Statistical Signal Processing
Prof. Prabin Kumar Bora
Department of Electronics & Electrical Engineering
Indian Institute of Technology, Guwahati

Lecture No 2
Probability and Random Variables

Hello students, in this lecture we will discuss the basic concepts of probability and random variables. These concepts are essential to build up the theory of statistical signal processing. We start with probability basics.

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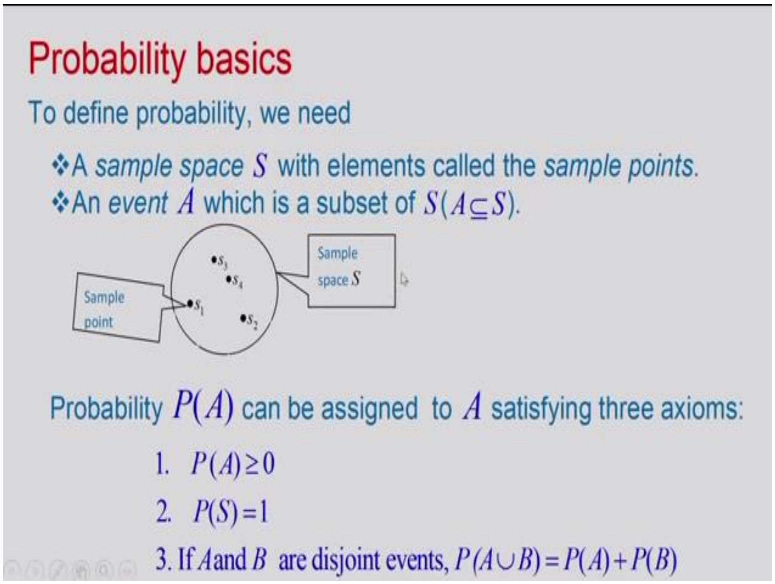

To define probability, we need a sample space S with the elements called the sample points and event A which is a subset of S. A is the subset of S. We are illustrating this with an example this is suppose, this is the sample space and it has four sample points S1, S2, S3 and S4. These are the sample points now probability PA can be assigned to A, satisfying three axioms. Number one is P of A is greater than equal to zero probability is always positive.

P of S is equal to 1 probability of this sample space is always equal to 1. If A and B are disjoint events, P of A union B is equal to PA+PB, this is the third axiom. So it says that, if two events are disjoint then there the probability of their union is the sum of their probabilities. To have a more consistent definition of probability the third axioms need to be modified.

However for that we need the concept of Sigma algebra therefore we will not discuss that form of the third axiom here, so we have defined probability, now let us see what is conditional probability.

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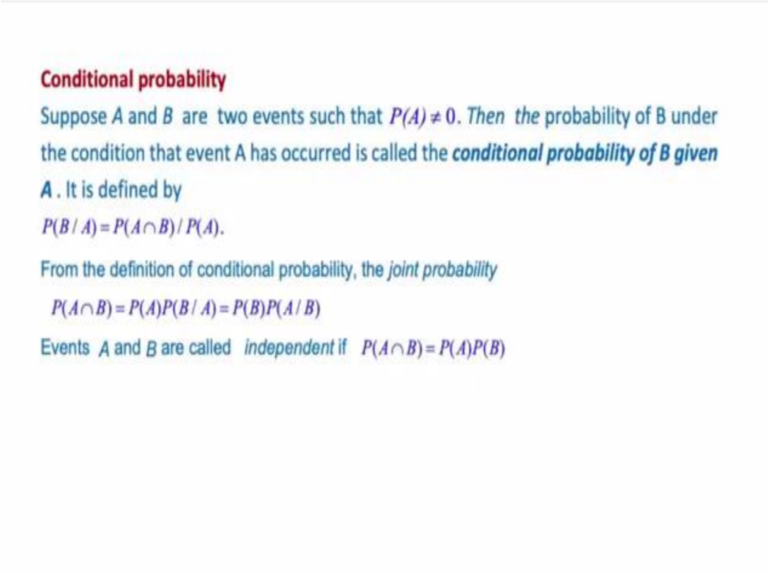

Suppose A and B are two events such that P of A is not equal to 0. Then the probability of B under the condition that event A has occurred is called the conditional probability of B given A. So it is defined by probability of B given A is equal to probability of A intersection B divided by P A. So what is the probability of the intersection of A and B that divided by P A, normalized by PA that will give us the conditional probability.

Similarly we can define probability of A given B from the definition of conditional probability. Now we can find joint probability P of A intersection B is equal to PA into B given A and that is also equal to P of B into P of A given B. Events A and B are called independent. If P of A intersection B is equal to P A into PB in the case of independent event, the joint probability is product of the individual probability.

This is very important concept independents. What does it imply; it is a property of probability that probability of joint event is the product of the probability of individual event.

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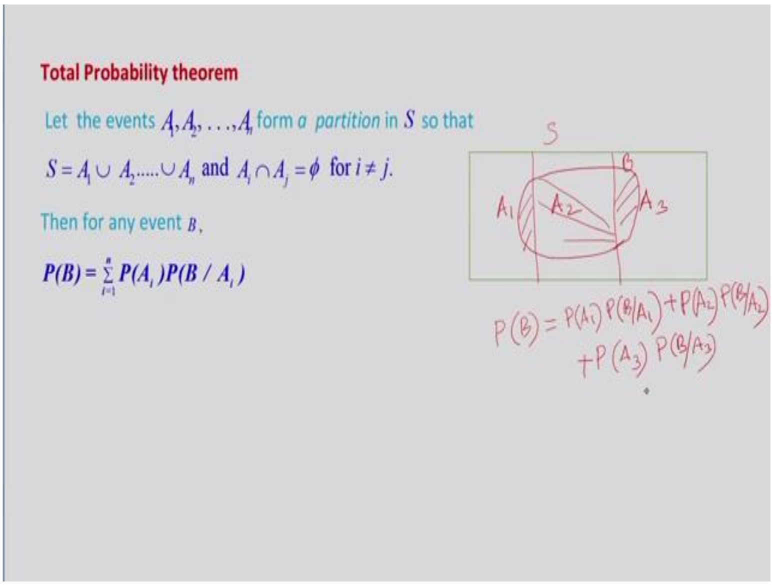

Total probability theorem, total probability theorem is one of the most important theorems in basic probability. Let us step this theorem, let the events A1, A2 up to An form a partition in S. So what is partition in S.? It means that S is equal to union of all these Ai's and Ai and Aj are mutually disjoint. So Ai intersection Aj is equal to Phi for i not equal to j. Then for any event B if we have a partition.

Then for any event B probability of B is equal to sum i going from 1 to N of P of Ai into P of B given Ai. Let us illustrate this suppose, this is my S and I have these events A1, A2 and A3. So A1, A2 and A3 forms a partition in S. Okay, because A1 and A2 are disjoint A1 and A3 are disjoint and all events are disjoint. Now you consider any event B. This is the event B. Now the probability of this event B is equal to the sum of the probabilities of this event.

Then, if I consider this event and then probability of this event. So that way probability of B is equal to probability of this, probability of this event is all the of A1 into a of B given A1 probability of this event is equal to probability of A2 into probability of B given A2 plus probability of this event that is equal to probability of A3 into probability of B given A3. So this illustrates the total probability theorem.

