Microwave Engineering
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Lecture 15
Transmission Line Resonators

Let us now consider Transmission Line Resonators where we will consider a section of a transmission line, either open or short and we will see that when the length of these sections of transmission lines are chosen appropriately there will exhibit resonance and near the resonant frequency we can model this transmission line resonators either in the form of a series RLC circuit or in the form of an equivalent parallel RLC circuit.

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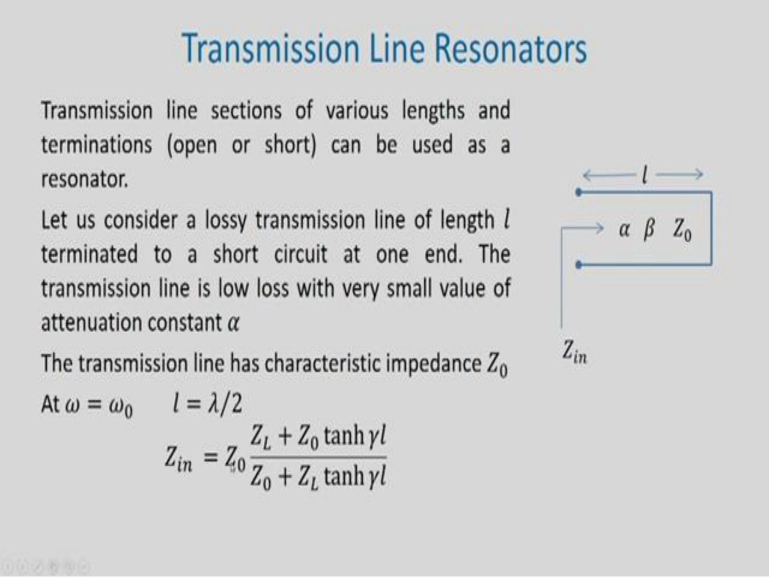
The transmission line has characteristic impedance Z_0

At $\omega = \omega_0$ $\quad l = \lambda/2$

$$Z_{in} = Z_0 \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l}$$



So, transmission lines sections of various lengths and terminations open or short can be used as a resonator. Now, let us consider a lossy transmission line of length l and let us assume that it is terminated to a short circuit at one end. We will also consider the transmission line to be of low loss with very small value of attenuation constant alpha. Now, this is shown in the figure we have a transmission line section of length l and alpha is the attenuation constant, Z naught is the characteristic impedance, and beta is the propagation constant.

We are assuming this transmission line to be of low loss type so that alpha is very, very small. Now, suppose we choose the line length l in such a way that at omega equal to omega naught, l is equal to lambda by 2. So, we are essentially considering a half wavelength short-circuited section of a transmission line. Please note that this half-wavelength will be only at a particularly frequency.

If we change the frequency of operation, the physical length of the transmission line will remain the same and it will be depending upon whether the frequency is higher or lower the line length will be longer than lambda by 2 or it will be shorter than lambda by 2. So, the line length l is lambda by 2 at omega equal to omega naught. Now, for such lossy lines we can write Z in to be equal to Z_L plus Z_0 tan hyperbolic gamma l divided by Z naught plus Z_L tan hyperbolic gamma l. Now, in our case this Z_L is 0 we have only these terms left because this become 0, this becomes 0 and these two Z zeros cancel out.

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For $Z_L = 0 \quad Z_{in} = Z_0 \tanh \gamma l = Z_0 \tanh(\alpha + j\beta)l$

