Advanced Topics in Probability and Random Processes
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Lecture - 08
Weak Convergence

In the last lecture we discussed about almost sure convergence. Today we will discuss some weaker sense of convergence, namely mean square convergence and convergence in probability.

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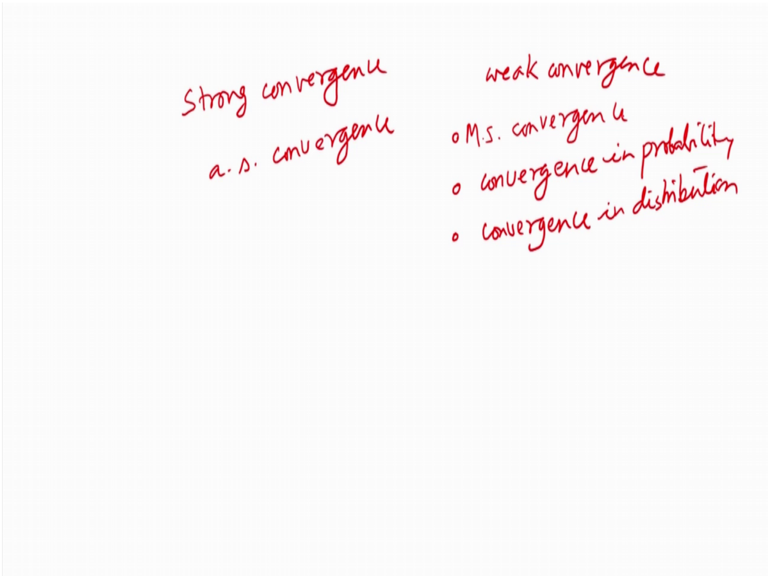

So, we discussed about that is strong convergence, we discussed under that almost sure convergence. We will discuss now above some weak convergence concept, weak convergence. There are several modes of weak convergence. They are like mean square convergence M.S. convergence, Mean Square Convergence, convergence in probability in probability and convergence in distribution. We will discuss 3 modes of convergence, convergence in distribution. So, out of these 3 we will cover m.s. convergence and convergence in probability today. These convergence concepts are called weak convergence because of the relax convergence conditions.

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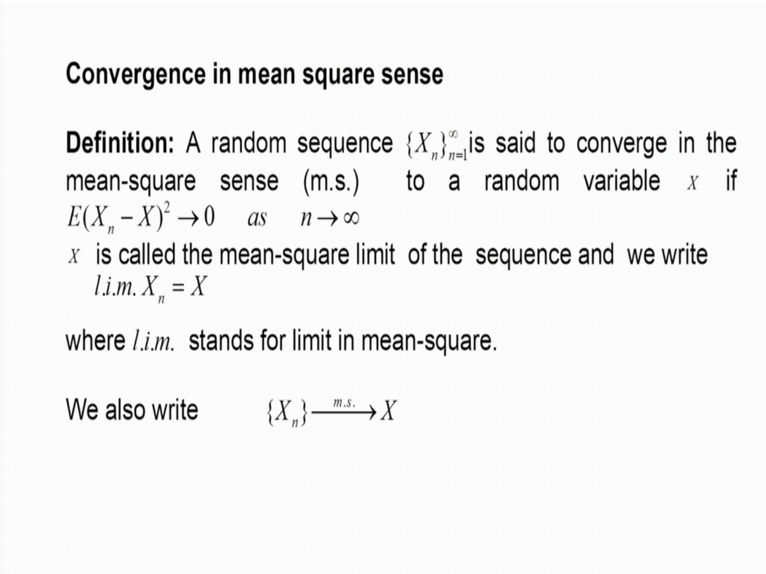

So, first we will discuss about convergent in mean square sense. We will first give the definition. A random sequence X n is said to converge in the mean square sense, this is abbreviated as m.s. to a random variable x, if E of X n minus X whole square that tends to 0, as n tends to infinity.

Now, these weak convergence concepts are considered in terms of some deterministic quantity, convergence of some deterministic quantity. In this case, this mean square error, E of X n minus X, whole square; this quantity is a deterministic quantity and we required at as n tends to infinity this quantity goes down to 0. Then we say that X n convergence in m.s. sense to X; where X is called a mean square limit of the sequence and it is written as just like in the case of limit in deterministic function, here also we write l.i.m. of X n as n tends to infinity is equal to X and what is l.i.m.? l.i.m is Limit in Mean Square.

Then this convergence that X n converges to X that we denote by this or this is one way or we can write in this way. So, therefore, what is mean square convergence? It is the convergence of the mean square error sequence. It converges to 0, if it converges to 0as n tends to infinity then we say that X n converges to X in the mean square sense.

