Statistical Signal Processing
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Lecture – 19
Noncausal IIR Wiener Filter

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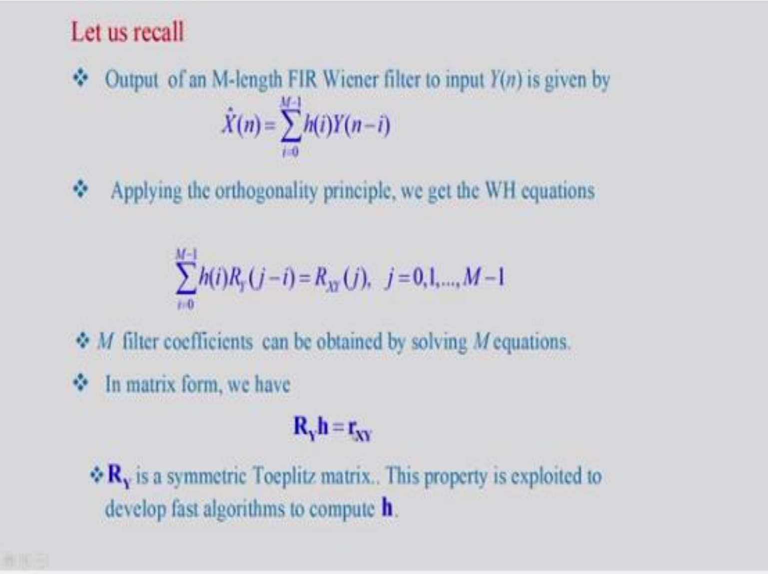

Hello students, welcome to this lecture on non-causal Wiener IIR filter, let us recall the output of an M length FIR Wiener filter to input Yn is given by this equation that is estimated signal X hat n is the summation hi Y n – i; i going from 0 to M – 1, this is the M length Wiener filter output, applying the orthogonality principle we get the Wiener Hopf equations that is given by this relationship; summation hi into Ry j – i; i going from 0 to M - 1 equals Rxy of j.

These are the Weiner Hopf equations for j is equal to 0, 1, etc., up to M – 1, there are M equations; therefore M filter coefficients can be obtained by solving these M equations. In matrix form, we write Ry matrix into h vector is equal to rxy vector, this Ry is the correlation matrix, h is the filter coefficient vector and this is the cross correlation vector and we also noted that Ry is a symmetric Toeplitz matrix.

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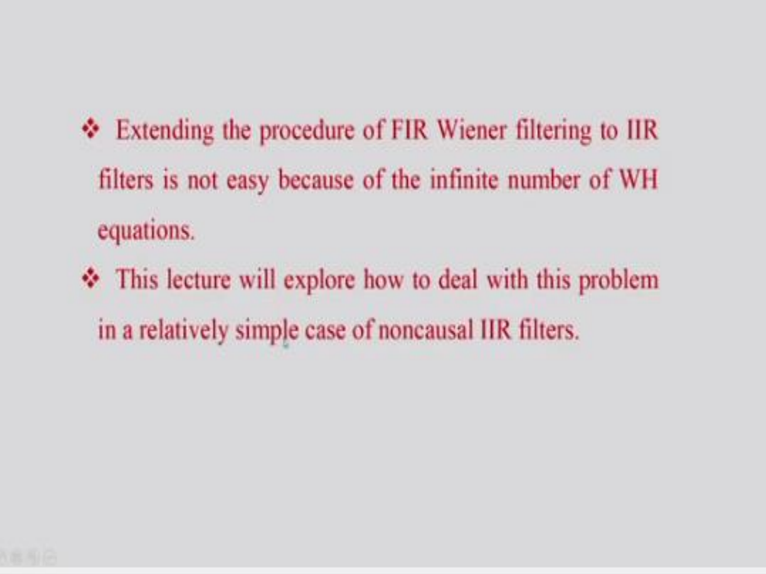

This property is exploited to develop fast algorithms to compute h, extending the procedure of FIR Wiener filtering to IIR filters is not easy because of the infinite number of Wiener Hopf equations, so we do not have any way to solve the infinite number of Wiener Hopf equations in the time domain. This lecture will explore how to deal with the problem in a relatively simple case of non-causal IIR filters.

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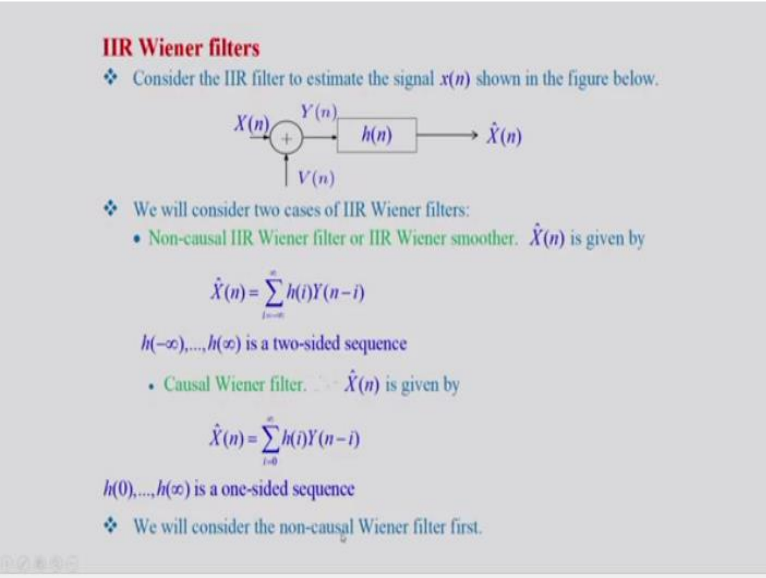