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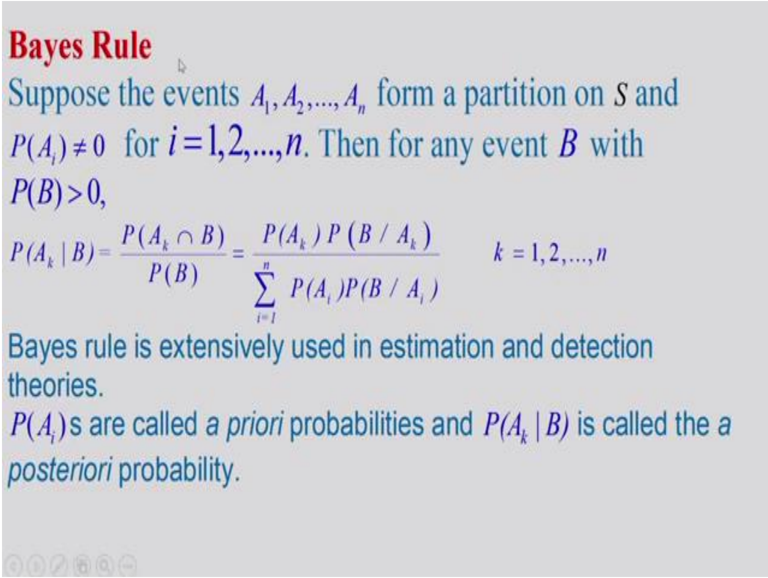

Now we will go to one of the very important result what is known as the Bayes rule. Suppose the events A1, A2 up to An form a partition on S. We know what is a partition and PAi is not equal to 0 for i is equal to 1 to n. So each of these probabilities are nonzero and these probabilities are called a priori probability. We know these events A1, A2 up to An and there are probabilities also that way these are a priori probabilities.

Then for any event B with P B probability of B greater than 0. We can write P of Ak given B. Suppose this event B is given then what is the probability of this prior event Ak. Probability of Ak given B and this by definition is equal to probability of Ak intersection B divided by PB. Now we apply the total probability theorem here, so this PB can be written in terms of the a priori probabilities and the conditional probability by this sum.

Similarly this intersection can be written in terms of the probability PAk into probability of B given Ak.This is true for k is equal to 1 to up to n. So this probability is now the a posteriori probability. A posteriori probability or posterior probability and it has extensive use in the estimation and decision theory. We have Bayesian estimation theory, we have Bayesian decision theory and we will be discussing Bayesian estimation theory

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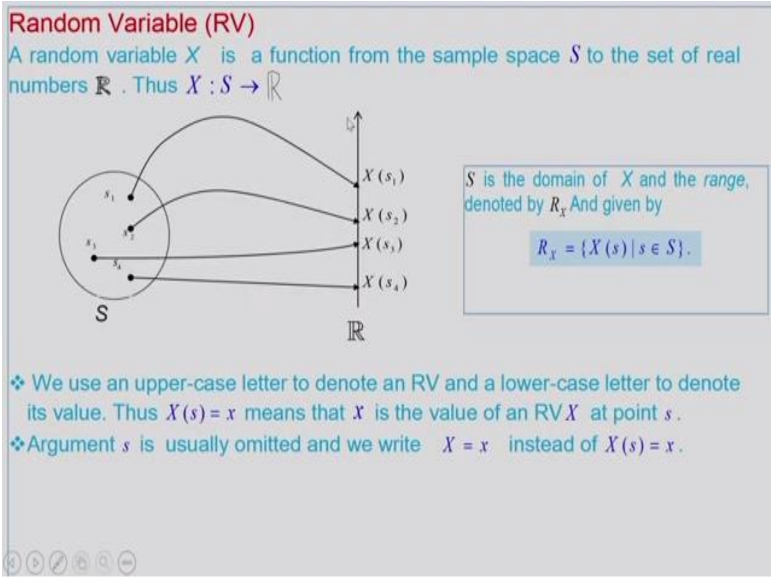

With this concepts of basic probability. Let us go to define random variable. Random variable it is abbreviated as RV. A random variable X is a function from the sample space S to the set of real number R. So actually it is a function. It is a mapping from the sample space S to the set of set of real numbers R. Thus X is a mapping from S to R. So since it is a mapping we have to define what is the domain?

S is the domain and the range Rx is a subset of R and given by Rx is equal to Xs. Such that X belongs to S. So those points are on R we said which has a corresponding point in s. So these points constitute the range. So we know what is the domain? Domain is S, range is R X which is a subset of R. We use an upper-case letter to denote a random variable and the lowercase letter to denote its value.

Thus X(s) is equal to x, a means that x is the value of a random variable X at a point s. Argument s is usually omitted and we simply write X is equal to small x instead of X s is equal to small x. So we have defined random variable. It is a mapping from the sample space to the real line.

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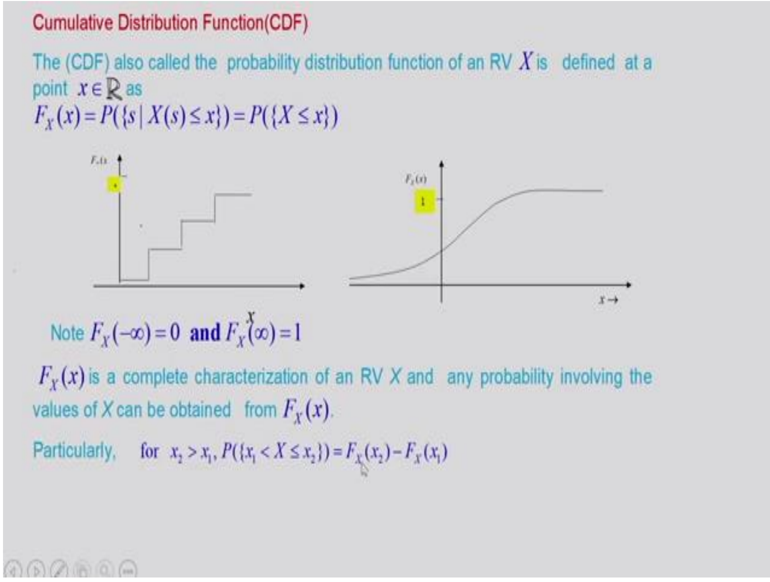

Now, how to describe probability in the real life .So we will define cumulative distribution function CDF. The CDF also called the probability distribution function of a random variable X is defined at a point X belonging to R as capital Fx of x. That is the CDF is the probability of those sample point s. Such that, X s is less than equal to small X. So we have to consider all s for which X s is less than equal to small X.

So that in that sense it is a cumulative probability. And instead of writing this symbol we write simply probability of X capital X less than equal to small X.This is the CDF. So since it is a cumulative probability.If we plot it if it is a sample space is finite then we may have this type of CDF staircase type of function. We says maximum below 1 and minimum below is always 0.

Similarly we may have a continuous CDF function. Where X will be ranging from minus infinity to plus infinity and we also observed that this this CDF here, here CDF is a non decreasing function also note that Fx of minus infinity is equal to 0 and Fx of plus infinity is equal to 1. CDF is a complete characterization of random variable X and any probability involving the values of x can be obtained from Fx of x.

Particularly suppose for x2 greater than x1, suppose we are interested to find out the probability what is the probability that X lies between x1 and x2. So this probability is equal to Fx of x2 - Fx of x1.

