Advanced Topics in Probability and Random Processes
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Lecture - 09
Weak Convergence 2

In the last lecture we discussed weak convergence, we covered 2 modes of convergence that is convergence in probability, and convergence in the mean square sense.

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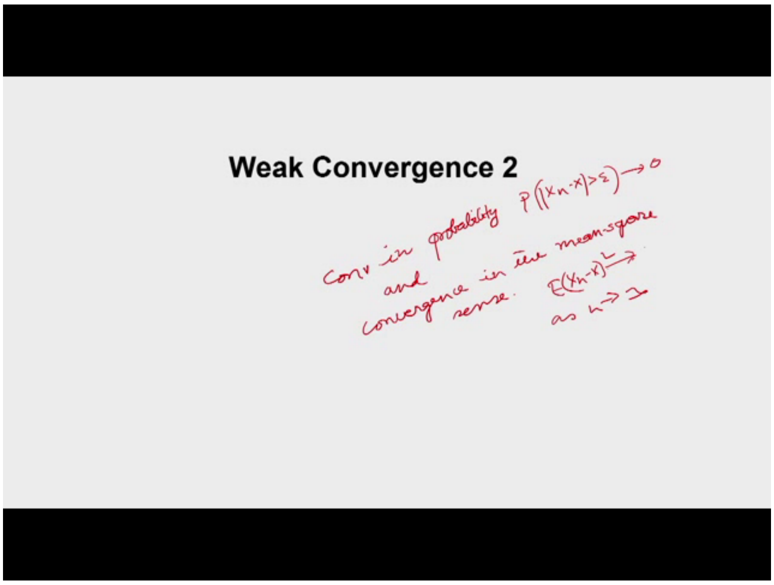

So, these are weak convergence modes because we consider some functions. For example, here we consider a probability; that X n minus X this is greater than any epsilon, arbitrarily small epsilon. So, these probabilities should go down to 0 as n tends to infinity. Similarly, here we consider E of X n minus X whole square this quantity goes down to 0 as n tends to infinity.

So, this is the convergence in mean square sense and convergence in probability and they are interrelation also we discussed. We will not discuss another weak convergence concept and this is in fact; this is the weakest concept. So, we will discuss about convergence in distribution.

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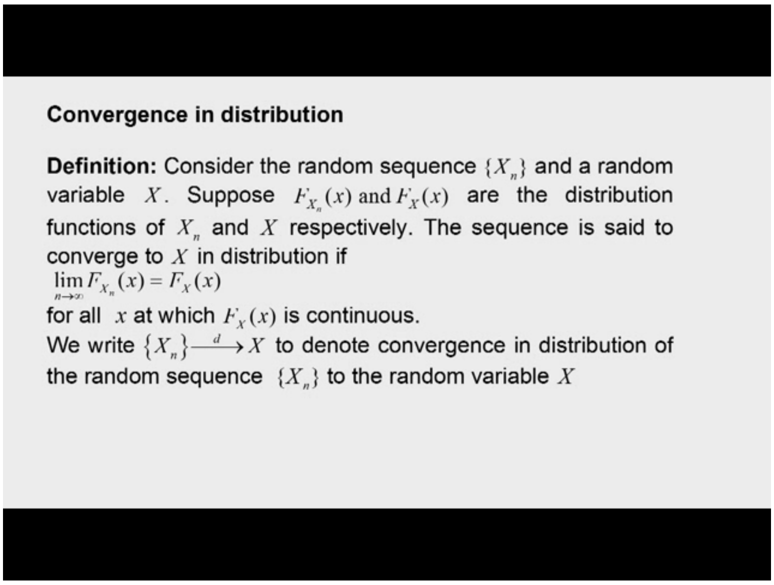

So let us see the definition considered a random sequence X n this is the sequence from n is equal to 1 to infinity suppose and a random variable X. Suppose F X n x and F X of x are the distribution functions of X n and X respectively. This sequence is said to converge to X, now the sequence is said to converge to X in distribution, if corresponding distribution function sequence converges. So, what does it say? If limit of F X n of x as n tends to infinity is equal to F X of x.

Now, this is for all x at which F X of x is continuous. So, this is another requirement that this limit should be equal to this for all X at which F X of x is continuous. And this convergence we write as X n converges in distribution to X, and sometimes it is also called X n converges in law to X. Now, essentially here the distribution functions, sequence of distribution functions converges to the distribution function of the random variable X at all points of continuity.

