Advanced Topics in Probability and Random Processes
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Lecture - 07
Convergence of a Sequence of Random Variables

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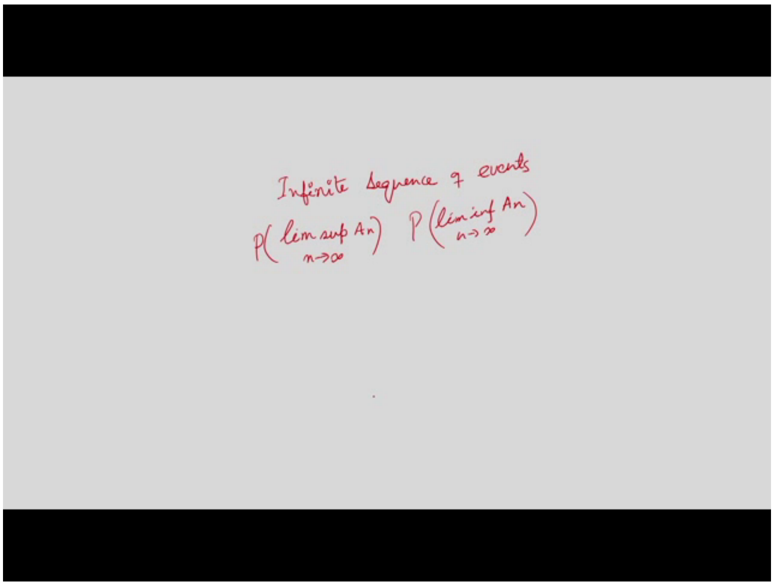

In the last lectures we discussed sequence of events that is infinite sequence of events and limits. Limits of these events, limits lim sup we define, lim sup and lim. Suppose, sequence is n, this event we defined. Similarly, lim inf as n tends to infinity, this type of limit we define and then how to find out the probability for this limiting events. We discussed different theorems and particularly Borel- Cantelli lemmas how to find out this probability, probability of lim sup A n under two different conditions.

Now, this application of convergence of infinite sequence of events is the convergence of sequence of random variables. Now, we will be discussing the convergence concepts for random variables and these concepts are useful in many practical applications, just like it is the basis of stochastic calculus. Also we will be discussing, very important applications like central limit theorem and we log up large numbers.

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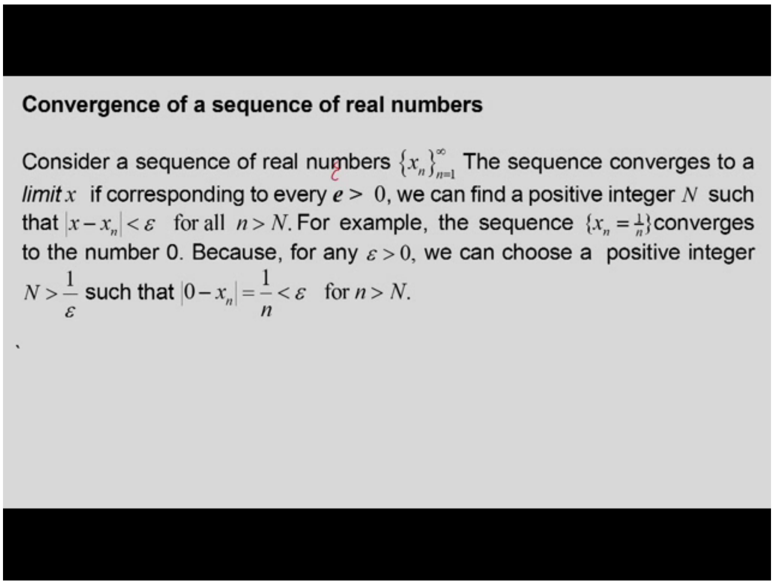

First of all let us see: what is convergence of a sequence of real numbers. Consider a sequence of real numbers x n, n going from 1 to infinity. This sequence converges to a limit x if corresponding every epsilon greater than 0, we can find a positive integer N such that x mod of x minus x n can be made smaller than epsilon for all n greater than N. So, what does it say that, this difference between the limit and the sequence can be made as small as possible where epsilon is an arbitrarily small number positive number, for sufficiently large N. Then we say that x n converges to x. So, for example, if I consider this sequence suppose x n is equal to 1 by n simple example.

Now, we know that limit of 1 by n as n tends to infinity 0. Now, how it confirms to this definition, suppose for any epsilon greater than 0, we can choose a positive integer N greater than 1 by epsilon. So, that in that case 0 minus x n mod of that will be 1 by that is equal to 1 by n and this will be less than epsilon. So, because we are considering N is greater than 1 by epsilon therefore, for any n greater than capital N this will be valid. So, that way this 0 will be limit of this sequence. And now we have to extend this definition of convergence or limit to sequence of random variables.

