion and take the part containing 1 - 0.8 z inverse as the denominator, depth part will be the positive part of the z transform because that part only will contain the non-positive powers of z, so therefore we will take the part and then simplify with this and ultimately, we will get Hz equal to 0.944 divided by 1 - 0.4 z inverse, so this is the transfer function of the IIR causal Wiener filter.

And correspondingly, hn also we can find out by taking the inverse transform hn will be 0.944 into 0.4 to the power n, for n greater than equal to 0, this will be defined for n equal to 0 to infinity, so that way we can find out the causal IIR Wiener filter.

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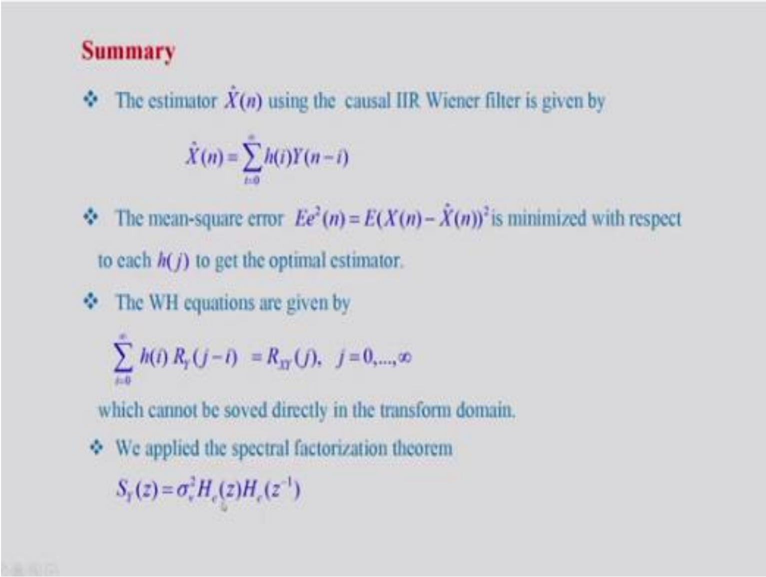

Let us summarize the lecture; the estimator X hat n and using the causal IIR Wiener filter is given by this relationship; X hat n is equal to summation hi into Y n – i; i going from 0 to infinity. The mean square error E of e square n that is equal to E of X n – X hat n whole square is minimized with respect to each hj to get the optimal estimator and the optimal estimator is given by the Wiener Hopf equations; summation hi into Ry j - i; i going from 0 to infinity is equal to Rxy of j; for j is equal to 0 to infinity.

And this set of Wiener Hopf equations cannot be solved directly in the transform domain, therefore we applied this spectral factorization theorem; Syz is equal to sigma v square into Hez into He z inverse.

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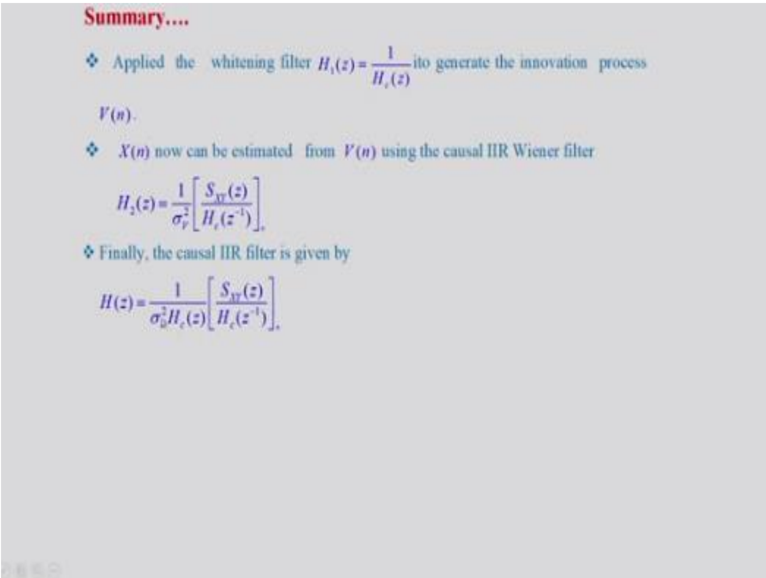

Then, we applied the whitening filter; H1z is equal to 1 by Hez to generate the innovation process Vn, Xn now can be estimated from Vn using the causal IIR Wiener filter which was derived as H2z equal to 1 by sigma v square into positive part of Sxyz divided by He z inverse finally, the causal IIR Wiener filter is given by Hz that is equal to 1 by sigma v square into He z into the positive part of Sxyz divided by He z inverse.

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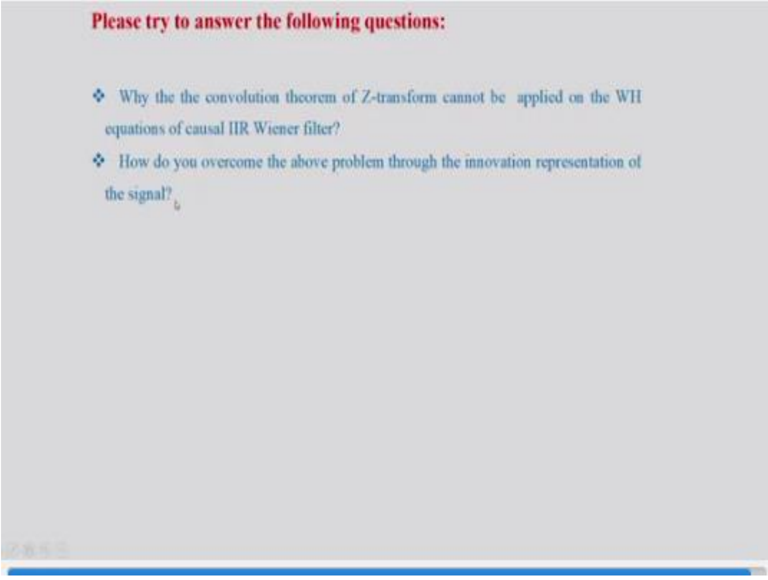

So, we have to find Sxy of z, He z inverse, He z and sigma v square, then we can find out Hz, please try to answer the following questions; why the convolution theorem of Z transform cannot be applied on the Wiener Hopf equations of causal IIR Wiener filter, so we noted that we cannot apply the convolution theorem of Z transform directly only Wiener Hopf equations of causal IIR Wiener filter, why? How do you overcome the above problem through the innovation representation of the signal Yn, so these 2 questions you try to answer, thank you.