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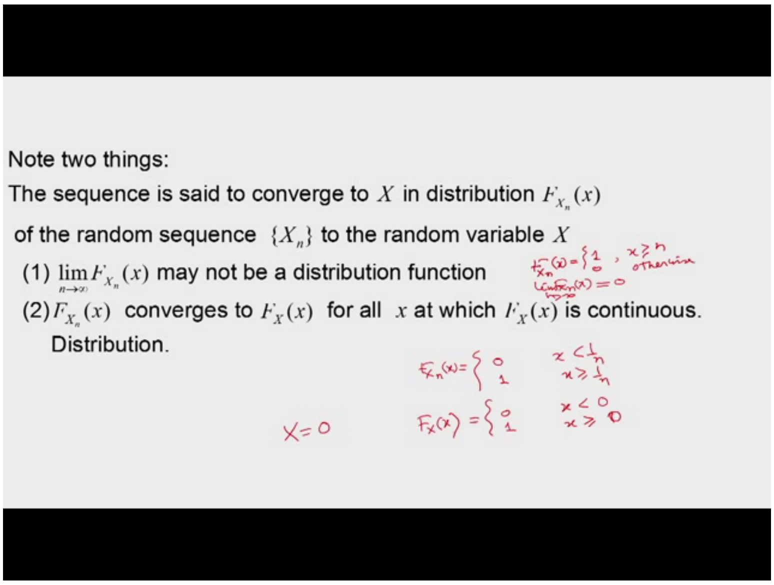

So, we have to note two things; first of all we are considering the limit of disciplines, distribution function sequence. So, this limit may not be a distribution function there is no guarantee that this limit will be a distribution function, but we will say that it is convergence in distribution only if this converges to a distribution function. For example, suppose you consider a function F X n of x is equal to 1 for x greater than equal to n suppose is equal to 0 otherwise.

So, this function now this function will convert to as n tends to infinity, so this function will converge to 0. So, this F X n a limit of F X n of x as n tends to infinity, it will become as n tends to infinity because x when we increase that; so it will 0 will extend, so it will become 0. So, which is not a distribution function therefore, we have to ensure that this distribution function sequence converges to a distribution function.

Second thing is F X n of x converges to F X of x for all x at which F X of x is continuous that is important. For example, if we consider suppose F X n of x is equal to 0 x less than 1 by n is equal to suppose 1 x greater than equal to 1 by n. Now, if I have suppose F X of x is equal to suppose 0, x less than equal to 0 suppose x less than 0 is equal to 1 x greater than equal to 1. Now this is this essentially mean the deterministic function that X is equal to this is this essentially means that X is equal to 0.

Now, if I consider this limit of this as n tends to infinity. So, this will become 1 for x greater than 1. And it will, but at n is equal to at x is equal to 0 it will be always equal to

0, unlike this function, so for this function for x is equal to 0 it is for x greater than equal to 1, it is x greater than equal to 0 it is always equal to 1. But at this point x is equal to 0 this is not converging because F X n of x as n tends to infinity, at point x is equal to 0 it will become 0. So, that way this convergence is they are only at points at which x is F X of x is continuous. So, at x is equal to 0 F X of x is not continuous. So, we do not bothered about the convergence at that point, but any other point where F X of x is continuous this F X n of x converges then we say that this sequence converges in distribution.

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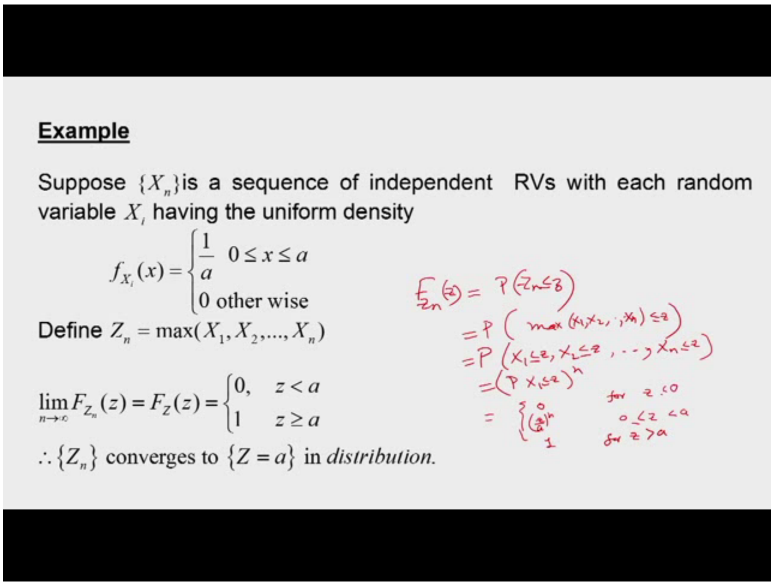