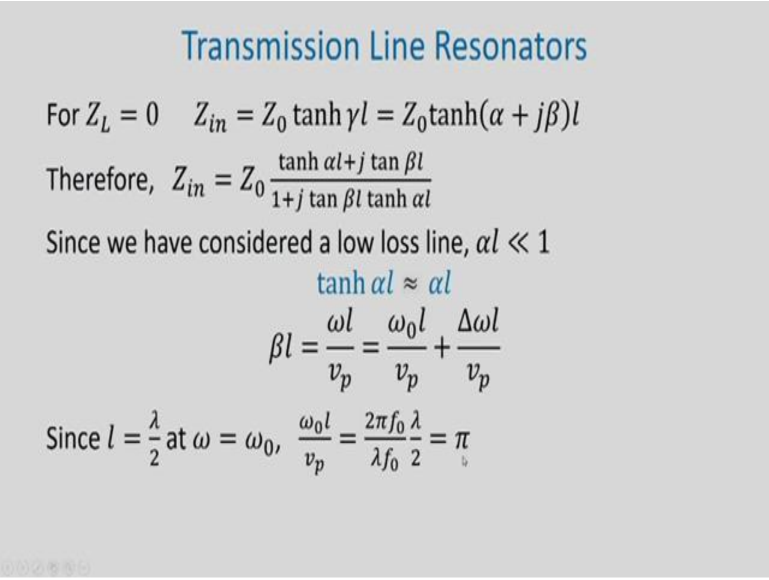
Therefore, $Z_{in} = Z_0 \frac{\tanh \alpha l + j \tan \beta l}{1 + j \tan \beta l \tanh \alpha l}$

Since we have considered a low loss line, $\alpha l \ll 1$

$$\tanh \alpha l \cong \alpha l$$

$$\beta l = \frac{\omega l}{v_p} = \frac{\omega_0 l}{v_p} + \frac{\Delta \omega l}{v_p}$$

Since $l = \frac{\lambda}{2}$ at $\omega = \omega_0$, $\frac{\omega_0 l}{v_p} = \frac{2\pi f_0}{\lambda f_0} \frac{\lambda}{2} = \pi$



And therefore when Z_L equal to 0 Z in is Z naught tan hyperbolic gamma l. Now, we can substitute gamma as alpha plus j beta and therefore Z_{in} becomes now Z naught tan hyperbolic alpha plus j beta multiplied by l and this equation can be expanded as Z_{in} is equal to tan hyperbolic alpha l and tan hyperbolic j beta l will become j tan beta l divided by 1 plus again j tan beta l tan hyperbolic alpha l.

Now, we have considered the transmission line to have very low loss. So, in that case we write alpha l to be much less compared to 1 and we can approximate tan hyperbolic alpha l as alpha l and similarly beta l can be written as omega l by v_p phase velocity and once we substitute omega equal to omega naught plus delta omega this can be written as omega naught l by v_p plus delta omega l by v_p.

Now, since we have l is equal to lambda by 2 at omega equal to omega naught this term omega naught l by v_p this becomes pi because v_p will be lambda into f naught omega naught can be written as 2 pi f naught and l is lambda by 2 so we will have omega naught l by v_p is equal to pi.

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Now, $\beta l = \frac{\omega_0 l}{v_p} + \frac{\Delta \omega l}{v_p}$ and $\frac{\omega_0 l}{v_p} = \pi$

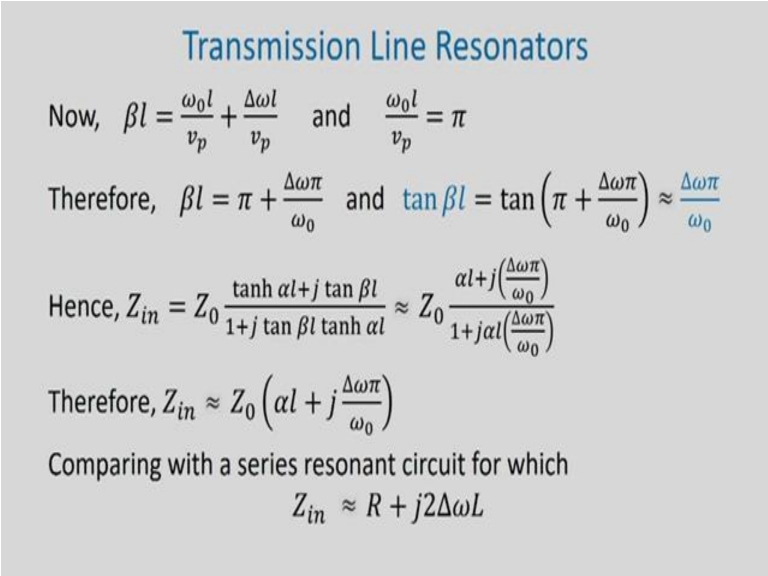
Therefore, $\beta l = \pi + \frac{\Delta \omega \pi}{\omega_0}$ and $\tan \beta l = \tan\left(\pi + \frac{\Delta \omega \pi}{\omega_0}\right) \cong \frac{\Delta \omega \pi}{\omega_0}$