Consider the IIR filter to estimate signal xn shown in the figure below, this is the filter suppose, this is Xn + Vn that is Yn and this is filtered by an IIR filter to get X hat n and this hn is an infinite duration sequence, we will consider 2 cases of IIR Wiener filters; first one is non-causal IIR Wiener filter or IIR Wiener smoother, it is also called a smoother. In this case, X hat n is given by X hat n is equal to summation hi into Y n – i; i going from minus infinity to plus infinity.

The filter coefficients h minus infinity up to h infinity constitute a 2 sided sequence, here the filter coefficients forming 2 sided sequence, next is causal Wiener filter here, X hat n is given by X hat n equals to summation hi Y n – i; i going from 0 to infinity, thus hi are 0 for i less than 0, here h0, h1 up to h infinity a constitute a one-sided sequence, the filter coefficients form a one-sided sequence.

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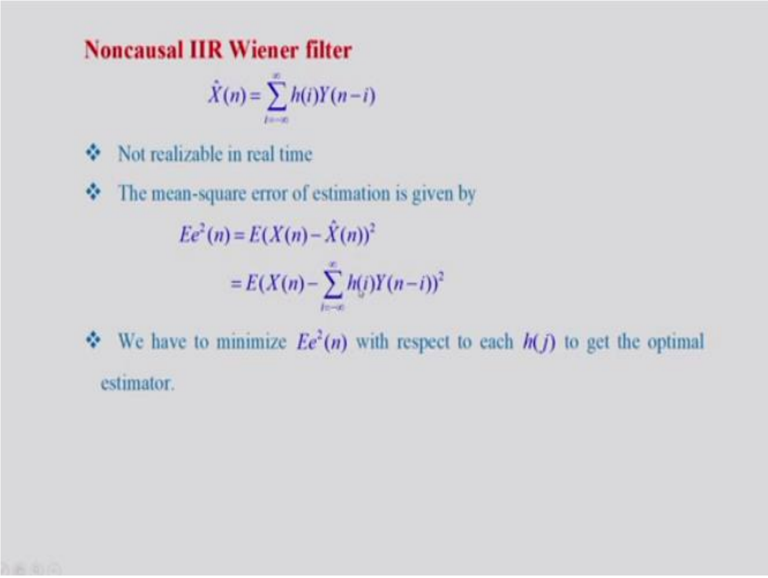

We will consider the non-causal Wiener filter first and now, in this case as I have told X hat n is given by summation hi into Y n – i; i going from minus infinity to plus infinity, this filter is not realizable in real time because the filtering involved future data here, there will be data corresponding to Y n + 1, Y n + 2 etc., therefore this filter is not realizable in real time. Now, the mean square error of estimation is given by E of e square n that is E of Xn minus X hat n whole square.

And this I can expand E of Xn minus summation hi into Y n – i; i going from minus infinity to infinity whole square, so this is the E of e square n, we have to minimize E of e square n with respect to each hj to get the optimal estimator, so we have to optimally estimate this hj's.

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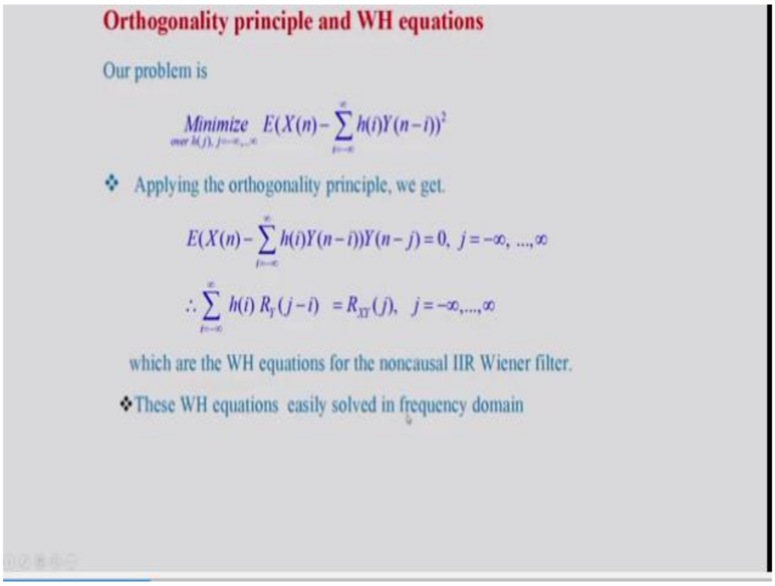

Now, we will see how we can apply the orthogonality conditions to get the Weiner Hopf equations, our problem is to minimize E of Xn minus summation hi into Y n – i; i going from minus infinity to infinity whole square over hj, j going from minus infinity to infinity, this is our minimization problem, we have to minimize this mean square error with respect to all hj. Applying the orthogonality principle, we get E of Xn minus summation hi into Y n – i; i going from minus infinity to infinity into Y n – j is equal to 0 for j is equal to minus infinity up to infinity.