We can consider another example suppose this is the pdf; that is X all Xs are uniformly distributed. The sequence of in the independent random variables with each random variable X i uniformly distributed between 0 to a. Now, in this case we define a sequence F Z n, so F Z n of Z now if I consider F Z n of z ok. So, what does it means this is that probability that Z n is less than equal to small z, what does it mean; that probability that maximum of X 1 X 2 up to X n, that is less than equal to Z.

So, this is now since maximum is less than that, so each of these variables random variables will be less than equal to Z. Therefore, this is same as the probability that X 1 is less than equal to Z, X 2 is less than equal to Z like that X n is less than equal to Z, so we can write like this. Now, we can use the independent property, so, that way this will be equal to probability of X 1 less than equal to Z to the power n. Now this is uniformly distributed, so therefore we know this is 1 by the pdf therefore, this quantity suppose this will be equal to 0 for z less than equal to 0 so less than 0 suppose.

Now, for that lying between for z lying between 0 and a; we can write this as because if I integrate then I will get that by a z by a to the power n, n is equal to 1, 1 for z greater than a this is the pdf. So, if I now this is the CDF, F Z n of z is given by this. If I do the limit take the limit then I will get this F z limit of F Z n of z is equal to F z of z is equal to 0 for z less than equal to a one for z greater than equal to a. So, therefore, Z n converges to z is equal to a in distribution. Now we have considered the concept of convergence in distributed distribution. How is it related to convergence in probability.

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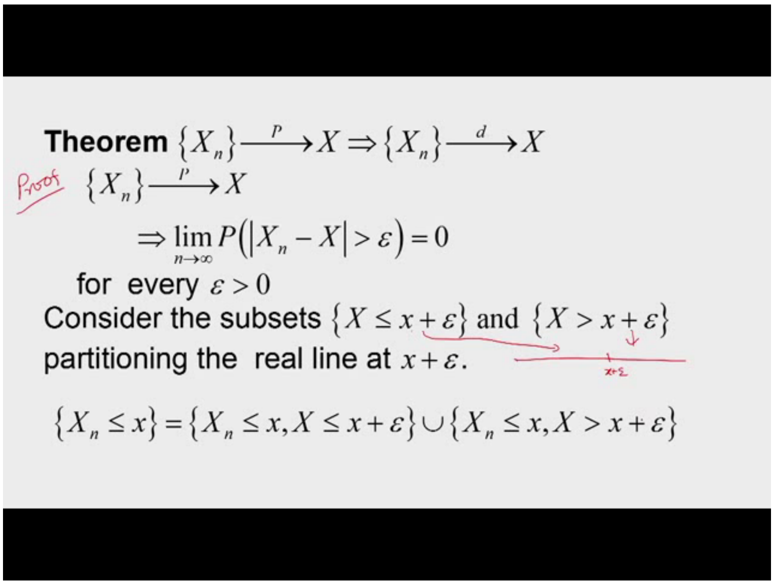

So, if X n converges to x in probability then it is it implies that X n also converges in distribution to X. We will see the proof suppose proof X n converges in probability to X, what does it say? It says the limit of probability of X n minus x absolute value of X n minus X greater than any epsilon arbitrarily small epsilon that is equal to 0.