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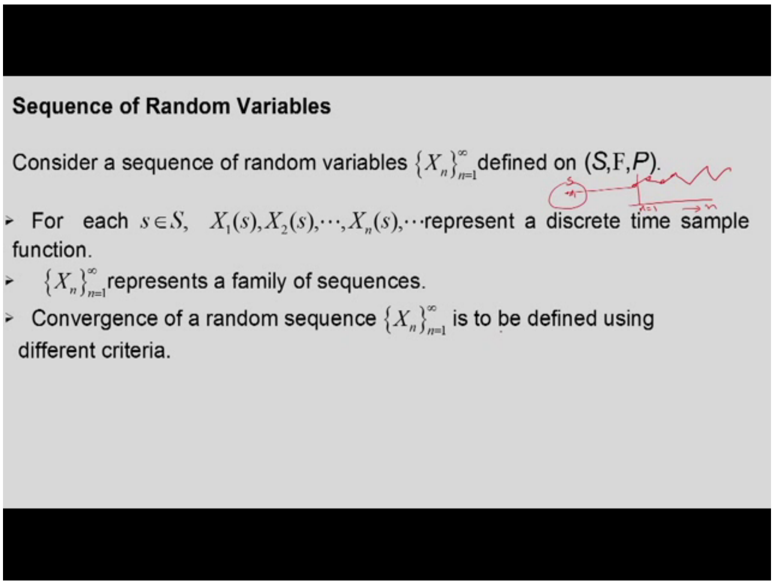

Consider a sequence of random variable X n, n going from 1 to infinity defined on S, F, P. We are defining suppose a sequence of random variables. Now what does it mean? That means, if we have a sample space suppose S, this is the sample space and now corresponding to suppose any element s 1 here. So, that will map to suppose, this is my time axis n, n is equal to 1 here. So, n is equal to 1 it will be 1 point. Similarly, n is equal to 2 it may be another point like that this s 1 will be mapped to different points at different instant of time. So, that way I may get a sequence, may be if I join this may be something like this.

So, that is that means, corresponding to every sample point I will get a sample functions like this. So, that means, fall is S belonging to sample space X 1 s, X 2 s up to X n s represent a discrete time sample function particular for each s. So, that will be, we will have a sample function like this. Therefore, since the sample space will contain other sample points therefore, corresponding to each sample point will get a sample function like this. So, unlike sequence of real numbers action sequence of random variables a represents a family of sequences.

So, each sample function is a sequence therefore, it is a family of sequences and here are the problem in defining the convergence for the sequence of a real numbers. So, convergence of a random sequence is to be defined using different criteria because, it is since it is not a single sequence, we have to use different criteria to, to the define the convergence of a sequence of random variables.

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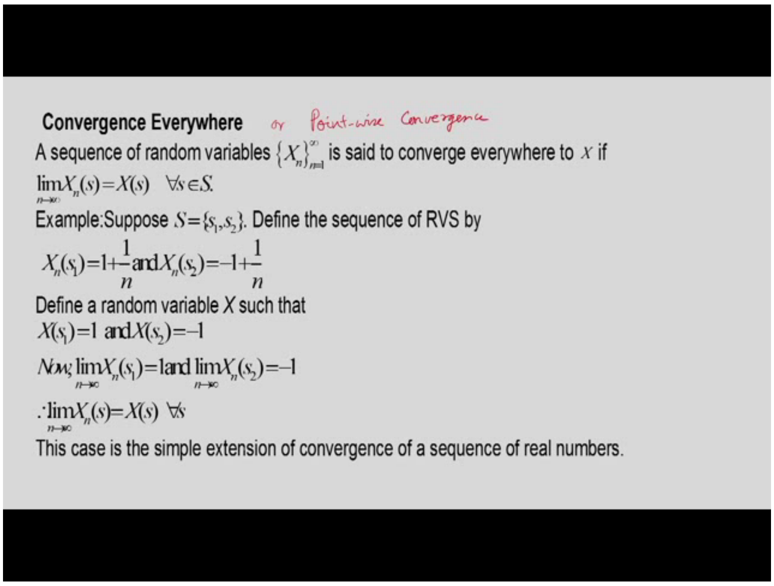

Therefore, there are different modes of convergence of a sequence of random variables. The most elementary convergence concept is convergence everywhere or point - wise convergence, it is also called point - wise convergence. A sequence of random variables X n is said to converge everywhere to x, if limit of X n s equal to X of s for all s belonging to S. So, suppose corresponding to each sample point we will find a limit. So, in that case this is the point wise convergent or convergent everywhere. We can consider one simple example. Suppose define a sequence of random variables suppose X only sample space comprises of two sample points s 1 and s 2.

Now, suppose X n s 1 is equal to 1 plus 1 by n and X n s 2 is equal to minus 1 plus 1 by n. We observe that as n tends to it infinity, this point will become 1 and this point will become minus 1. So, that way we can now define the a random variable X such that suppose X of s 1 is equal to 1 and X of s 2 is equal to minus 1. So obviously, limit of X n s 1 will be 1 and limit of X n of s 2 will be minus 1. Therefore, in this case the sequence X n, this X n sequence converges point wise or converges everywhere to random variable X. So, that way convergence everywhere is this just an extension of convergence of a sequence of real numbers. So, here we are not, not considering the property of the random variable, we are simply considering this as a sequence of function.