Now, this equation can be rewritten as summation hi into Ry j – i; i going from minus infinity to infinity is equal to Rxy j because E of Xn into Y n - j will be equal to Rxy j, similarly E of Y n - i into Y n - j will be Ry of j – i, therefore we get this equation; summation hi into Ry j – i; i going from minus infinity to infinity that must be equal to Rxy of j, for j going from minus infinity to plus infinity.

So, therefore this is the equation which relates the autocorrelation of the data with the cross correlation now, this infinite set of equations are the Wiener Hopf equations for the non-causal IIR Wiener filter, this Wiener Hopf equations are easily solved in the frequency domain.

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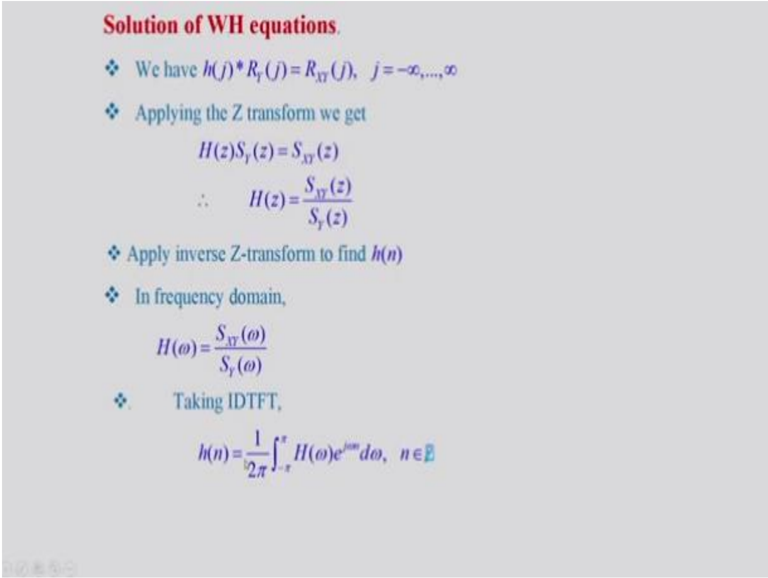

Thus we have hj convolve with Ryj is equal to Rxy j; j going from minus infinity to plus infinity, so this is the relationship we have that is Rxyj is the convolution of hj and Ryj, applying this Z transform we get, Hz, this is Z transform of hj sequence into Syz is equal to Sxy z, this is the relationship in the Z transform domain, from which will get Hz is equal to xyz divided by Syz, so this is the power spectral density in terms of Z and this is the power spectral density in terms of Z.

Therefore, the transfer function or system function is the ratio of cross spectral density and power spectral density, therefore we have found out Hz and we can apply inverse Z transform to find hn sequence. In the frequency domain we can write that is H Omega is equal to Sxy omega divided by Sy omega in terms of Omega, we can write like this. Taking the IDTFT, inverse DTFT, we get hn is equal to 1 by 2 pi integration minus pi to pi H Omega e to the power j omega n d omega, where n belongs to Z.

So, for all integers we can find out the filter coefficient, so we have found out the filter coefficients corresponding to non-causal IIR Wiener filter.

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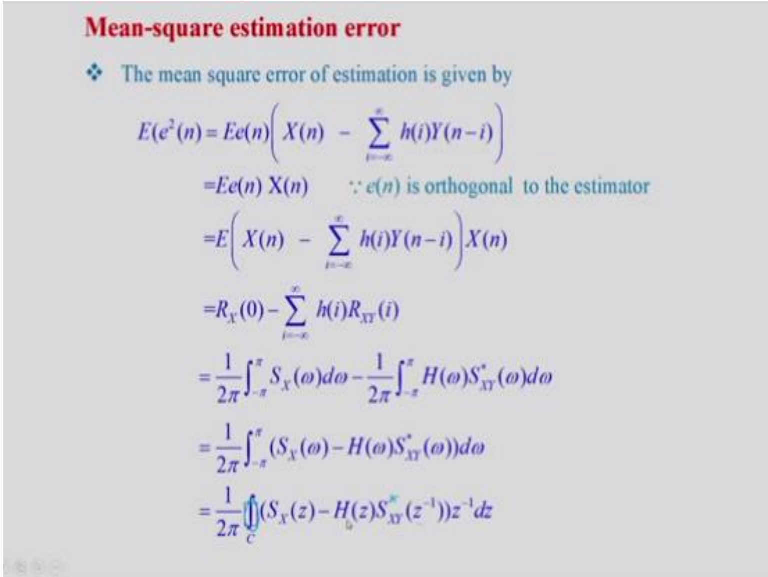

Let us find out the mean square error of estimation; the mean square error of estimation is given by E of e square n that is equal to E of en; en we are writing like this, Xn minus summation hi into Y n – i; i going from minus infinity to infinity and since en is orthogonal to this part, en is orthogonal to the estimator we can write E of e square n is equal to E of en into Xn.