Now, we will consider the second random variable this is X n sequence and X this random variable. Suppose at x plus small epsilon, that is x plus epsilon we will consider a partition. Consider these subjects suppose I have this real line. So, this is x plus epsilon. So, I will consider 2 events that is X greater than, this side is X greater than small x plus epsilon. And this side is X less than this is X less than so this is X less than x plus epsilon. So, this set is this subset and this is another subset at point x plus epsilon we have a partition on the real line

So now we want to find out the distribution function. So, for that let us consider the event X n less than equal to small x. This we can consider by using the total probability theorem we can find out this probability. So, considering these 2 partition. So, this is X n less than equal to small x, and X less than equal to small x plus epsilon. Union X n less than equal to small x, X greater than small x plus epsilon. So, this event we have considered as the union of 2 disjoint events.

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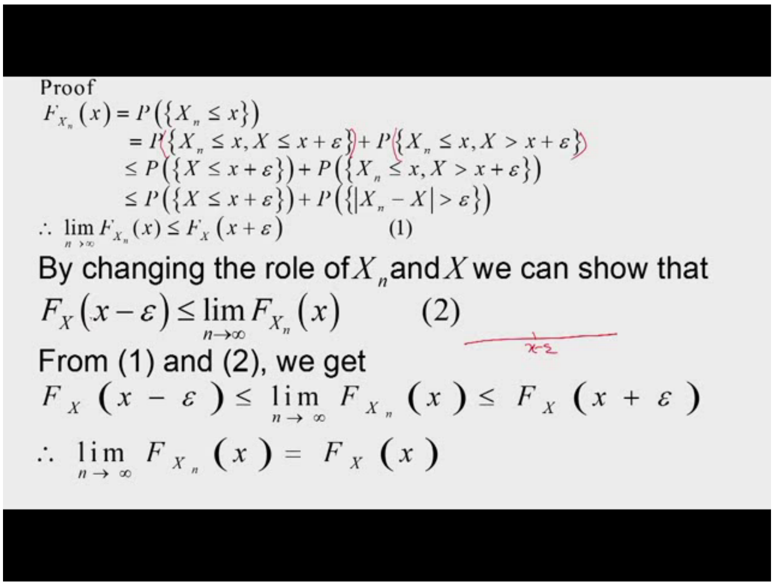

So, therefore, F X n of x we can write that that is equal to probability of X n less than equal to small x. Now considering that partitions we write that this is equal to probability of X n less than equal to small x this is comma; that means, and X less than equal to small x plus epsilon plus probability of X n less than equal to small x, capital X is greater than x plus epsilon. So, these two probability we have considered yeah this is.

Now, this is a probability of joint event probability of X n less than equal to small x and X less than equal to small x definitely if I consider only one event, then we can write that this is less than equal to probability of X less than equal to small x plus epsilon. Now considering this event, so we consider this event X n less than equal to small x X greater than equal to small x plus epsilon. So, this is a subset of this event mod of X n minus X. Here we see that if I take the difference between X and X n, that is that will be greater than epsilon. So, that way this event is a subset of this event therefore, again we will have less than equal to this probability. So, therefore, F X n of x is less than equal to this probability plus this probability ok.

Now, what happens as n tends to infinity, we are given that the sequence converges in probability. So, this term will become 0 as n tends to infinity. Therefore, what we get is that limit of F X n of x is less than equal to F X of x plus epsilon. So, this is that is CDF of X n that is at point x is less than equal to the CDF of x at point x plus epsilon. Now, we can consider again we can consider a partition on R F point x minus epsilon suppose x minus epsilon. Now we will consider the partition in terms of the random variable X n.

So, that X n is less than equal to small x minus epsilon and X n is greater than small x minus epsilon. So, that way we can consider a partition and in the same way we can continue. And then we can prove that F X of x small x minus epsilon that is, less than equal to limit of as n tends to infinity F X n of similarly, we can consider we can consider a partition on the real line at point x minus epsilon, involving the random variable X n. So, that X n is less than equal to small x minus epsilon and X n is greater than equal to small x minus epsilon, these are the 2 event suppose.

So, then we can show that F X of F X of x minus epsilon that is less than equal to limit of F X n of x as n tends to infinity. Here we show that limit F X n of x as n tends to infinity it is less than equal to small capital F X of x at x plus epsilon. So, this inequality we showed earlier, now we show that F X of x minus epsilon is less than equal to limit of F X n of x as n tends to infinity.

So, from this one and 2 we get that that limit F X n of x lies between F X at point x minus epsilon and F X at x plus epsilon, where epsilon is arbitrarily small number. Therefore, as n tends to infinity this limit F X n of x will be equal to F X of x. So, this says that if sequence X n is convergent in probability it will be convergent in distribution also.