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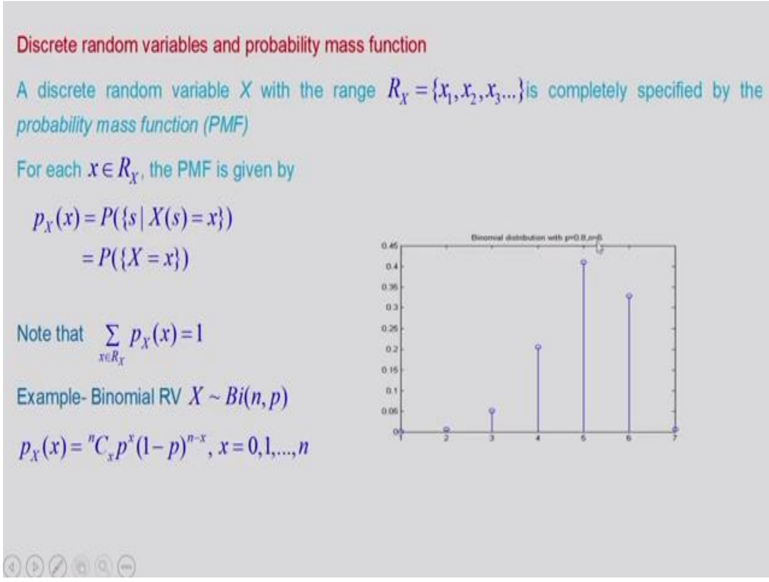

So we have discussed CDF which is a complete description of the random variables. We may have simpler description for example in the case of discrete random variables. We have the probability mass function. A discrete random variable X with this range, because discrete means it range is only discrete points comprises of discrete points. So such a random variable is completely specified by the probability mass function.

What is the probability mass function? That is we use the symbol small px of X. That is equal to probability. That those s such that X s is equal to small x. So that is equal to we write it in simple notation probability that X is equal to small x. So since probability mass represents a probability function therefore clearly we get summation Px of x over x belonging to Rx is equal to 1.

If we sum up all the probability mass function, this sum must be equal to 1.We can give an example binomial random variable, all random variables have specific symbol. This is Bi, n, p, binomial random variable with parameter n and p. So the probability mass function is described by small px of x, that is equal to n Cx P to the power x into 1- P to the power n-x.. For x equal to 0, 1 up to n.

So here we have plotted the PMF card, for P is equal to 0 it is a binomial random variable with P is equal to 0.8 and n is equal to 6. So this is the plot. So that way we can describe a discrete random variable in terms of probability mass function.

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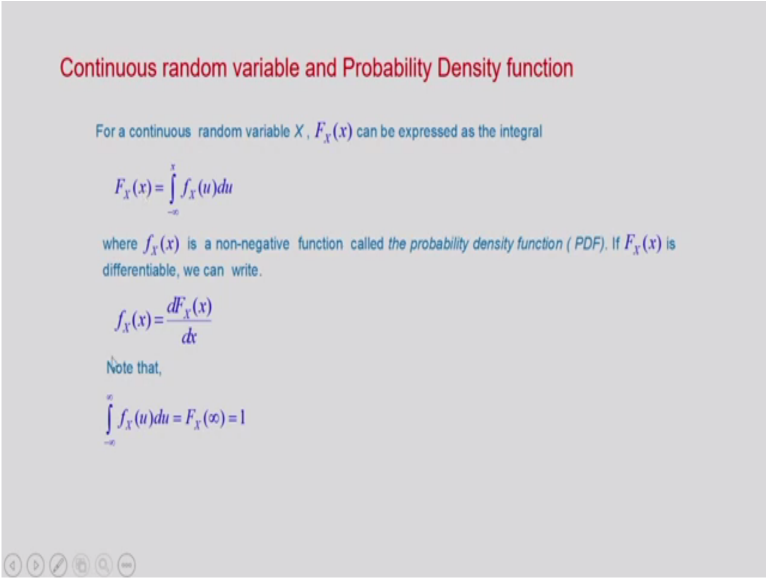

Next, continuous random variable and probability density function. PDF for a continuous random variable X ,the CDF fx of x can be expressed as the integral, this is the basic result. That CDF Fx of x can be expressed as the integral from minus infinity to x of fxu du. Now this quantity fx of x is a non-negative function and it is called the probability density function PDF.

If fx of x is differentiable,we can write PDF small fx of x derivative of CDF of x dFx of capital Fx of x. So if the PDF is the differentiable curve in that case we can find out the PDF at the derivative of the CDF and also we know that integration of because if I put a minus infinity and x infinity here this will be infinity and that is equal to 1. So that way integration of PDF over the entire range from minus infinity to infinity is equal to one.

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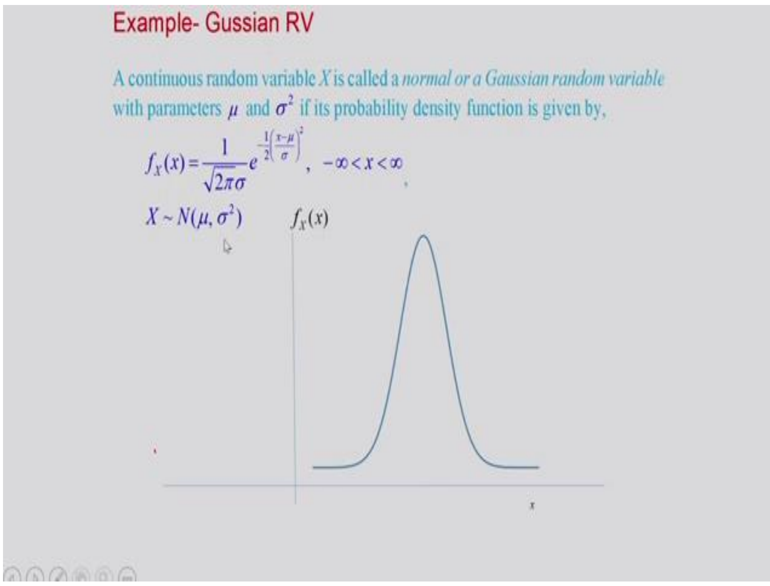

We will give an example Gaussian random variable or normal random variable. This random variable is the most important continuous random variable and it is described by two parameters mu and sigma square and the PDF is given by this expression small fx of x PDF at point x is e to the power minus this quadratic x- mu by sigma whole square divided by 2, and this term is normalized because it is a PDF normalized by root over 2 pi into sigma.

So this distribution occurs in many practical situation necessarily and we denote the normal distribution by X distributed as normal with parameters mu and sigma square. Here is a plot of this PDF.it is a bell-shaped curve and the parameter sigma will determine the extent of this distribution and mu is the central point. That is this, this is the mean or even mode of this curve or median of this curve is given by mu.

So we have defined a single random variable and how to characterize this random variable in terms of CDF and if it is continuous in terms of PDF and if it is described in terms of PMF. Now let us go to multiple random variables.

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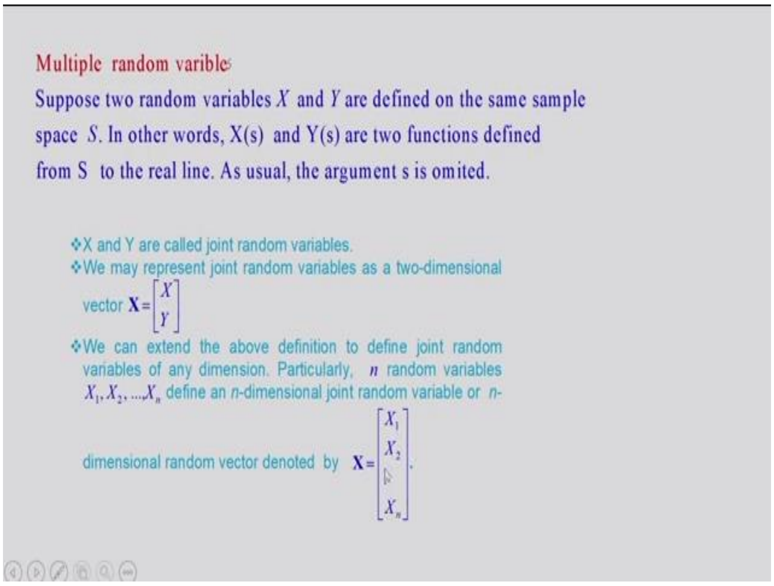

Suppose two random variables X and Y are defined on the same sample space S. In other words, Xs and Ys are two functions defined from S to the real line. As usual the argument s is omitted. Now X and Y are called joint random variables. We may also represent a joint random variable as a 2-dimensional vector. We can this is the vector notation, usually vectors are represented by a column matrix and here two component x and y.