Again expanding en, we can write MSE is equal to E of Xn minus summation hi Y n – i; i going from minus infinity to infinity into Xn, this is the mean square error. Now, E of Xn into Xn will be equal to Rx of 0 similarly, E of Y n - i into Xn will be equal to Rxy of i so that way, E of e square n MSE is equal to Rx of 0 minus summation hi into Rxy of i; i going from minus infinity to infinity.

And now, we can write this expression in terms of the DTFT, so that way Rx of 0 is 1 by 2 pi integration minus pi to pi Sx omega d omega - 1 by 2 pi integration minus pi to pi of H omega into Sxy star omega d omega and we can simplify this expression because we can take this integral outside, so that way it will be equal to 1 by 2 pi integral from minus pi to pi Sx omega - H omega into Sxy star omega whole into d omega.

So, this is the expression for mean square error and similarly, if we use a closed contour other than the unit circle, then we can write the E of e square n in terms of Z transform like this, so this is the expression for mean square error.

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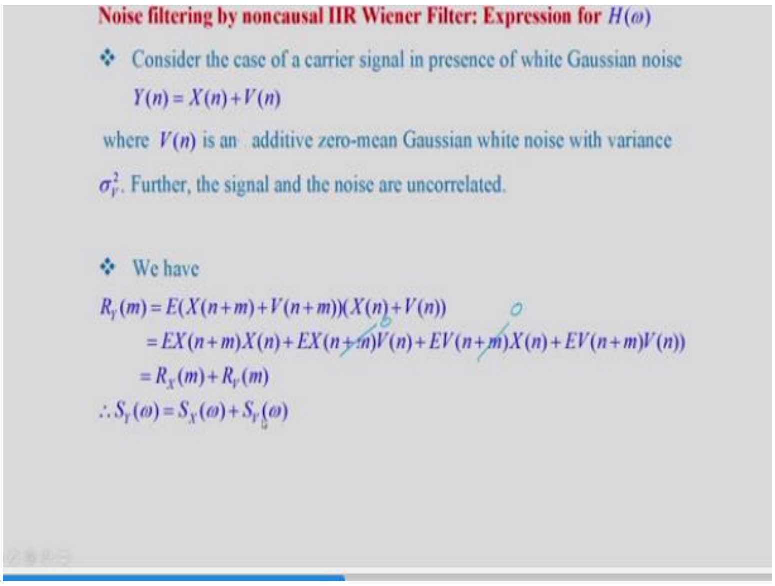

We will consider the case of noise filtering by non-causal IIR Wiener filters; consider the case of a carrier signal in presence of white Gaussian noise that is Yn is equal to Xn; Xn is a sinusoid + Vn, where Vn is an additive 0 mean Gaussian white noise with variance sigma v square, so this variance is sigma v square, further the signal and the noise are uncorrelated, this we assumed.

Now, we have Rym by definition E of X n + m + V of n + m into Xn + Vn, now we can expand this term, we will get E of X n + m into Xn, E of X n + m into Vn, E of V n + m into Xn + E of V n + m into Vn, now signal and noise are uncorrelated, so this part will become 0, similarly this part will become 0, therefore we will have, this is Rx of m plus this quantity is Rv of M.

So, therefore Rym is equal to Rx of m + Rv of m, taking the Fourier transform we will get Sy of omega is equal to Sx of omega + Sv of omega, so this way we can compute the output power spectral density.

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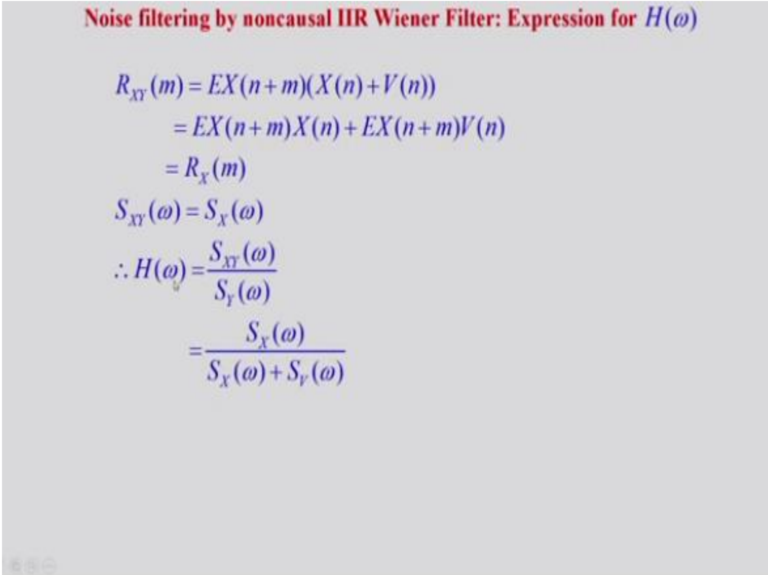

Similarly, Rxym that is the cross correlation between X of n + m and Yn and Yn we will write as Xn + Vn, therefore Rxy of m will be equal to E of X of n + m into Xn + E of X of n + m into Vn, now again this part is uncorrelated therefore, we will get simply Rxy of m is equal to Rx of m, so cross correlation between X of n + m and Xn is equal to Rx of m, it is same as the autocorrelation of Xn sequence.