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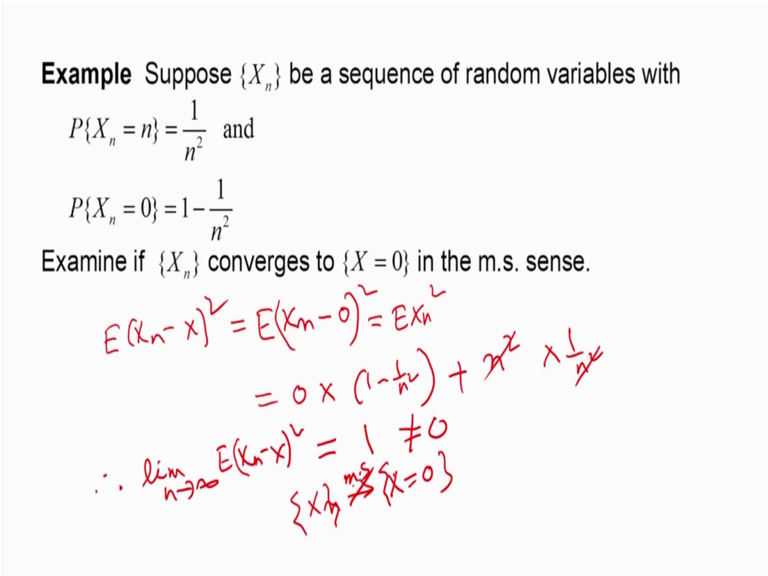

We will see one example X n is defined by this; X n are takes n with probability 1 by n square and X n takes 0 value with probability 1 minus 1 by n square. So, X n is a random sequence of random variable; where each X n, n can take 2 values 0 and n with this probabilities. We see that this probability as n tends to infinity this probability will go down to 1 and this will go down to 0.

Now, question is whether X n converges to X is equal to 0. So, this probability becomes 1 as n tends to infinity, but we want to examine that whether it converges in the mean square sense? So, what we will do? E of X n minus X whole square. So, this is same as E of X n because X is 0, this is a this type of random variable, it is a deterministic quantity, it is known as the degenerate random variable. So, E of X n minus X whole square is same as E of X n square. That is equal to E of X n square.

Now, what is E of X n square? That is equal to X n square can take 0 square with this probability 0 into 1 minus 1 by n square plus, it can take X n square so, it can take n square with probability 1 by n square. So, these 2 get cancel so, this is equal to 1. So, limit as n tends to infinity of E of X n minus X whole square is equal to 1, which is not equal to 0. Therefore, X n does not converges to X is equal to 0 it does not convert to X is equal to 0, in the mean square sense. So, it does not converge, because E of X n minus X whole square as n tends to infinity is not equal to 0.

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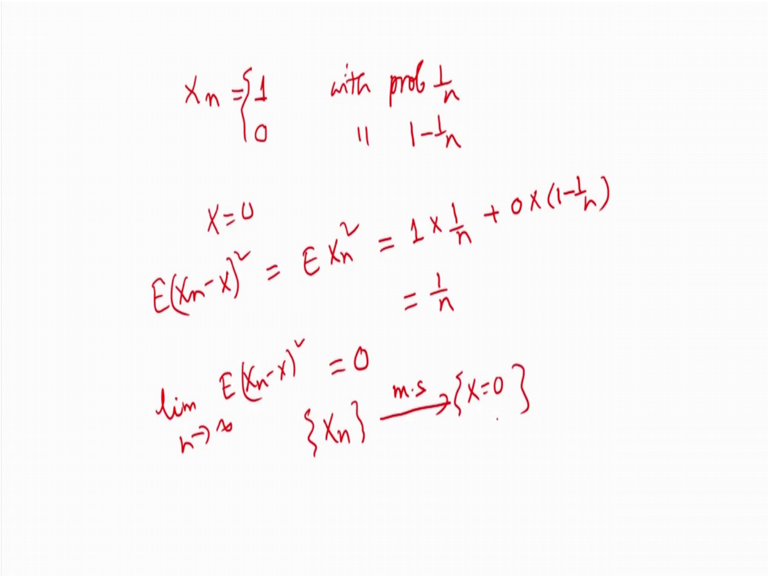

We will consider another example X n is equal to 1 with probability 1 by n, n equal to 0 with probability 1 minus 1 by n. We see that 0 probability of 0 is increasing as n tends to infinity it will approach one. So, question is whether X n converges to X is equal to 0 in mean square sense? So, how we can prove this? E of X n minus X whole square that is equal to E of X n square because X is equal to 0; is equal to 1 into 1 by n plus 0 into 1 minus 1 by n. So, this is equal to 1 by n.