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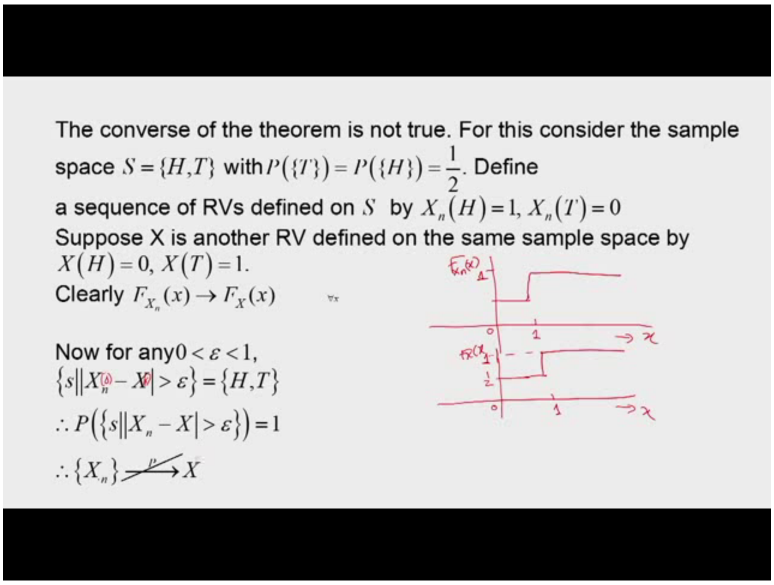

Now the converse of this theorem is not true; that means, convergence in distribution did not imply that convergence in probability. We considered a sample space S is equal to HT, head and tail suppose with probability of tail and probability of head equal that is equal to half. Now, we define a sequence of random variables suppose X n of H is equal to 1, X n of T is equal to 0. Another random variable we define X of H it this other way X of H is equal to 0, X of T is equal to 1. In the case of X n X n of H is equal to 1; here X of H is equal to 0.

Now, if I have to draw the distribution function of both suppose this is n this is x F X n of x. Now X n a of x is equal to 1. So, probability of head suppose at 1 this is the 1, and so this is 0 so up to here it will be half. So, here half and then this will go to 1 like that this is 1. So, this is the CDF of X n of S or lCDF of X n that is if I consider X this side F X of x that at X is equal to 0 that will be corresponding to tail. So, that the half probability at 1 that is a circulative probability will be equal to 1

Similarly, in the case of F X of x also, if I consider this is suppose x F X of x here also this will be equal to F 0 it will be equal to half and then this is at 1 again it will be equal to 1. So, now these two distributions functions are coinciding, but here we have to remember that these mappings are different. Therefore, let us see what happened to the probability of the event, where X n x are different, considerably different. So, that is we consider the probability of those sample points where X n and x X n minus x is absolute value of that is greater than epsilon.

So, for that now let us see what is the events of S such that mod of X n, X n s minus x s mod of mod of that is greater than epsilon. Now whenever s is equal to head X n of head is equal to 1 and X of H is equal to 0 therefore, their difference will be greater than epsilon. So, head will be included here. When S is equal to T, suppose X n of T will be equal to 0, but X of T will be equal to 1. Therefore, their difference is also greater than epsilon therefore, T is also included here.

So, the probability of this set will be equal to 1 therefore, this does not go down to 0; that means, X n does not converges in probability to X. So, this is a counter example which shows that convergence in distribution does not imply convergence in probability. So, that way we have discussed now convergence in distribution. So, earlier we considered convergence in probability convergence in mean square sense now another we concept convergence in distribution. Also if suppose X is continuous this is also theorem, but we will not prove this just these results we will state.