Hence, $Z_{in} = Z_0 \frac{\tanh \alpha l + j \tan \beta l}{1 + j \tan \beta l \tanh \alpha l} \cong Z_0 \frac{\alpha l + j\left(\frac{\Delta \omega \pi}{\omega_0}\right)}{1 + j \alpha l \left(\frac{\Delta \omega \pi}{\omega_0}\right)}$

Therefore, $Z_{in} \cong Z_0 \left(\alpha l + j \frac{\Delta\omega\pi}{\omega_0} \right)$

Comparing with a series resonant circuit for which

$$Z_{in} \cong R + j2\Delta\omega L$$



So, we have now beta l is equal to omega naught l by v_p plus delta omega l by v_p and omega naught l by v_p is equal to pi and therefore we can write beta l to be equal to pi plus delta omega pi by omega naught and tan beta l can be written as tan pi plus delta omega pi by omega naught. Now, delta omega being very small compared to omega naught we can use the approximation first of all tan pi plus theta will become tan theta and then we can use the approximation for small theta tan theta equal to theta.

So, we can write tan beta l to be approximately equal to delta omega pi divided by omega naught. Hence we can now write Z_{in} which is Z naught tan hyperbolic alpha l plus j tan beta l divided by 1 plus j tan beta l tan hyperbolic alpha l. Now, if you substitute tan hyperbolic alpha l as alpha l and tan beta l as delta omega pi by omega naught we can write Z_{in} appropriately equal to Z naught alpha l plus j delta omega pi by omega naught divided by 1 plus j alpha l delta omega pi by omega naught.

Now, here you can see in the denominator we have the product of two small terms one is alpha l and another is delta omega pi by omega naught. So, this term can be neglected with respect to 1 and therefore we can write Z_{in} approximately equal to Z naught alpha l plus j delta omega pi by omega naught. Now, what we can do? We can compare it with the expression for input impedance of a series resonant circuit near it is resonant frequency so which is given by Z_{in} equal to R plus j2 delta omega L.

So, if we compare these two this for a lambda by 2 short-circuited transmission line sections and this Z_{in} is for a series RLC circuit operating near it is resonant frequency.

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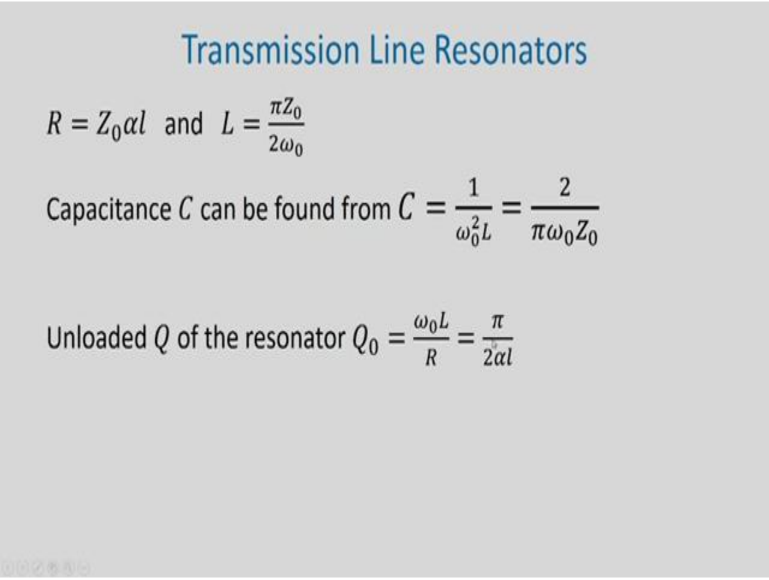
$R = Z_0 \alpha l$ and $L = \frac{\pi Z_0}{2\omega_0}$