By taking the DTFT, we will get Sxy of omega will be simply Sx of omega, therefore (()) (15:13) per function of the IIR non-causal Wiener filter that is H Omega and it is given by Sxy omega divided by Sy omega and Sxy omega is equal to Sx omega and Sy omega is equal to Sx omega + Sv omega, therefore H omega will be equal to Sx omega divided by Sx omega + Sv omega.

So, this way we can determine the system function or transfer function of the non-causal IIR Wiener filter in terms of the power spectral densities of the signal and the observed data.

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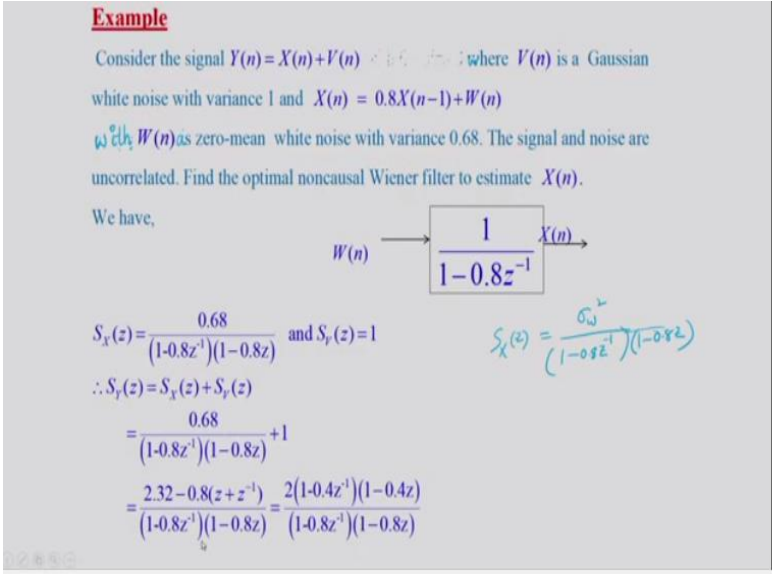

Let us consider one example; consider the signal Yn equal to Xn + Vn, where Vn is a Gaussian white noise with variance 1 and Xn is equal to 0.8 X n - 1 + Wn, with Wn as zero mean white Gaussian noise with variance 0.68, this signal and noise are uncorrelated, find the optimal non-causal Wiener filter to estimate Xn? So, we have observed signal model, signal plus noise, noise variance is given and signal model is given.

Therefore, we can find out the optimal non-causal Wiener filter to estimate Xn, now this signal model is like this, Wn is the input it passes through a filter of transfer function 1 by 1 - 0.8 z inverse and Xn is the output, this is the signal model because this is a white noise we can find out the power spectral density of the signal Sxz is equal to variance of the white noise that is 0.68 divided by 1 - 0.8 z inverse into 1 - 0.8 z.

So, we have use the relationship that is Sx of z is equal to sigma W square that is the variance of the white noise divided by 1 - 0.8 z inverse into 1 - 0.8 z, 1 by this quantity is the causal transfer function and 1 by this quantity will be the corresponding non-causal transfer function and since Vn is a Gaussian white noise with variance 1, we can write Sv of z is equal to 1, so these are the power spectral density of the signal and the noise.

Now, Syz is equal to Sxz + Svz, we will write this expression and this expression we will get this result; 2.32 - 0.8 z + z inverse divided by 1 - 0.8 z inverse into 1 - 0.8 z, now the numerator part also we can factorise, we can take 2 common and then we can factorize like this; 2 into 1 - 0.4 z inverse into 1 - 0.4 z divided by 1 - 0.8 z inverse into 1 - 0.8 z, so this is the power spectral density of the output.

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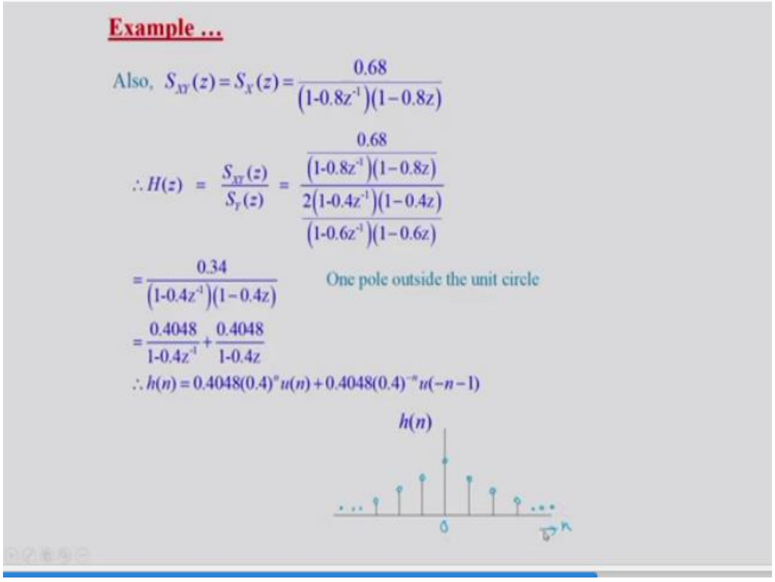