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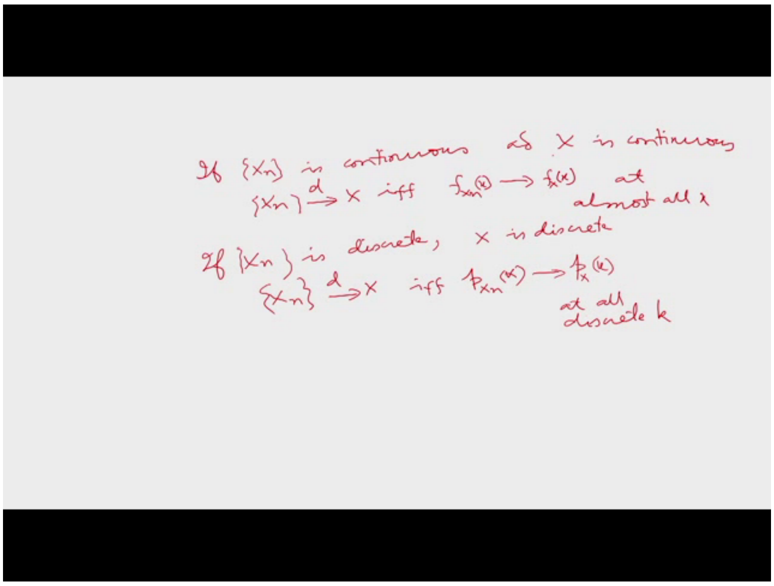

Suppose if X n sequence is continuous and X is also continuous. Then X n converges in distribution to X this is if and only if F X n of x this is pdf converges to F X of x that is pdf of x at almost all x. So, this is a theorem; that means, if X n converges in distribution then corresponding density function also converges in the case of case of continuous random variables. If we consider a sequence of continuous random variables if this sequence is convergent in distribution, then they are pdfs will also converge. And this is if and only if condition therefore, if suppose pdf sequence is converges then also the sequence will converge in distribution. That means, corresponding distribution sequence, distribution function sequence will also converge.

Similarly, we can also write in terms of probability mass function if X n converges, suppose if X n is discrete. So, in that case an X is also discreet then X n converges in distribution to X, if and only if that probability sequence that is probability mass function at point suppose k, converges to p x of k at all discrete k at all discrete k. So, what does it say that if a sequence of random variable convergence in distribution then this convergence can be studied in terms of their pdf and pmfs.

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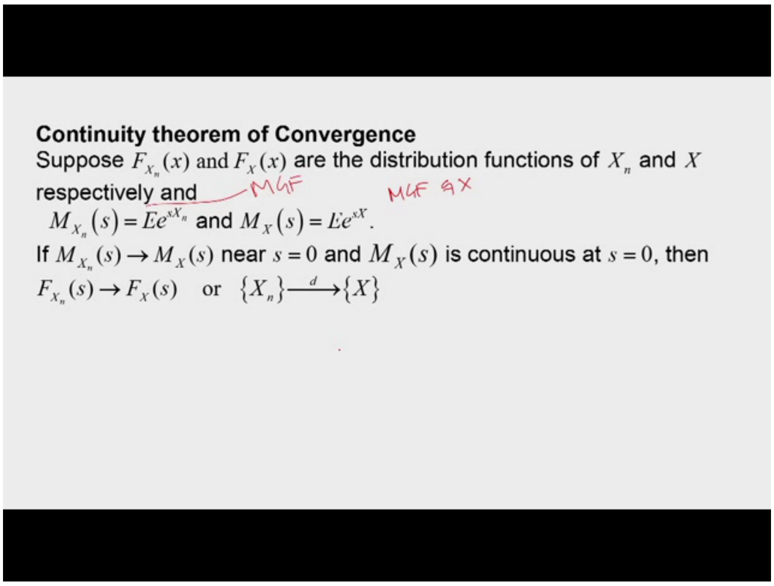

Next we will state another important theorem without any proof, this is required when we study the application of this concept this mode of convergence. We will study the central limit theorem to prove that it is required. So, this is the continuity theorem of convergence what does it say, suppose F X n of x and F X of x are the distribution functions of X n and x respectively and M X n of s is equal to E of e to the power s n of X n. So, what is it is? This is the MGF moment generating function. So, this is moment generating function MGF.

Similarly, M X s is E of e to the power s X that is the MGF of MGF of x. So, now we have a CDF sequence and we have a MGF sequence. If this sequence M X n of s, that is the MGF sequence for X n if it converges to M X of s near s is equal to 0. And M X of s is continuous at s is equal to 0 this is another requirement. Then corresponding this CDF sequence will also converge or in other words X n will converges to X in distribution.