Capacitance C can be found from $C = \frac{1}{\omega_0^2 L} = \frac{2}{\pi \omega_0 Z_0}$

Unloaded Q of the resonator $Q_0 = \frac{\omega_0 L}{R} = \frac{\pi}{2\alpha l}$



Then we can write R equal to Z naught alpha l and L equal to pi Z naught by 2 omega naught, and since omega square is equal to 1 by LC we can find the capacitance C as 1 by omega naught square L, and we have already had the expression for L and this becomes equal to 2 by pi omega naught Z naught. Now, unloaded Q of the resonator it is given by Q naught is equal to omega naught L by R.

So, if we substitute L and R here R equal to Z naught alpha l and L equal to pi Z naught by 2 omega naught then we get Q naught the unloaded Q of the resonator to be equal to pi by 2 alpha l. So, smaller, the value of alpha larger will be the value of Q, and we will have a sharp resonance at omega equal to omega naught. So, you see that if we have a lambda by 2 sections of a transmission line having small amount of loss and the line is short-circuited at one end then the transmission line section can resonate at a frequency where the line length corresponds to half the wavelength.

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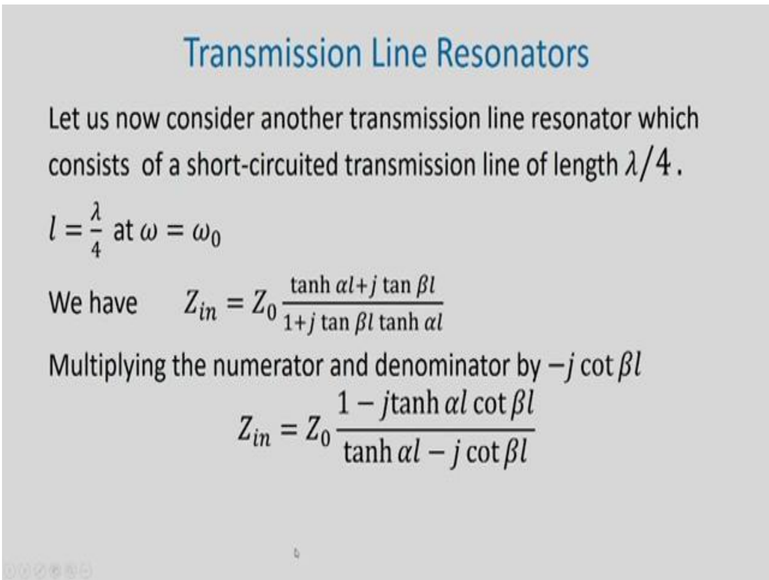
Let us now consider another transmission line resonator which consists of a short-circuited transmission line of length $\lambda/4$.

$l = \frac{\lambda}{4}$ at $\omega = \omega_0$

We have $Z_{in} = Z_0 \frac{\tanh \alpha l + j \tan \beta l}{1 + j \tan \beta l \tanh \alpha l}$

Multiplying the numerator and denominator by $-j \cot \beta l$

$$Z_{in} = Z_0 \frac{1 - j\tanh \alpha l \cot \beta l}{\tanh \alpha l - j \cot \beta l}$$



So, we continue our discussion on transmission line resonators. We consider another type of transmission line resonator. Here, we consider a short-circuited transmission line of length lambda by 4. So, for this type of transmission line resonator we have at omega equal to omega naught. The physical length of the transmission line l is equal to lambda by 4 which means it is a quarter-wave sections, short-circuited at one end.