So, what happen? Limit of E of X n minus X whole square as n tends to infinity is equal to limit of this as n tends to infinity is equal to 0. Therefore, X n sequence in this case converges in mean square sense to X is equal to 0.

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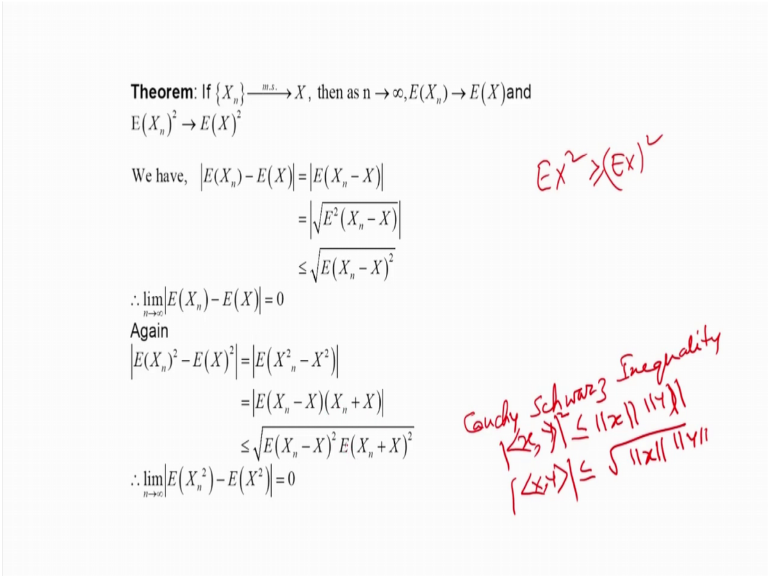

Now, suppose X n is a sequence of random variable and X is a random only one random variable. So, that we have considered that X n this sequence converges to X in the mean square sense.

Then we want to show that E of X n that mean, it is a deterministic quantity now, it is a deterministic sequence of real number; it converges to E of X. Similarly, E of X n square this is the mean square value, it will converge to E of X square. How we can prove this? Consider the absolute the difference between E of X n minus E of X.

Now, I can take expectation outside. So, that way it will be of X n minus X. Now this I can consider this square root of square of this quantity. So, square root of E square of X n minus X. Now I know that suppose E of X square is greater than equal to E of X whole square, why? Because I know that E of X square minus EX whole square it is the variance. So, that way now this quantity is less than equal to square root of E of X n minus X whole square.

Now, what happen as n tends to infinity? Because I know that this sequence is m.s. convergence. So, this term as n tends to infinity it will go down to 0. Therefore, if I have to take the limit of this, E of X n minus EX as n tends to infinity that will become 0. Because this quantity goes down to 0. So, that way if X n converges to X in mean square sense, then it is mean sequence will also converge to mean of X. The same is true for E of X n square.

Now, let us consider E of X n square minus E of X square. That is E of X n square minus X square, we took E outside. Now if I factorize this, that is equal to mode of this mode is there, E of X n minus X into X n plus X. Now we have a famous inequality what is known as the Cauchy Schwarz inequality. What does it say? Suppose I have a vector X and vector Y, it is magnetic if I consider the inner product and then take the magnitude then it is less than equal to norm of into norm of Y.

So, in this case, that inner product operation is defined through this joint expectation operation. So, here we have a square here. So, inner product of XY square is less than equal to norm of X into norm of Y. If I take this square root then inner product of XY that is less than equal to mode of inner product of XY is less than equal to square root of norm of X into norm of Y. So, that we apply here, so, what we get?

So now, it is a norm of E of X n minus X into X n plus X. So, that is the joint expectation of 2 quantities. Therefore, we can apply the Cauchy Schwarz inequality. So, it will be square root of norm of this quantity E of X n minus X whole square. Similarly, norm of this quantity is E of X n plus X whole square.

Now, what happen as n tends to infinity, limit n tends to infinity? Now this quantity will go down to 0. Therefore, whatever this value ultimately this expression will become 0. So, that we have what we have learned? Therefore, that E of X n square this sequence also converges to E of X square. So, if X n sequence converges to X in the mean square sense, then the corresponding mean sequence and the mean square sequence they will also converge. So, these 2 important properties are very useful property for application point of view.