So, ah; that means, if we have to prove that X n is converges and if X n converges to X in distribution then we can start with the MGF sequence. If we take the moment generating function of X n, and if it converges to the moment generating function of x at near s is equal to 0. And M X sub s is continuous at s is equal to 0 then we can prove that X n converges in distribution to X.

So, starting the moment generating function; if the moment generating function sequence that will be another sequence, if that sequence converges to the moment generating function of X then we can assume that according to this theorem, X n will converge in X to this in distribution. So, this is a continuity theorem of convergence and this is one important result which we will be using.

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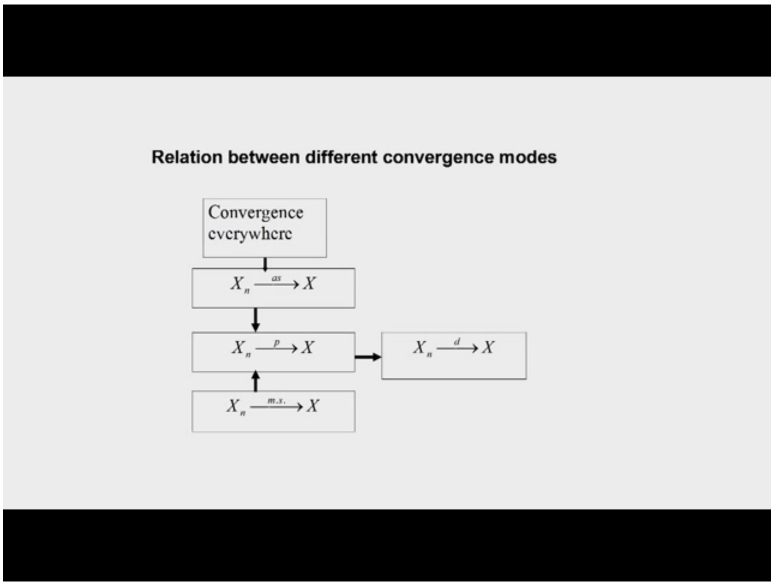

Let us see the entire relation between different convergence modes. We have considered convergence everywhere of course, that is convergence for every sample point. That is a very strong sense of convergent, convergence this is a very strong sense of convergence. So, this will imply convergence in convergence with probability 1 or convergence almost sure. So, this convergence everywhere because this is the strongest it will imply convergence almost sure. This in turn will imply convergence in probability and finally, convergence in probability implies convergence in distribution.

There is another type of convergence that is convergence in mean square. So, this can be consider this neither of this may imply this. So, that therefore, this is considered independently and this converges suppose X n converges in mean square sense. This will always imply convergence in probability and convergence in probability implies convergence in this division.

So, that where we have considered this 5 modes of convergence everywhere, convergence almost sure, convergence in probability, convergence in distribution and convergence in mean square sense. So, convergence in distribution is the weakest mode of convergence. Now, we will see some application of this convergence concept. We are interested particularly about two results two important results.

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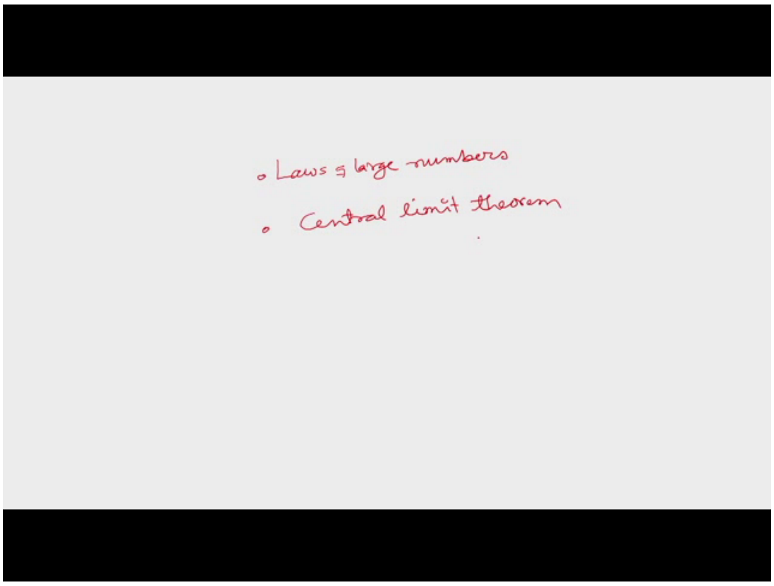

They are known as a laws of large number, and second one is central limit theorem. So, here we have weak law of large numbers, strong law of large numbers. So, that way these will apply the corresponding weak and strong modes of convergence, then central limit theorem this is a this is an application of the convergence in distribution.