Now, Sxyz as we have shown it is equal to Sx of z and therefore it will be equal to 0.68 divided by 1 - 0.8 z inverse into 1 - 0.8 z, therefore Hz that is the transfer function of the corresponding filter and that is given by Sxyz divided by Syz and we will substitute the value of Sxyz and Syz, we will get this expression, so 0.34 divided by 1 - 0.4 z inverse into 1 - 0.4 z.

So, one pole will be inside the unit circle, other pole will be outside the unit circle and this we can expand like this; 0.4048 divided by 1 – 0.4z inverse + 0.4048 divided by 1 - 0.4 z and this part will give the causal solution, this is given by this, 0.4048 into 0.4 to the power n un; un is 1 for n greater than equal to 0 and this part will give an anti-causal sequence, 0.4048 into 0.4 to the power - n because n is negative u of - n – 1.

So, if we plot hn versus n, we will get a plot like this, so this hn will be an infinite duration sequence extending from minus infinity to plus infinity.

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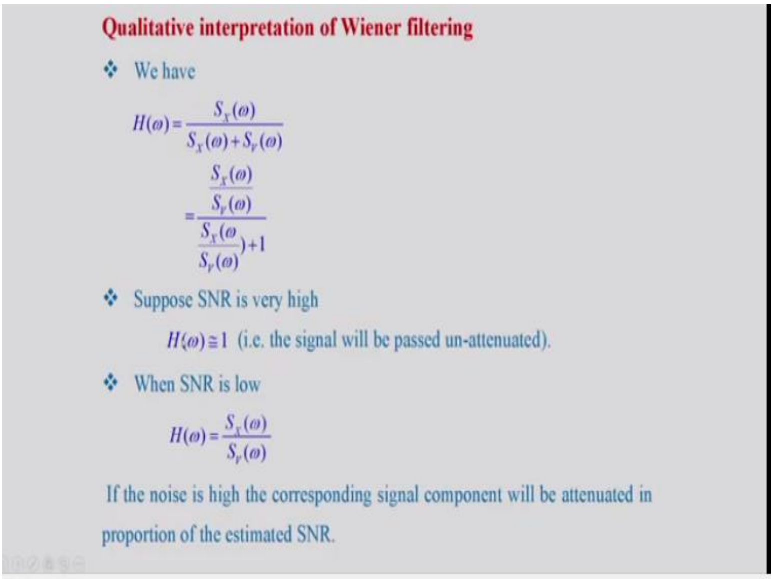

Let us give a qualitative interpretation of Wiener filter, we have established H Omega is equal to Sx omega divided by Sx omega + Sv omega and this we can write dividing both numerator and denominator by Sv omega, we get Sx omega divided by Sv omega, whole thing divided by Sx omega by Sv omega + 1. Now, this quantity is the signal to noise ratio, so this is SNR, here SNR + 1.

Suppose, SNR is very high in that case, this part will be dominant compared to 1 and therefore, this ratio will be almost equal to 1 and this means that signal will be passed un attenuated that is when SNR is very high, signal quality is very high, now this is very less in that case, H omega is approximately equal to 1. When SNR is low this quantity will be approximately equal to 1.

So, therefore H Omega will be Sx omega divided by Sv omega in other words, if the noise is high the correspondingly signal component will be attenuated in proportionate to the SNR because signal spectrum that is observed data spectrum will be multiplied by this quantity, this quantity is nothing but the signal to noise ratio therefore, if the noise is high the corresponding signal component will be attenuated in the proportion of the estimated SNR.

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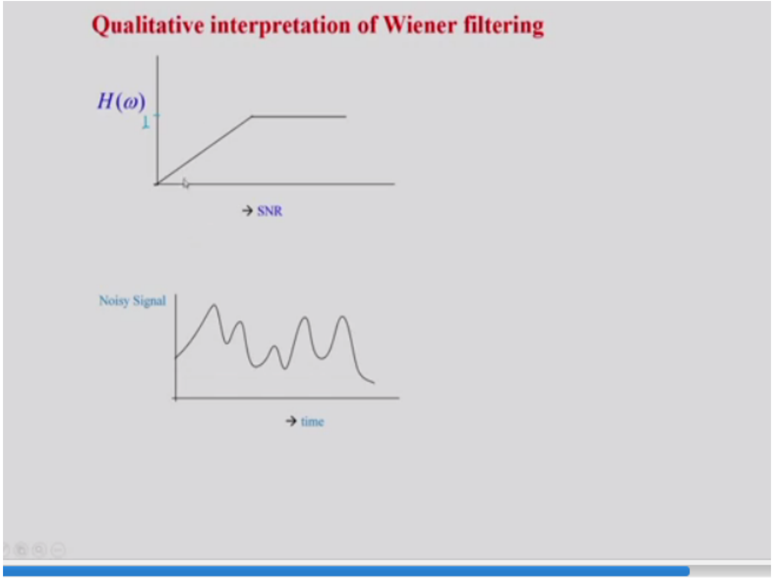

We can plot H omega versus SNR suppose, when SNR is very low in that case, H omega is equal to SNR, so that way this is a line of slope 1 and when SNR is high, H omega will be equal to 1, so that the signal will be passing un attenuated, so that way we can show that suppose, we are considering the case of sinusoid in the presence of noise in this part of the data, this signal quality is good therefore, my H omega will be equal to .