We can extend the above definition of two random variables to any dimension. Particularly, n random variables X1, X2 up to Xn define n-dimensional joint random variables or n dimensional random vector denoted by X is equal to this is the these are the components X1, X2 up to Xn. This is a random vector with component X1, X2, up to Xn. Now how to characterize multiple random variables?.

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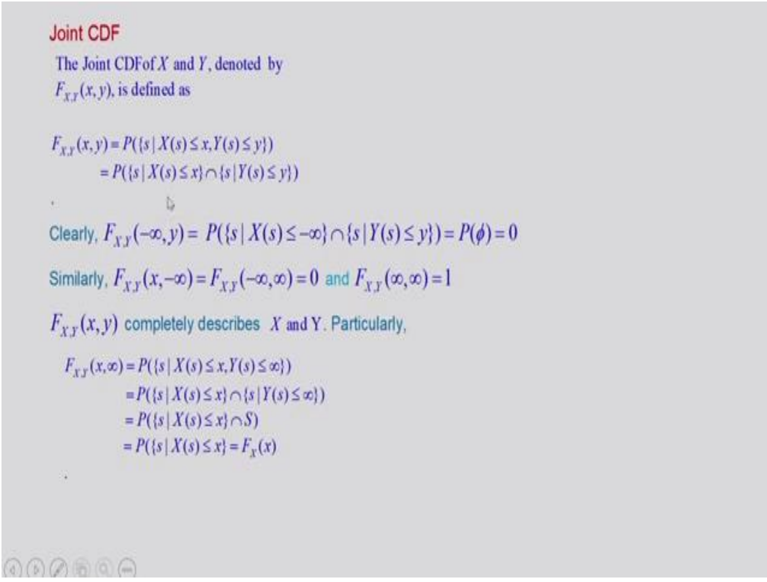

We have the joint CDF. The joint CDF of X and Y, denoted by this is the notation capital Fxy at point x,y is defined as how do we define this is the probability that s probability of those s for which X s is less than equal to small x and Ys is less than equal to small y. This come up in fact means and and we can write this in terms of this intersection event probability of the event s.

Such that Xs is less than equal to small x intersection s such that Ys is less than equal to small y. So this is the definition of joint CDF and this joint CDF completely described tool and random variables x and y. Now if we see Fxy at minus infinity y that is the intersection of this event s. Such that Xs is less than equal to minus infinity and s such that Ys is less than equal to Y.

Since this part is Phi therefore this intersection will be equal to Phi and therefore corresponding probability will be equal to 0. So joint CDF at minus infinity y is equal to 0. Similarly joint CDF at X minus infinity equal to joint CDF at minus infinity minus infinity is equal to 0. Also we can saw that the joint CDF at infinity infinity is equal to 1. As I said, the joint CDF completely described two random variables x and y.

we can determine any probability involving x and y using this CDF. Particularly if we consider f x y at point x infinity by definition this is equal to probability of s. Such that X s is less than equal to small X and Ys is less than equal to infinity. This Y s less than equal to infinity it will corresponding to it will correspond to this sample space. Therefore we will get probability of s.

Such that X s is less than equal to small x intersection with the sample space and and this will be any event if we intersect with s you will get that event only therefore it will be probability of s. Such that Xs is less than equal to small x and which is equal to C the F of x at Point small x. so therefore from the joint CDF we can find out the individual CDF effects of X and it is called the marginal CDF.

So the marginal CDF can be obtained from this joint CDF. Now we will go to continuous random variables, If the joint CDF is continuous.

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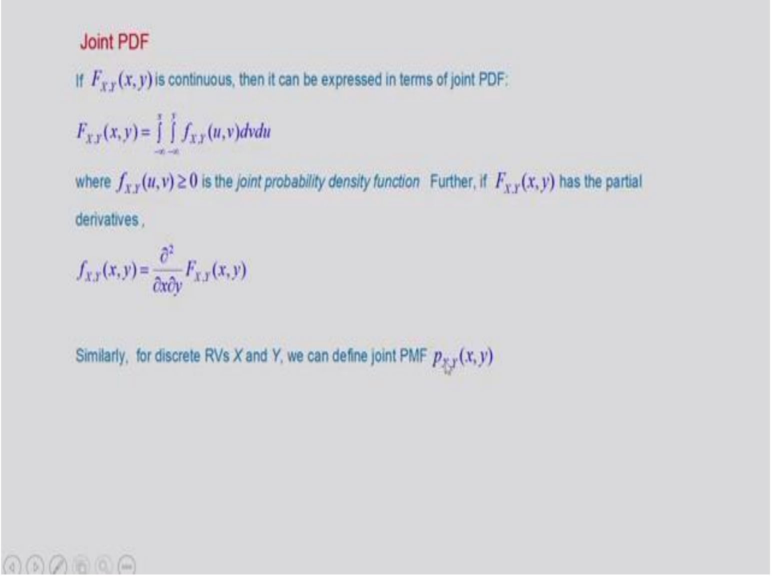

Then it can be expressed in terms of the joint PDF. This is the basic expression the joint PDF can be expressed as a double integral minus infinity to x and minus infinity to y of small F x y uv dv du. So this is the limiting expression between the CDF and the PDF and this PDF joint PDF is a non-negative quantity and if suppose CDF has the partial derivatives.

Then we can write the joint PDF at point x y as the second order mixed partial derivative. del square del X del Y of this CDF at point x y.Thus two continuous random variables x and y can be represented in terms of they are joint PDF small effects of y at point x y defined by this relationship. Similarly for discrete random variables x and y we can define joint probability mass function and it is the denoted by small Px y at point x y.

So same definition of suppose probability mass function can be extended to two dimension and to get the joint probability mass function.

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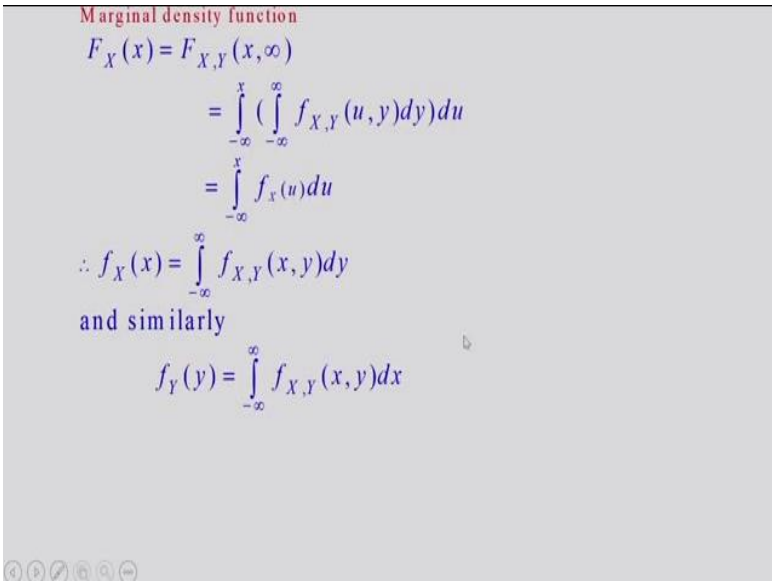

Now as I said joint CDF or joint PDF is a complete description of two random variables and from design CDF. We can get the individual CDF in individual PDF and so on. For example, FX of x that is the CDF of X and we saw that it is equal to joint CDF of XY at point X infinity. If we write in terms of the PDF this will be d integral. Integral minus infinity to X and since it is infinity integral from minus infinity to infinity of this PDF, PDF of the PDF.