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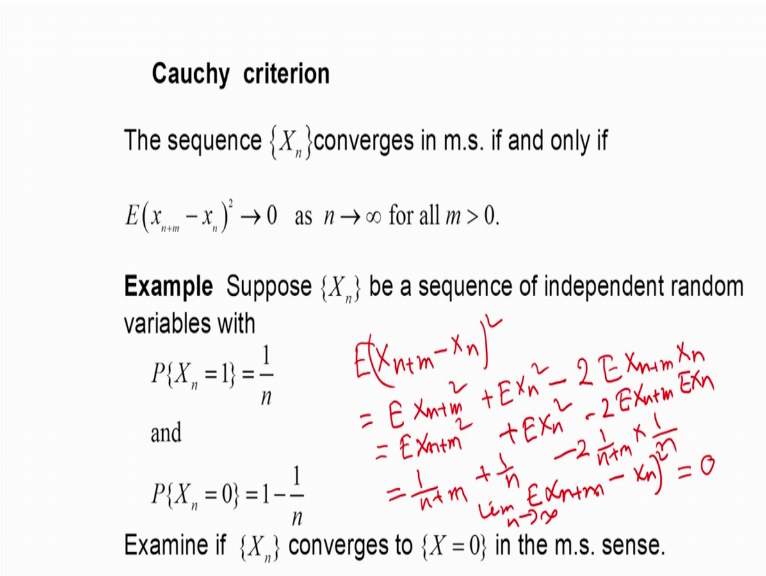

Now, we discussed about the convergence of mean square convergence of this sequence. Now in the case of deterministic sequence, the sequence of real number we have one important result that is Cauchy criterion of convergence. So, the same concept can be adapted here. If X n is a sequence of random variables, then X n converges in m.s. if and only if E of X of n plus m minus X n whole square tends to 0 as n tends to infinity for all m greater than 0. So, this is the concept.

Now, here without knowing the limit what is the mean square limit of X n, without knowing that also we can prove whether this sequence X n converges to some unknown random variable X or not. So, the result is we have consider, the mean square value of difference between X of n plus m and X n. So, this mean square value should go down to 0 as n tends to infinity. So, that means, if we consider for any m, we consider the mean square error; suppose X n X of n plus m minus X n whole square.

Now, this quantity will go down to 0 as n tends to infinity, then only we can say that X n converges in m.s. to some unknown random quantity X. For example, we can consider this problem again, probability of X n is equal to 1, that is suppose is equal to 1 by n, probability of X n is equal to 0 is equal to 1 minus 1 by n. And we recently assume that it is a sequence of independent random variable.

So, we do not know suppose that it converges to X equal to 0. Without that also we can prove this; how we will prove? E of X n plus m X of n plus m, minus X n whole square, this quantity is equal to E of X of n plus m whole square plus E of X n square minus 2 E of X n plus m into X of n. So, this quantity we can write that is E of X of n plus m whole square, X of n plus m square plus E of X n square minus now this one I can write as E of X n plus m into E of X n ok.

So, this quantity now, what is this quantity? X of n plus m, suppose it takes value it will take 1 with this probability. So, 1 into 1 by n plus m, because X of n plus m whole square. So, this will be 1 divided by n plus m. Because other quantity is 0; similarly, this will be equal to 1 by n. This quantity will be E of X of n plus m that is 1 into 1 by n. So, 2 into 1 by n plus m and similarly this quantity E of X n it will be equal to 1 by n.

So, this is the expression and what happens? Limit of this quantity limit E of X n plus E of X n plus m minus X n whole square, as n tends to infinity because n is in denominator for all the expression, therefore, this quantity will go down to 0. Therefore, we can say that this sequence is a convergent sequence in the mean square sense. So, that way we can apply the Cauchy criterion to see whether a sequence is mean square convergent or not.

Now, mean square convergence is a very important concept, because we are E of X n minus X whole square; suppose X is the actual quantity X n is some observed quantity suppose, then E of X n minus X whole square will represent the mean square error. So, that way in iterative algorithm suppose whether that mean square error converge or not that we can prove using this concept of mean square convergence.

Next we will be considering about convergence in probability. So, how do I define this?

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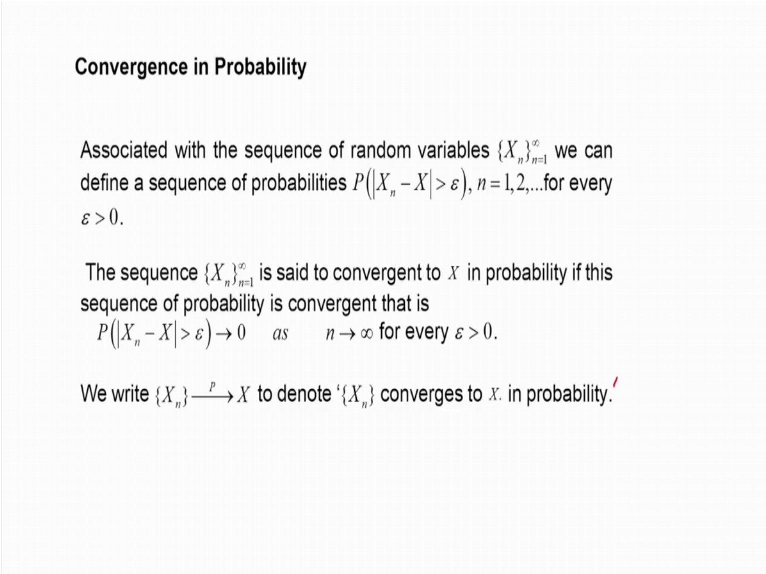

Suppose X n is a sequence of random variable. Now we can define a sequence of probability. Suppose, there is an epsilon, epsilon greater than 0, it is a arbitrarily small number. Suppose we are considering this probability, probability of X n minus X, it is deviation. So, absolute value of this deviation is greater than epsilon.