Now, if the line as before is assumed to be slightly lossy that means having an attenuation constant of alpha and phase constant beta and characteristic impedance Z naught. So, for such line we can we have already seen that we can write Z_{in} equal to Z naught tan hyperbolic alpha l plus j tan beta l divided by 1 plus j tan beta l tan hyperbolic alpha l. Now, what we do? We multiply both the numerator and the denominator by minus j cot beta l.

So, in that case when it is multiplied by minus j cot beta l j tan beta l minus j cot beta l will give 1 and therefore we can write Z_{in} is equal to Z naught 1 minus j tan hyperbolic alpha l cot beta l and then this term will become tan hyperbolic alpha l minus 1 into j cot beta l will give minus j cot beta l. So, Z_{in} becomes after multiplying by minus j cot beta l Z_{in} becomes Z naught 1 minus j tan hyperbolic alpha l cot beta l divided by tan hyperbolic alpha l minus j cot beta l.

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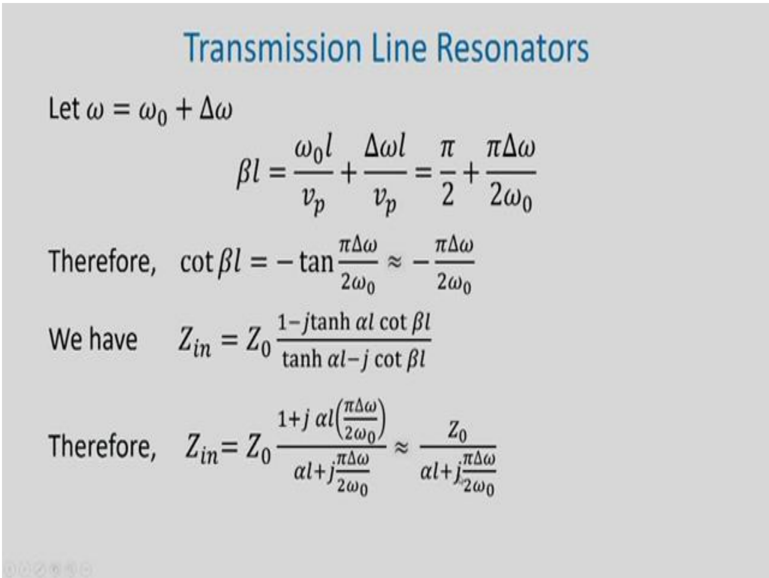
Let $\omega = \omega_0 + \Delta\omega$

$$\beta l = \frac{\omega_0 l}{v_p} + \frac{\Delta\omega l}{v_p} = \frac{\pi}{2} + \frac{\pi\Delta\omega}{2\omega_0}$$

Therefore, $\cot\beta l = -\tan\frac{\pi\Delta\omega}{2\omega_0} \cong -\frac{\pi\Delta\omega}{2\omega_0}$

We have $Z_{in} = Z_0 \frac{1 - j\tanh\alpha l \cot\beta l}{\tanh\alpha l - j\cot\beta l}$

Therefore, $Z_{in} = Z_0 \frac{1 + j\,\alpha l\left(\frac{\pi\Delta\omega}{2\omega_0}\right)}{\alpha l + j\frac{\pi\Delta\omega}{2\omega_0}} \cong \frac{Z_0}{\alpha l + j\frac{\pi\Delta\omega}{2\omega_0}}$



Once again, we consider omega to be omega naught plus delta omega close to the resonant frequency, and in this case, when it is a lambda by 4 transmission line section beta l can be written as pi by 2 plus pi delta omega divided by 2 omega naught. Now, once we have this expression for beta l we find that cot beta l because of this pi by 2 terms it becomes minus tan pi delta omega divided by 2 omega naught.