So this is equal to the joint CDF of X Y at X infinity and which is the double integral from minus infinity to x and y infinity there from minus infinity to infinity of f xy that is the PDF at point u,y dy du. This is the expression for the CDF fx of x, now also see therefore fx of x has this expression this is the integration from minus infinity to x of the PDF at point u du. So therefore if we compare this expression and this expression then this bracketed term must be equal to fx of u.

The CDF fx of x is given by integration from minus infinity to infinity here minus infinity to infinity of the joint PDF with respect to dy. So if we have to find out fx of x, see the PDF at point x then it is the integration of the joint PDF with respect to the other random variable y.

Similarly if we have to find out the PDF of y in that case it will be the integration with respect to x. So that way marginal density concerns can be obtained from the joint density function.

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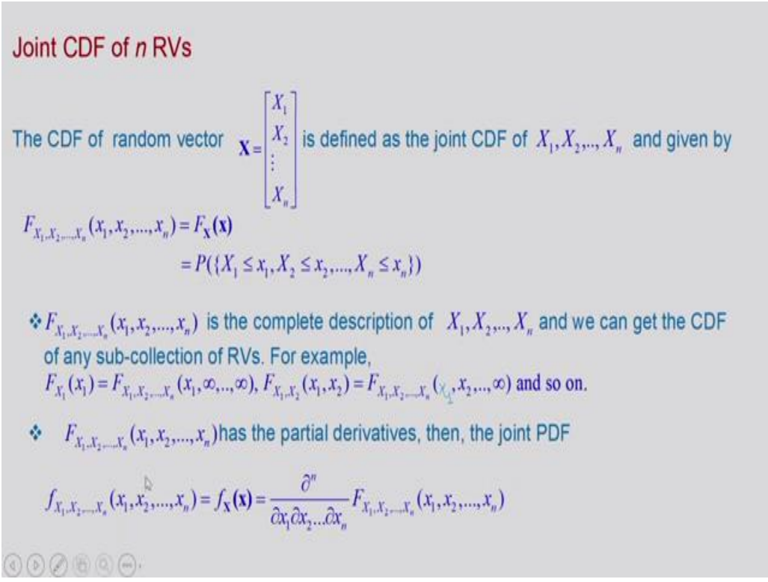

Now let us go to a more general case joint CDF of n random variables.The CDF of random vector x denoted by X1, X2 up to Xn is defined as this joint CDF. This is a CDF, because we are considering a random vector single quantity at the joint CDF of X1, X2 up to Xn and given by this is the expression of join CDF at point X1, X2 up to Xn and this is the CDF of the random vector and this is by definition probability.

That X1 is less than equal to small X1, X2 is less than equal to small X2 and up to Xn is less than equal to small Xn. So this is the joint event and probability of the disjoint event is this joint CDF, like in the two dimensional case the joint CDF is the complete description of random variable X1, X2 up to Xn and we can get the CDF of any sub collection of random variables from this set.

For example if we have to find out only CDF of X 1 so this is the joint CDF at point X 1 and rest of the point at infinity. Similarly if we have to find out this joint CDF of X1, X2 at Point small X1, X2 then this is the n-dimensional CDF at point X1, X2 and rest of the point set in this way we can find out the joint CDF of any order. If the joint CDF at point X1, X2 up to Xn has the partial derivatives and design PDF.

Now we can defend it on PDF can be written as this is the symbol small F X1, X2 up to Xn at point X1, X2 up to Xn and this is the PDF of the random vector at points small X vector and this is the NS order mixed partial derivative. del n del X 1 del X 2 up to del Xn of the CDF. In this way we can relate the PDF and the CDF of multiple random variables.

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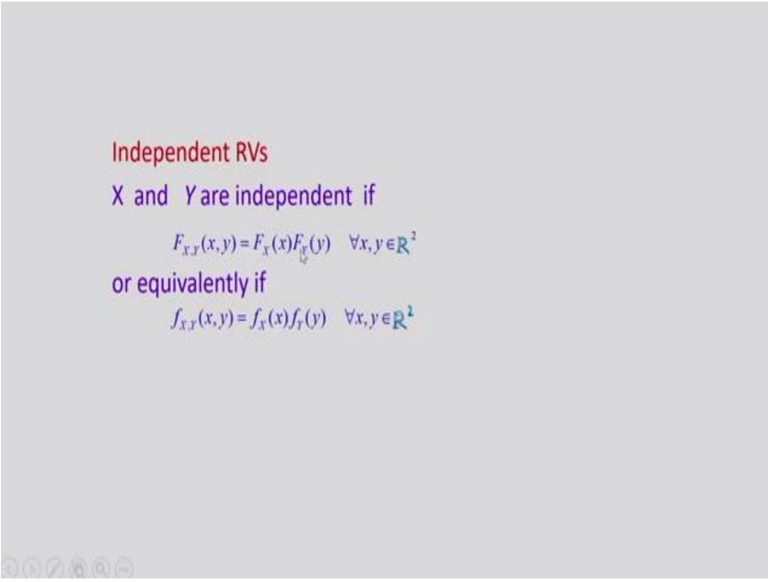

Independent random variables, we know what are independent events. The same definition is generalized here x and y are independent if the joint CDF at point x y is equal to Fx of x into Fy of y . This is true for for all XY belonging to R2. So that way this definition is a very strict definition because this relationship should be valid at every pair of point xy belonging to r 2 or equivalently in terms of PDF joint PDF is the product of marginal PDF.

This definition of independent random variables can be extended to n random variables in that case n dimensional CDF will be the product of n marginal CDF and n dimensional PDF will be the product of and marginal PDF.

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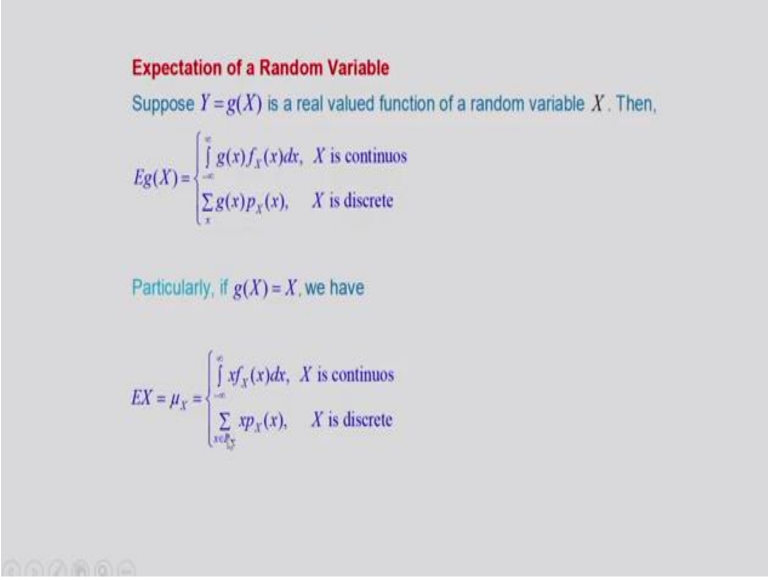

So we have seen how to characterize random variables in terms of they are CDF, PDF, PMF or in the case of multiple random variables in terms of joint CDF, joint PMF and joint PDF. Next we'll be discussing how to partially represent a random variable in terms of expectation operation. Suppose Y is equal to gX is a real valued function of a random variable X. Then, E of gX is defined by this integral g x into CDF PDF of x dx.