What is the probability? Now this probability for a given epsilon is a function of n. So, it is a function of n, therefore, it is a real sequence. Now, if this sequence is convergent; that means, probability of X n minus X absolute value of that is greater than epsilon, if that probability sequence this probability sequence if it goes down to 0 as n tends to infinity, then we say that X n converges to X in the sense of probability. So, this sequence X n is said to converge to X this is a random variable in probability, if this sequence of probability this sequence of probability is convergent that is probability of X n minus X absolute value of that is greater than epsilon.

So, the probability that absolute value of X n minus X is greater than epsilon goes down to 0 as n tends to infinity. We write this is the notation sequence X n converges in probability to X. And so, that way we denote the convergence in probability. So, X n converges to X in probability that is denoted by this notation. So, this is a weaker sense of convergence, but how it is related to other concepts like almost sure and mean square convergence that we will see.

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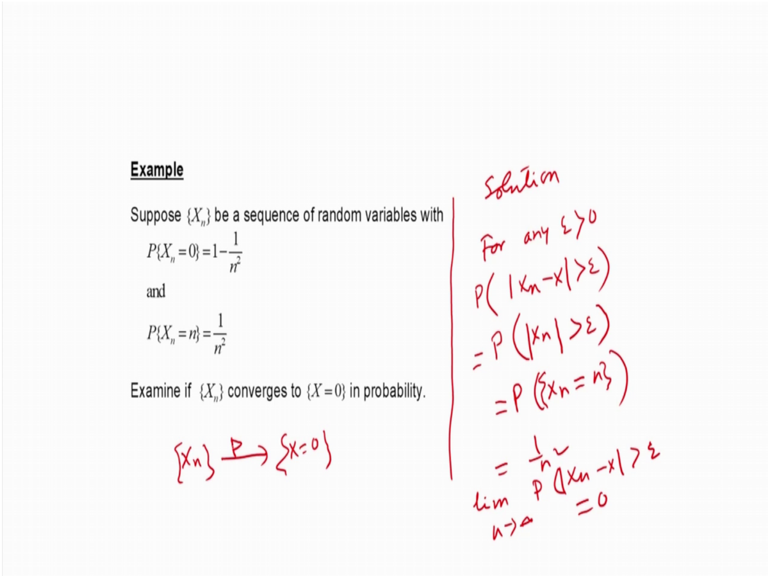

Now, we will see an example first suppose similar example X n takes 0 with probability 1 minus 1 by n square and X n takes value n with probability 1 by n square. Now, examine if X n converges to X is equal to 0 in probability. So, we have to consider solution; suppose for any epsilon greater than 0, what we can write? Probability of absolute value of X n minus X that is greater than epsilon. That will be probability of X is equal to 0, probability of X n mod of X n is greater than epsilon.

So now whatever epsilon we consider that mod of X n greater than epsilon, that will be the probability that X n is equal to n. Because other value of X n is equal to 0, that is equal to probability of X n is equal to n. So, what is this probability? So, this is equal to 1 by n square. So, therefore, limit of this probability sequence probability of X n minus X that is greater than epsilon. So, that probability is equal to as n tends to infinity that will become 0. Therefore, we say that X n sequence that X n sequence converges to X is equal to 0, this is a random variable in the sense of probability.

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Theorem $\{X_n\} \xrightarrow{m.s.} X \Rightarrow \{X_n\} \xrightarrow{P} X$

$+ve$

$P(Y > a) \le \frac{EY}{a}$

Proof: Suppose $\{X_n\} \xrightarrow{m.s.} X$

Then $\lim_{n\to\infty} E(X_n - X)^2 = 0$

Now, for $\varepsilon > 0$,

$$P(|X_n - X| > \varepsilon) = P(|X_n - X|^2 > \varepsilon^2)$$

$$\le E(X_n - X)^2 / \varepsilon^2 \qquad \text{(Using Markov Inequality)}$$

$$\therefore \lim_{n\to\infty} P(|X_n - X| > \varepsilon) \le \lim_{n\to\infty} E(X_n - X)^2 / \varepsilon^2$$

$$= 0$$

$\therefore \{X_n\} \xrightarrow{p} X$

Now we will examine the relation between mean square convergence and convergence in probability. This is the theorem if X n converges to X then X n converges to X n probability. X n converges to X in the mean square sense implies that X n converges to X in probability. So, this is the theorem suppose if we can. So, that X n is convergent in mean square sense then X n will be convergent in probability sense also.