Once again since we are assuming delta omega to be small compared to omega naught we can write tan pi delta omega by 2 omega naught to be approximately equal to minus pi delta omega divided by 2 omega naught. And then we already have the expression for Z_{in}, we can now make appropriate substitutions tan hyperbolic alpha l can be substituted by alpha l and cot beta l we can make substitution minus pi delta omega by 2 omega naught.

So, once we do that we get Z_{in} to be equal to Z naught 1 plus j alpha l pi delta omega by 2 omega naught. So, this minus and this is minus will make it plus and alpha l again plus j pi delta omega by 2 omega naught. Now, here in the numerator once again we have the product of two small terms alpha l and pi by 2 delta omega by omega naught. So, this term can be neglected in comparison with 1, and Z_{in} finally can be approximated as Z naught divided by alpha l plus j pi by 2 delta omega by omega naught.

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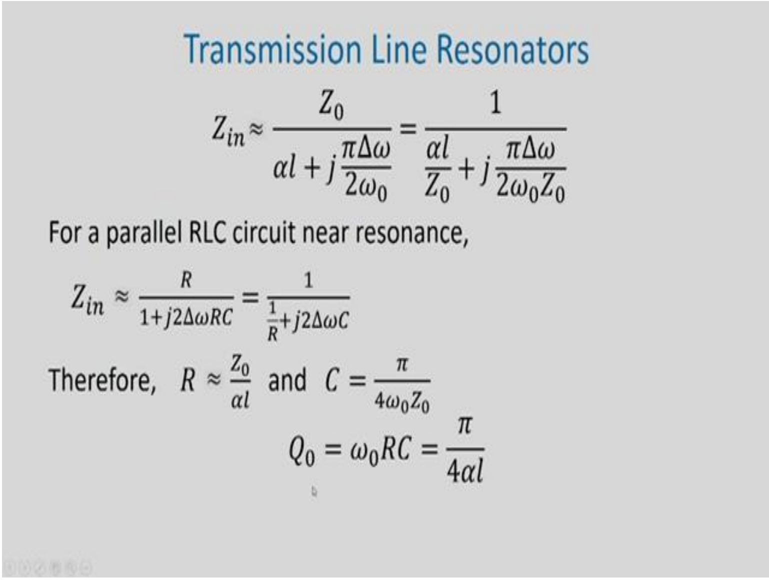
$$Z_{in} \cong \frac{Z_0}{\alpha l + j\frac{\pi\Delta\omega}{2\omega_0}} = \frac{1}{\frac{\alpha l}{Z_0} + j\frac{\pi\Delta\omega}{2\omega_0 Z_0}}$$

For a parallel RLC circuit near resonance,

$$Z_{in} \cong \frac{R}{1 + j2\Delta\omega RC} = \frac{1}{\frac{1}{R} + j2\Delta\omega C}$$

Therefore, $R = \frac{Z_0}{\alpha l}$ and $C = \frac{\pi}{4\omega_0 Z_0}$

$$Q_0 = \omega_0 RC = \frac{\pi}{4\alpha l}$$



So, this can be further written in this form 1 by alpha l by Z naught plus j pi delta omega by 2 omega naught Z naught. Now, we identify that this form is of the input impedance is that of a parallel RLC circuit operating near it is resonance. For a parallel RLC circuit near resonance we have Z_{in} to be approximately equal to R by 1 plus j 2 delta omega RC, which can be put of this form 1 by 1 by R plus j2 delta omega C.

Now, if we equate this corresponding terms then we get R equal to Z naught by alpha l and C equal to pi by 4 omega naught Z naught and once we have the values for R and C we can calculate the value of L if required and we can also calculate the unloaded Q. Q naught is equal to omega naught RC and which is given by pi by 4 alpha l. Once again, we see that low alpha l we have high values of unloaded Q.

So, we find that a short-circuited lambda by 4 sections of a transmission line is essentially it can be modeled as a parallel RLC circuit, and the resonant frequency will be determined by the length of the line at which it is lambda by 4. In this lecture we have studied different types of transmission line resonators. In the next lecture, we will consider another form of resonator, which is waveguide resonators.