If x is continuous and this is a summation if X is discrete. So in the case of discrete summation of our X of gX into PMF of X. Particularly if gX equal to X, we have gX is equal to X. Then we have Ex is equal to that is called the mean mu X and this is by definition this is equal to integral minus infinity to infinity X FX of X DX. If X is continuous and summation over X of X px of X if X is discrete.So we know what is the mean of a random variable.

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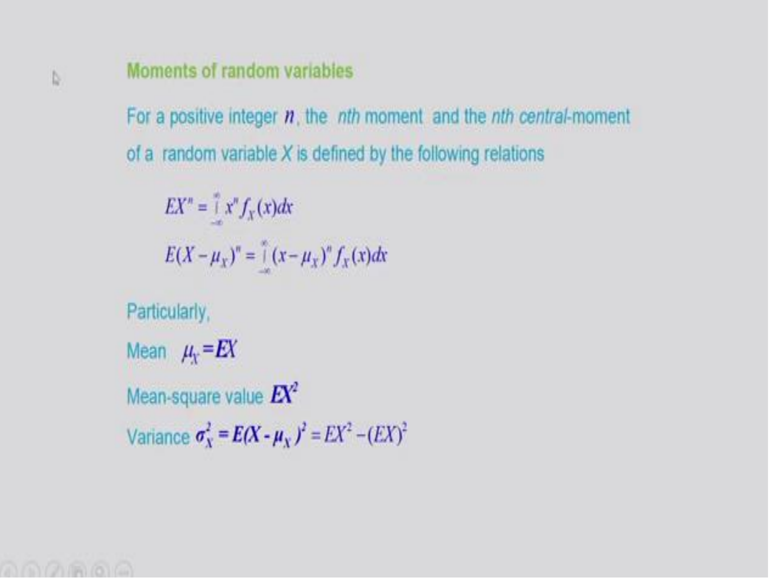

We also defined a moment's of random variables. Moments and central moment will be defining for a positive integer n, the n'th moment and the n'th central moment of a random variable X is defined by the following relations. E of X to the power n that is the nth moment and I know the definition of expectation. So this will be integration minus infinity to infinity X to the power n PDF of x into dx.

Similarly, central moment that is the NS order moment about the mean. So we have to consider E of X minus mu X to the power of n. So that way we will get the central moment and particularly these moments are important for us mu X that is the first order moment that is equal to EX. Mean square value E of X square and similarly variance is Sigma x squared that is E of X minus mu X whole square and this is also equal to we can use the property of expectation and establish that this is equal to E of X square minus E of X whole square.

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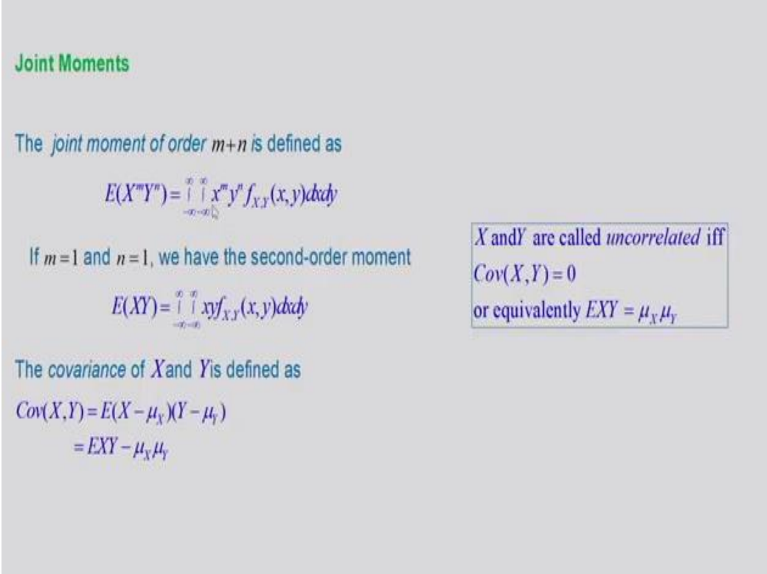

Similarly in the case of multiple random variables for example joint random variables X and Y. We can define the joint moment of order m+ n and this is given by E of X to the power m into Y to the power of n and you are taking the mth power of X into a nth power of Y and then taking the average value and this is given by this integral. Particularly if m is equal to 1 and n is equal to 1, we have the second order moment E of X Y.

So here it will be x y and multiplied by the joint PDF and even integrate from minus infinity to infinity. The second order moment E of X Y, it is also called the correlation between X and Y. So now we can define the second order central moment that is the covariance of X and Y and it is defined as covariance of X Y is equal to E of X - mu x into Y - mu y, and it can be shown that this is equal to E of XY - mu X into mu Y.

This is the covariance of random variable X and Y. Now covariance is a very important operation and it leads us to the definition of uncorrelated random variables X and Y are called uncorrelated, iff covariance of X Y is equal to 0 or equivalently E of XY is equal to mu x into mu y. So this is a relationship or this is a definition in terms of the average value unlike independence which is defined in terms of PDF or CDF. Here uncorrelated, this is defined in terms of covariance.

This is an average value so that that way this definition is more realist and it has a lot of importance in any signal processing application and also we can show that if random variables are independent. Then they will be uncorrelated. But the converse is not generally true. Now let us see how to characterize random vectors in terms of mean correlation and covariance.

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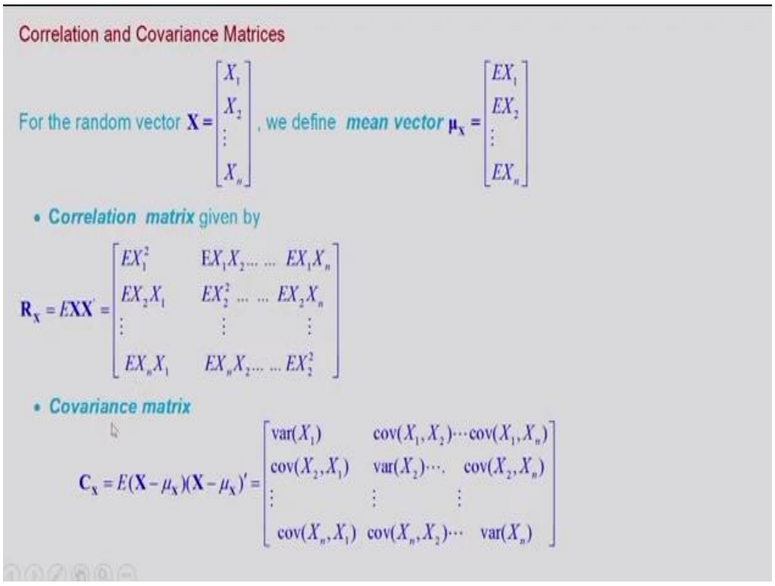

So for random vector X, this is the random vector. We defined a mean mu x. mu x is again a vector and its components are individual mean. So mu x is a vector consisting of EX1, EX2 up to EXn and correlation. Now in the case of two random variables you can find out correlation. Now since there are n random variables we have the correlation matrix. So how to get the matrix.