So, proof is simple what we what do I proof that is given that limit of X n minus X whole square, if we take the expected value of X n minus X whole square and take limit as n tends to infinity. That will become 0, because X n converges to X in the mean square sense. So, this is given.

Now, for any epsilon greater than 0, what we will get? Probability of mod of X n greater than epsilon. So, that will be equal to because we can consider it in terms of the square quantity. So, that way probability of X n minus X whole square is greater than epsilon square, because just we are squaring both quantity. So, this probability and this probability is same.

Now this is a positive quantity E of X n minus X whole square. So, this is a positive quantity. So, this positive probability that this positive quantity Y is suppose greater than some quantity a. So, probability of Y, Y is a positive quantity it is positive; Y is positive Y greater than a. So, that will be less than equal to E of Y by a. So, this is the Markov inequality. So, if I apply here it will be E of less than equal to E of X n minus X whole square divided by epsilon square. So, this is the relationship.

Now, what we know that this quantity as n tends to infinity this goes down to 0. Therefore, if I consider limit as n tends to infinity of this probability sequence, then it will become limit of E of X n minus X square that is equal to 0, so, less than equal to 0 probability cannot be a negative. So, this is equal to 0. Therefore, what is my conclusion? The sequence of X n converges to X in probability also. Mean square convergence imply convergence in probability, very important result.

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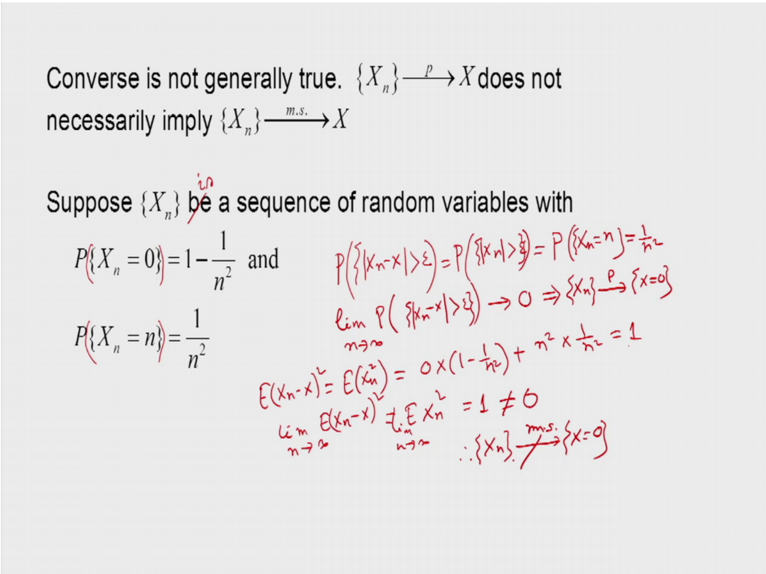

The converse of the theorem is not generally true, X n convergence in probability to X does not necessarily imply that X n converges in mean square sense to X. Suppose X n is a sequence of random variables with p of probability of X n is equal to 0, probability of X n is equal to 0 is equal to 1 minus 1 by n square. And probability of X n is equal to n is equal to 1 by n square.

So, in this case, we see that this probability sequence probability that X n is equal to 0 this is converging into 1. So, if we see if 0 is the X equal to 0 if the limit in probability. So, that way probability that X n minus X mod of that that is greater than epsilon. So, in this case X, X is equal to 0. So, probability of X n mod of X n greater than epsilon. So, this is equal to probability of X n equal to n, that is equal to 1 by n square. Therefore, limit probability of X n minus X that is greater than epsilon. So, this probability will go down to 0; imply that, this is sequence convergence in probability converges in probability. So, by that X n converges in probability to X is equal to 0.

Now, considering the mean square convergence we see that E of X n minus X whole square, that is equal to E of X n minus X is equal to 0. So, that way it will be E of X n square since X is equal to 0. So, that is equal to 0 into 1 minus 1 by n square plus n square into 1 by n square so, this is equal to 1. Therefore, limit E of X n minus X whole square as n tends to infinity. Therefore, limit therefore, limit of E a E of X n minus X whole square as n tends to infinity, that is equal to E of X n square limit n tends to infinity that is equal to 1 which is not equal to 0.