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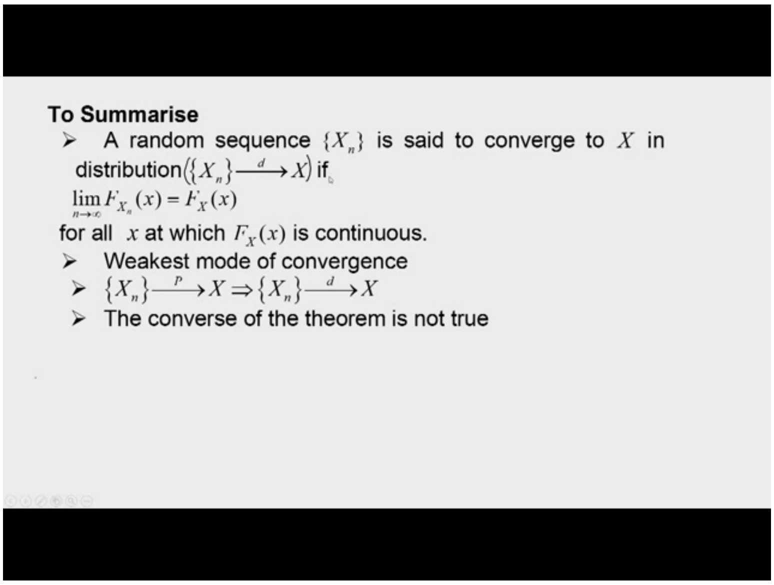

Let us summarize the key points of the lecture, a random sequence X n is said to converge to X in distribution that is; X n converges in distribution to X. If limit of F X n of x equal to F X of x for all x at which F X of x is continuous. This is the condition we impose. So, that way we define convergence in distribution. And it is the weakest mode of convergence, also we proved the theorem X n converges in probability implies that X n convergence converges in distribution. With a counter example we show that the converse of the theorem is not true, that is convergence in distribution does not imply convergence in probability.

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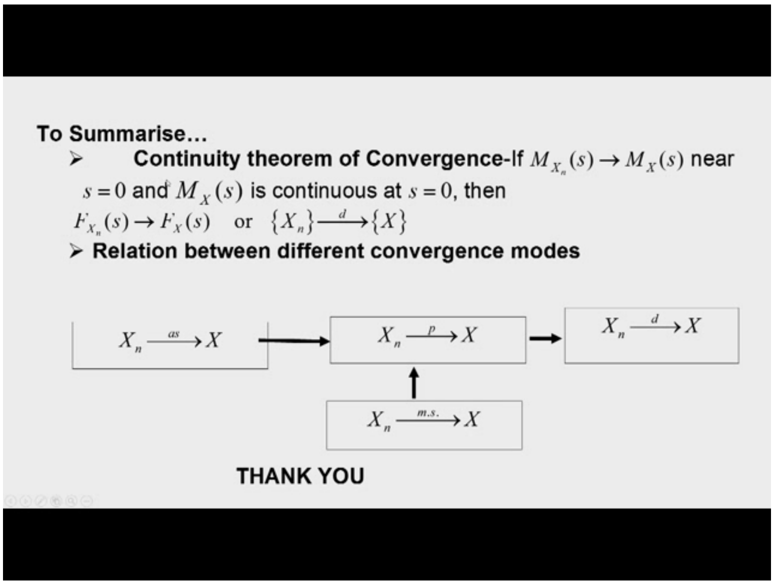

Then we stated the continuity theorem of convergence, if the moment generating function M X n of s converges to M X of s, near s is equal to 0. And M X of s is continuous at s is equal to 0, then F X n of s converges to F X of s, that is distribution function of X n converges to distribution function of X. So, this is one important theorem that is the continuity theorem of convergence

Now, let us summarize the relation between different convergence mode of random sequence, that is X n converges almost sure implies that X n converges in probability; implies that X n converges in distribution. Similarly, X n convergence in mean square sense implies that X n converges in probability; this in turn implies that X n converges in distribution.

Thank you.