So we multiply X by X transpose and then we will take the expectation of the individual components. So that way the diagonal elements will be the mean square value E of X1 square, E of X2 square like that E of Xn square. This is E of Xn square and the off diagonal elements will give the correlation E of X1, X2, Similarly E of X2, X1. Similarly we can define the covariance matrix.

Covariance matrix again to get the matrix we have X minus mu X vector into X minus mu X vector transpose and then we will take the expectation of the individual terms. So that way we get a matrix with diagonal elements as the variances. Variance of X1, here variance of X 2, here here variance of Xn and of diagonal elements are the covariance values. So that way we get the in the case of random multiple random variables the correlation structure can be expressed succinctly in terms of the correlation matrix and the covariance matrix.

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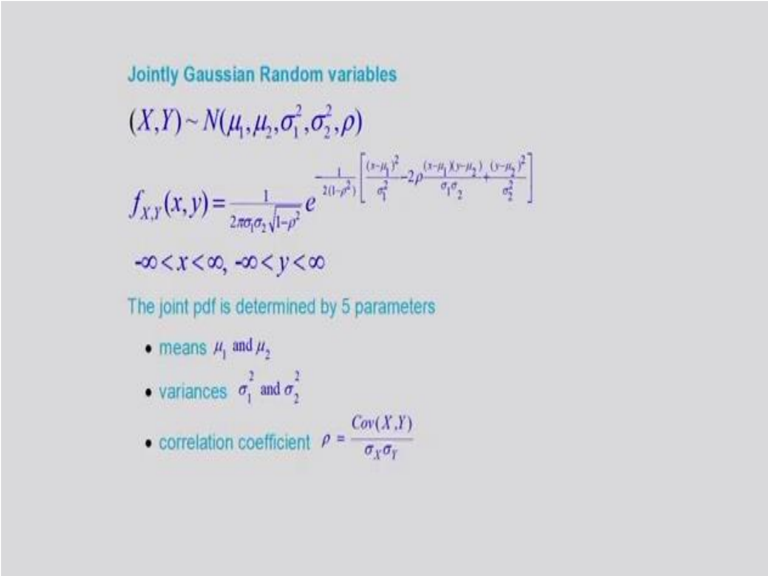

We will define jointly Gaussian random variables.X and Y are jointly Gaussian random variables distributed as normal with parameters mu1, mu2, sigma1 square, sigma2 square, raw and the joint PDF is given by this expression it is exponential with negative quadratics in x and y and there is a normalizing factor 2Pi, Sigma1, Sigma2 into root over 1 minus raw square and thus this joint PDF joint gaussian PDF is this determined by Pi parameters.

Means mu1 and mu2, it can be shown that mu1 is the mean of X X and mu2 is the mean of Y variance is Sigma1 square, Sigma2 square. Sigma 1 square is the variance of X and Sigma 2 square is the variance of Y and the correlation coefficient raw. That is covariance of XY divided by Sigma X into Sigma Y in this case it is Sigma 1 into Sigma 2. So these five parameters describe this jointly gaussian random variables x and y. We can also define the gaussian random vector X.

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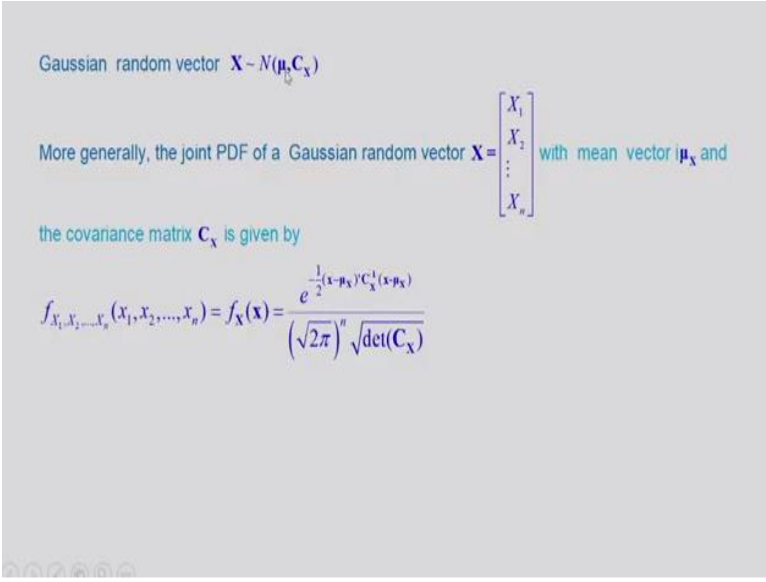

It is distributed as normal with parameter mu vector, mean vector and it covariance matrix. So if suppose this is the random vector and mean is mu X and the covariance matrix is CX then this is the expression for day joint gaussian PDF and this is again this is X minus exponential minus 1/2 of X minus mu X transpose C X inverse.This CX inverse is the inverse of the covariance matrix into X minus mu X.

Now normalizing factor is root over 2 pi to the power n into determinant of the matrix CX, CX is a matrix and we can find the determinant of CX so that way this is the joint gaussian PDF of n random variables.

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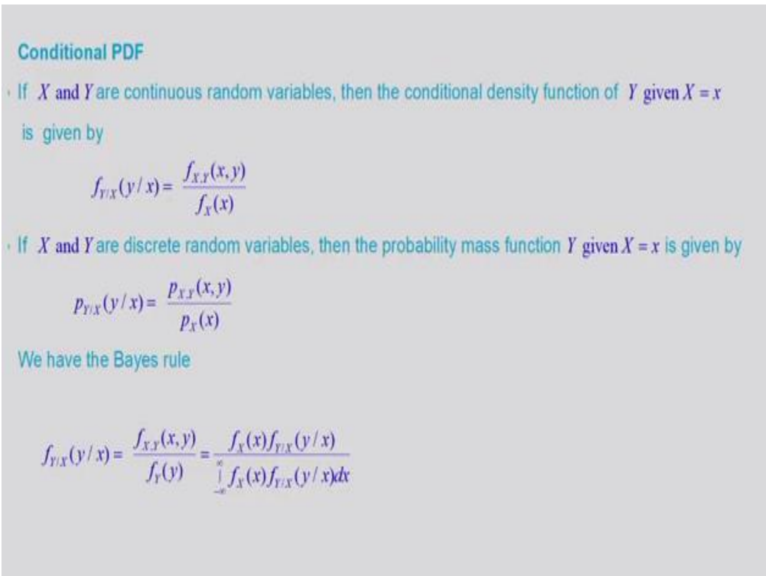

We have characterized multiple random variables in terms of joint CDF joint PDF joint PMF etc. Now let us see how to describe the conditional probabilities, suppose if x and y are continuous random variables then the conditional density function of Y given X is equal to small x is given by this this is the conditional PDF of Y given X and this is a joint PDF divided by the marginal PDF at point X of course we need the condition that fx of x not equal to 0, fx of x not equal to 0.

Similarly if X and Y are discrete random variables then the probability mass function of Y given X is equal to small x is given by this expression this is the conditional probability mass function again this is the ratio of the joint probability mass function divided by the marginal probability mass function and we can use this definition of conditional PDF to derive the Bayes rule for continuous random variable.