Therefore X n does not converge in the mean square sense to X is equal to 0. So, we see that in this case X n converges in probability to X is equal to 0, but X n dos not converges in mean square sense to X is equal to 0. So, what we have established therefore?

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Theorem $\{X_n\} \xrightarrow{a.s.} X \Rightarrow \{X_n\} \xrightarrow{P} X$

as convergence
$\Rightarrow$ convergence in probability

Proof: Suppose $\{X_n\} \xrightarrow{a.s} X$

Then for any $m \geq 1$

$$P\left(\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\left\{s \,\middle|\, \left|X_j(s)-X(s)\right| \geq \frac{1}{m}\right\}\right)=0$$

$$\therefore \lim_{n\to\infty} P\left(\left\{s \,\middle|\, \left|X_n(s)-X(s)\right| \geq \frac{1}{m}\right\}\right)$$

$$\leq \lim_{n\to\infty} P\left(\bigcup_{j=n}^{\infty}\left\{s \,\middle|\, \left|X_j(s)-X(s)\right| \geq \frac{1}{m}\right\}\right)$$

$$= P\left(\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\left\{s \,\middle|\, \left|X_j(s)-X(s)\right| \geq \frac{1}{m}\right\}\right)=0$$

$$\therefore \{X_n\} \xrightarrow{P} X$$

Convergence in mean square sense implies convergence in probability. Similarly, we have suppose if X n is convergent to X in the almost sure sense convergence X n converges to X almost sure, then X n converges to X in probability also. So, almost sure convergence, imply convergence in probability, as convergence implies convergence in probability. So, this also we can prove. Now here in place of epsilon, we can we consider 1 by m; where m is arbitrary large positive integer.

Now, we know that X n is almost sure convergence to X. What does it means? Probability of this l.i.m. sup of this deviation. So, X j s minus X s greater than 1 by m as such that absolute value of this deviation is greater than m. So, that you will take the union from j is equal to n to infinity. And then this sequence we will take the intersection of all from n is equal to 1 to infinity. So, this is the actually this is the l.i.m. sup l.i.m. sup of this deviation sequence.

Now, I know that that inside part that union part this is the decreasing sequence. So, we can apply the continuity theorem. So, that way what we will get is that, this limit of this, suppose this will be same as what you will show here is that probability of this l.i.m. sup is same as probability of the inner event as n tends to infinity. So, this concept we will use now let us see that probability of this quantity is greater than 1 by m.

Now this is single event therefore; this probability will be less than if we consider some union of event because this is one event there. So, that way that probability will be bigger this right hand side probability is bigger therefore, this probability is less than this quantity.

Now, I know that this quantity is same as this quantity, because this if I consider this sequence as I told earlier this is a decreasing sequence. Therefore, probability of this will be same as this probability as n tends to infinity. So, therefore, this quantity I can substitute by a quantity in terms of l.i.m. sup. But given that X n converges almost sure to X therefore, this quantity is equal to 0. Therefore, this quantity that probability that this absolute deviation from the random variable X is greater than 1 by m is going down to 0 as n tends to infinity. Therefore, this sequence X n sequence converges to X in probability. So, almost sure convergence implies convergence in probability.

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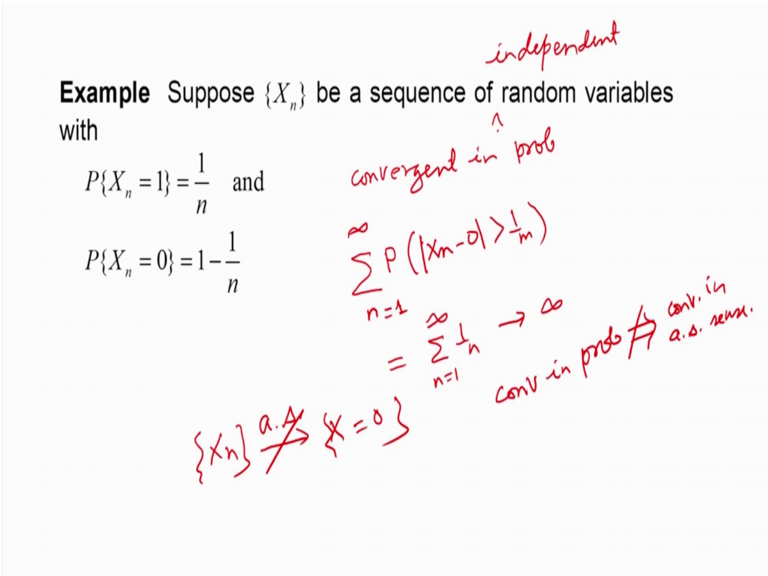

We will consider one example X n be a sequence of random variables with probability of X n is equal to 1 is 1 by n and probability of X n is equal to 0 is equal to 1 minus 1 by. So, this sequence we can prove that this sequence suppose this sequence is convergent in probability in probability because probability that X n is positive because this is 0 is the random variable. So, that X n is positive that is only one case X n can take one and that probability is 1 by n. And this probability will go down to 0 as n tends to infinity. Therefore, this sequence is convergent in probability.