And this part of data, the signal quality is low, SNR is low therefore, this transfer function will be equal to SNR, so that way we can qualitatively interpret the Xn of non-causal IIR Wiener filter.

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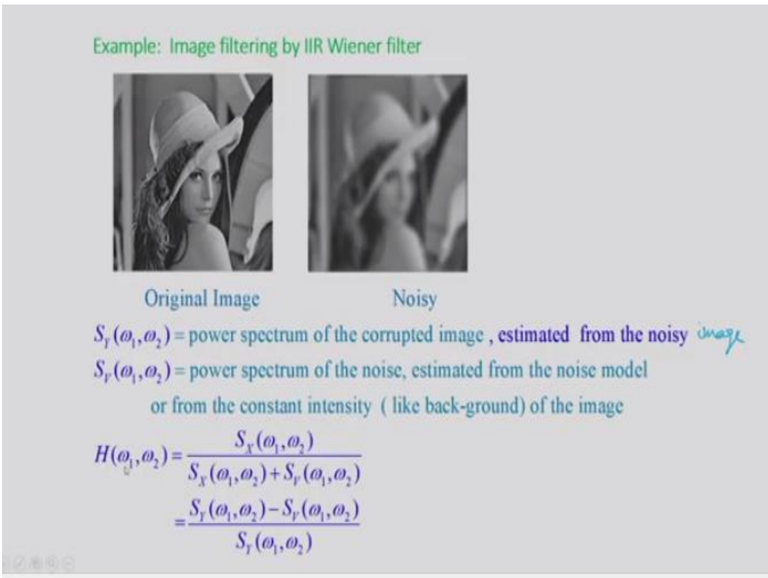

Let us consider one example that is the image filtering by IIR Wiener filter in this case, this is the original image and this is the noisy image suppose, this noisy image is obtained by adding white Gaussian noise. Now, in this case because image is a 2 dimensional signal we have to apply the 2 dimensional Wiener filter and entire image is processed at a time, therefore we have the entire data suppose, if we consider a point here to the left of this point to the right of this point above this point or below this point all that are available.

Therefore, we can apply non-causal IIR Wiener filter, suppose Sy omega 1, omega 2 is the power spectrum of the corrupted image for this image which is generally estimated from the noisy image. Now, similarly we required the power spectrum of the noise so, Sv of omega 1, omega 2 that is the power spectrum of the noise which can be estimated from the noise model or from the constant intensity portion of the image.

In the image some portion are there, where pixel intensity is uniform in that case from the variance of the noisy data we can find out the power spectrum of the noise and now the transfer function will be H omega 1, omega 2 equal to Sx omega 1 omega 2 that is signal power spectrum divided by the power spectrum of the signal plus noise Sx omega 1 omega 2 + Sv omega 1 omega 2.

But what is available for us is; Sy omega 1 omega 2 that is the power spectrum of the observed data, which is usually estimated, this part we can write as Sy omega 1 omega 2 - Sv omega 1 omega 2 divided by the power spectrum of the observed noisy data, so that way we can determine the transfer function H omega 1 omega 2 by this relationship.

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Once we have H omega 1 omega 2 we can determine the Fourier transform X hat omega 1 omega 2 as H omega 1 omega 2 that is the transfer function multiplied by the Fourier transform of the output noisy image, Y omega 1 omega 2, taking the inverse DTFT or inverse transform we get the denoised image, this is example, so this is the noisy image. After this step, we get this Wiener filtered image.

So, this is an application of Weiner filter in 2 dimensions and here a non-causal Wiener filter is used because entire image data is thought out and then we process.

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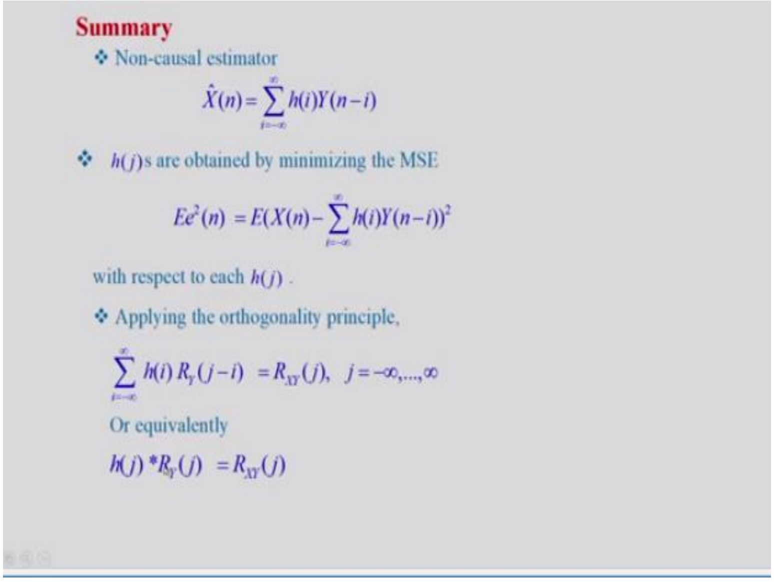