This is given by F Y given X at point y given small X. That is by definition, this is the joint PDF of X Y divided by the marginal PDF of Y and now if we can write joint PDF in terms of conditional PDF.We will get the numerator and again this marginal PDF can be written in terms of the joint PDF. This is given by this so that way we get the Bayes rule for continuous random variables with the definition of conditional PDF and conditional PMF. We can also get,

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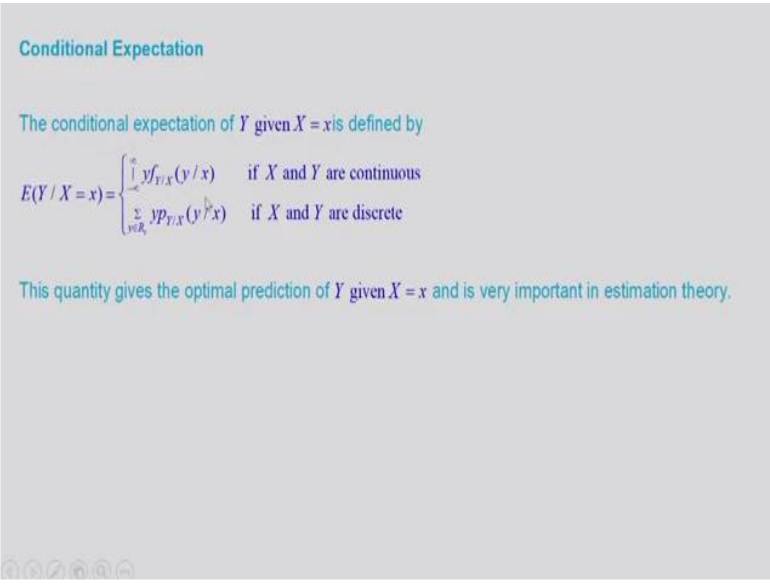

The conditional expectation, the conditional expectation of Y given X is equal to small x is defined by, E of Y given X is equal to small x.We are finding out the conditional expectation of Y given X is equal to small x. So this is defined by this integral in the case of continuous random variables, this is the integration, it is expectation with respect to Y therefore it is integration of Y multiplied by the conditional PDF and then you integrate with respect to dy.

Similarly this if we have to take the sum in the case of discrete random variable this is y multiplied by the probability mass function of Y given that X is equal to small X. so that way we can determine the conditional PMF, we can determine the conditional expectation and it has importance in estimation theory because this conditional expectation gives the optimal prediction of Y given X is equal to X. What is optimal we have to explain further and therefore it is very important in estimation theory.

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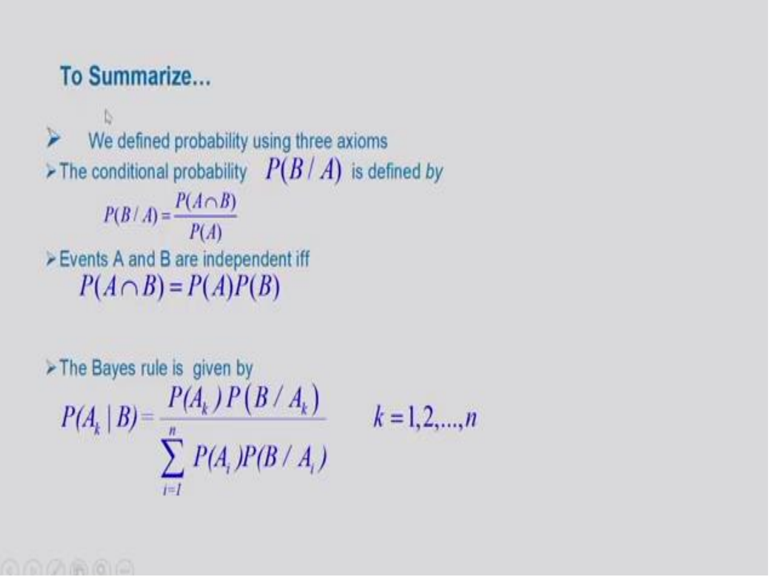

Let us summarize what we have done today we defined probability using three axioms.Then we defined conditional probability by this expression conditional probability of B given a is equal to probability of A intersection B divided by P A, events A and B are independent.If and only if probability of a intersection B is equal to PA into PB and finally in the probability case we defined a Bayes rule.

By this relationship, this is the posterior probability and it can be found in terms of prior probabilities. Probability of Ak into probability of B given Ak divided by sum of i going from 1 to n of P Ai into probability of B given Ai.This is the expression we obtain through total probability theorem.we also noted that a random variable X is characterized by CDF, fx of x.

That is defined by probability of those sample point for which X s is less than equal to small x. Similarly we defind probability mass function; probability of s such that X s is equal to small X and this is for discrete case. Similarly we defined the PDF probability density function; it is given by this expression.We also noted that joint random variables X and Y are characterized by joint CDF, joint PMF and joint PDF.

These are the corresponding functions we have defined joint CDF, joint PMF and joint PDF and we introduced one very important concept x and y are independent if and only if the joint CDF is equal to product of marginal CDF for all XY belonging to R2. Similarly the same can be defined in terms of joint PDF joint PDF is the product of the marginal PDF for all XY belonging to R2.

So this is the definition of a independent random variables.X and Y will be independent if join if joint CDF is product of marginal CDF or equivalently joint PDF is the product of marginal PDF.

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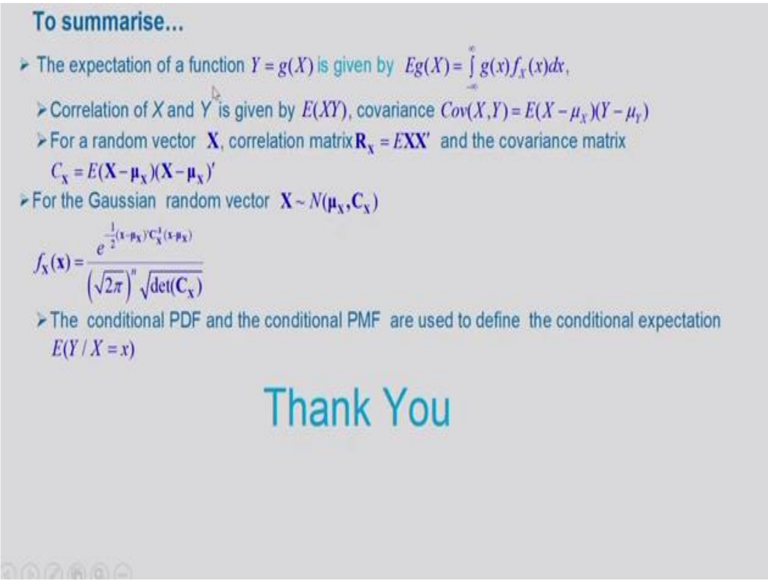

How we define the expectation of the function by this relationship E of gX is integration from minus infinity to infinity of gx fx of x dx. Then correlation of X Y is given by E of X Y covariance of X Y is given by E of X minus mu x into y minus mu y. We define what is called lesson?Whatis covenants for a random vector X? The correlation matrix Rx and the covariance matrix.

Correlation matrix Rx is equal to E of X X transpose and the covariance matrix Cx is equal to E of X minus mu X into X minus mu X transpose. These are the description of correlation structure in the case of random vectors.And we introduced the n dimensional Gaussian random vectors which is given by X it is a normal distribution with mean mu vector and covariance Cx matrix and this is the PDF of the Gaussian random vector.

Conditional PDF and the conditional PMF are used to define the conditional expectation finally we define the conditional expectation E of Y given X is equal to small x. It can be defined in terms of the conditional PDF and conditional PMF. In the next lecture we will discuss the linear algebra of random variables. Thank you.