But, whether it is convergence almost in almost sure sense? That we can see this condition, suppose here we will consider the sequence of independent random variable. That also we will be considering. So, in that case we can so that that deviation sequence probability that mod of X n minus X is 0 X 0 that is greater than 1 by m. So, if I consider this as from n is equal to 0 to n is equal to 1 to infinity, if I consider this. So, this now this is the what is this probability that is equal to 1 by n. So, summation 1 by n going from n is equal to 1 to infinity.

So, now this will diverge to infinity. Therefore, this probability that there will be this divergence ok, that probability will be equal to 1; that means, this sequence X n does not converge in almost sure sense to X is equal to 0. So, it does not converge. So, what we have seen? That, though we have the theorem that convergence almost sure imply convergence in probability, but other way it is not true. Convergence in probability does not imply convergence in almost sure sense, ok. So, this is another important result that convergence in probability does not generally does not imply convergence in almost sure sense.

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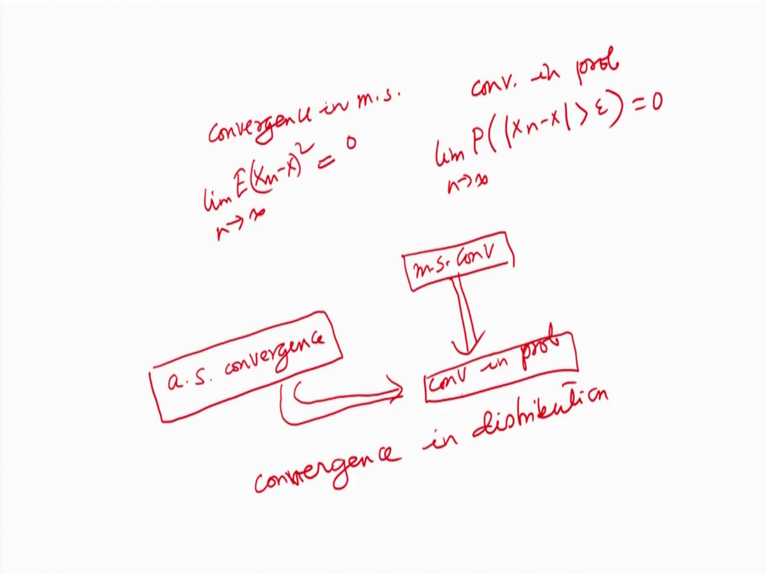

So, we have considered 2 important weak convergence concept convergence in mean square. What does it say? It says that E of X n minus X whole square goes down to 0 as limit of this as n tends to infinity that will be equal to 0. Convergence then X n is convergent in mean square sense.

Then convergence in probability what does it means? That it says that, probability sequence probability that X n minus X mod of that that is greater than an epsilon. So, this probability sequence limit of this probability sequence as n tends to infinity is equal to 0.

So, in the case of convergent in mean square sense that, mean square error sequence that goes down to 0 as n tends to infinity. So, this is equal to 0 as n tends to infinity. In the case of convergence in probability, we are considering the probability of the absolute deviation that is mod of X n minus X is greater than epsilon. This also goes down to 0, for any epsilon as n tends to infinity. Then we say that X n converges to X in probability. And there is the implication result what we have shown that suppose this is convergence m.s. convergence and then we have convergence in probability. So, this will imply this m.s. convergence we will always imply convergence in probability.

And earlier we discussed about almost sure convergence that is the stronger concept. So, this almost sure convergence here, suppose m.s. convergence and convergence in probability. So, this will also imply that convergence in probability m.s. convergence will always imply convergence in probability m.s. convergence will also imply convergence in probability. So, that way and this concept will imply this concept will imply this, but other way it is not true.

So, other way it is not true means generally convergence in probability will not imply m.s. Convergence or convergence in almost sure sense. So, we discuss about toward the weak convergence concept. And the inter relation between as convergence m.s. convergence and convergence in probability.

We have to consider another weak concept that is convergence in convergence in distribution. So, in this case, this distribution function sequence that will be again a real sequence that sequence will be converging to some distribution function of some random variable X. So, that way we will discuss about this weak concept also weak convergence concept convergence in distribution. And then we will see some of the application of all these convergence concept as convergence m.s. convergence in probability and convergence in distribution. That we will be discussing in the next class and what is the implication between these 2 that also we will be discussing.

Thank you.