Let us summarize the lecture that is non-causal IIR estimator is given by this X hat n is equal to summation hi Y n – i; i going from minus infinity to plus infinity, hj’s are obtained by minimizing this mean square error, this is the mean square error with respect to each of hj, so we have to minimize E of X n - summation hi into Y n – i; i going from minus infinity to infinity whole square with respect to hj.

Now, we applied the orthogonality condition and obtain the Wiener Hopf equation that is summation hi into Ry j – i; i going from minus infinity to infinity that is equal to Rxy of j, for any integer j or equivalently this Rxyj is the convolution of hj and Ryj, this is the result.

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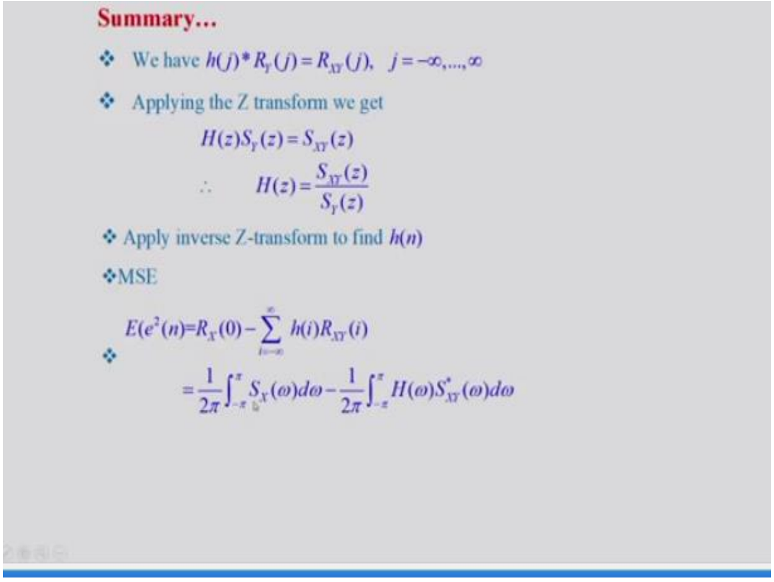

And we can apply this Z transform now, Hz into Sxz that is equal to Sxyz, therefore Hz that is the system transfer function corresponding to IIR non-causal Wiener filter will be given by Sxyz divided by Syz and we can apply the inverse Z transform to find hn, so Hz is given therefore, we can find out hn and MSE that is mean square error is given by E of e square n that is equal to Rx of 0 minus this quantity summation hi into Rxy i; i going from minus infinity to infinity.

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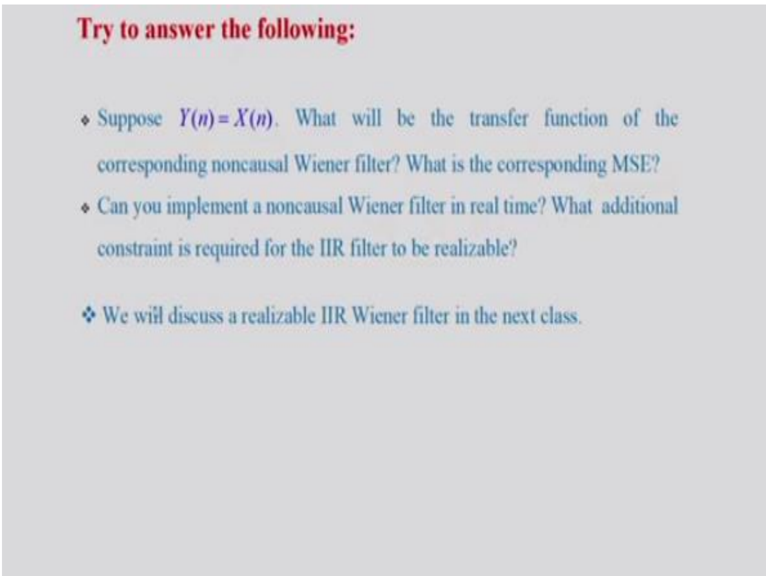

This we can write in the frequency domain like this, try to answer the following questions; suppose Yn is equal to Xn, what will be the transfer function of the corresponding non-causal IIR Wiener filter, what is the corresponding MSE? This you can find out from the expressions for non-causal IIR Wiener filter and corresponding mean square error. Can you implement a non-causal Wiener filter in real time, what additional constraint is required for the IIR filter to be realizable?

So, obviously answer to this question is no, you find out why it is no and then what additional constraint is required for IIR filter to be realizable, we will discuss a realizable IIR filter in the next lecture, thank you.