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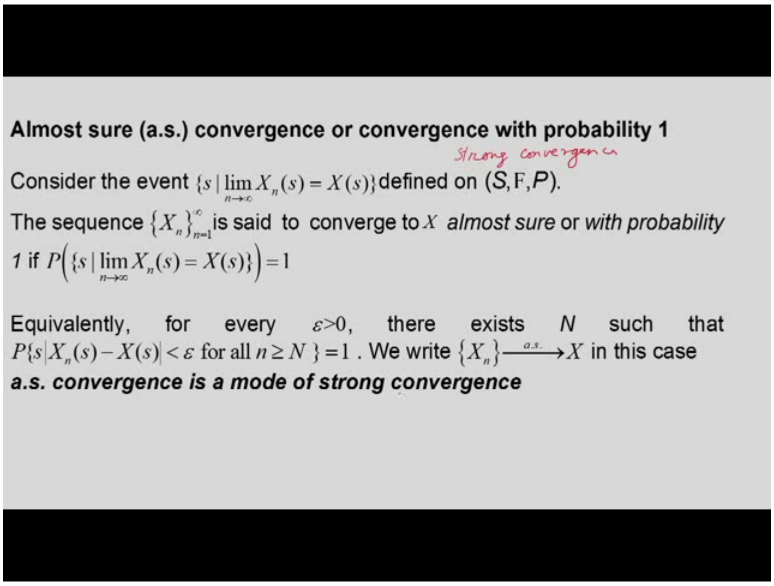

Next we will, consider one concept which is known as convergence almost here or convergence with probability 1, this is also known as the strong mode of convergence, this is strong convergence. Let us consider an event that is it is comprising of points S such that limit of X n s is equal to X s. So, we are considering all those sample points for which limit of X n s is equal to X s there may be some sample points where this limit is not equal to Xs ok, but we are considering only those simple points for which this limit is equal to X s.

Since, decision even define on the on S, F, P we can find out the probability of this event. This sequence now X n is said to converge to X almost here or with probability 1, if probability of this event that is probability of the event s such that limit of X n s equal to X s is equal to 1. So, probability of this event is equal to1. So, that is the definition of almost your convergence; that means, wherever there is a convergence. So, those sample points if we consider then the corresponding event should have probability 1.

Equivalently for every epsilon greater than 0 there exists N such that a probability of this event is less than epsilon for all n greater than N, this is the event we are considering that is S such that X n s minus X s mod of that is less than epsilon for all n greater than certain N. Suppose epsilon corresponding to each epsilon we will get a big N number and suppose where this convergence takes place for all n greater than equal to N, that probability is equal to 1. We write X n converges to X almost here this is the symbol we use, as I have told this is a, mod of strong conversions.

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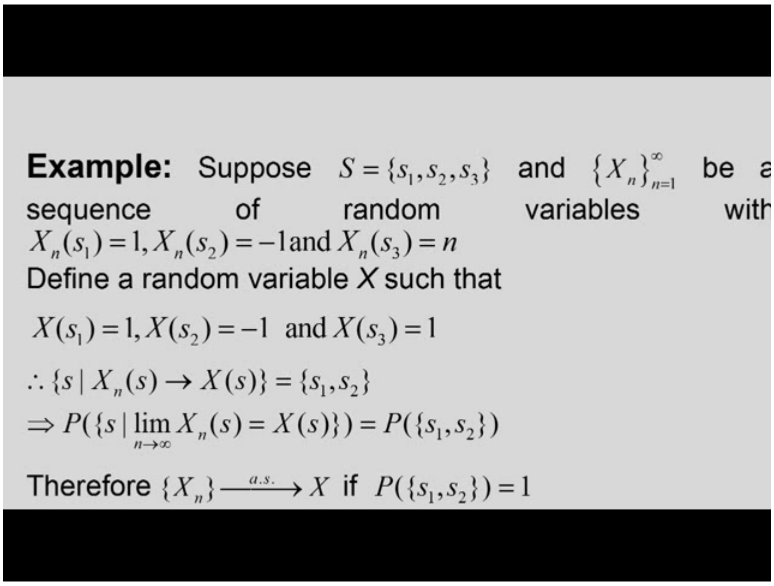

Let us consider one example, suppose sample space is s 1, s 2, s 3, it comprises of three elements and the X n be a sequence of random variables with X n of s 1 is equal to 1, X n of s 2 is equal to minus 1. And, suppose this is a divergent sequence X n of s 3 is equal to n and now, we will define another random variable on the same sample space X s 1 is equal to 1, X s 2 is equal to minus 1 and X of s 3 is equal to 1. So, here it is divergent, here it is 1.

Therefore, if I consider the event for where, which this convergence is taking place I, I we see that X n s 1 is 1 and here also 1, X n s 2 is minus 1, here also minus 1, but X n s 3 is 1, here it is 1 therefore, it is not converging for s 3. So, that way this event now where that convergence is taking place that is that comprises of s 1 and s 2 therefore, probability of this limiting event now is equal to probability of s 1 s 2. So, when X n will converge to X almost here if probability of this event is equal to 1, so that way the basic concepts of almost sure convergence is introduced.

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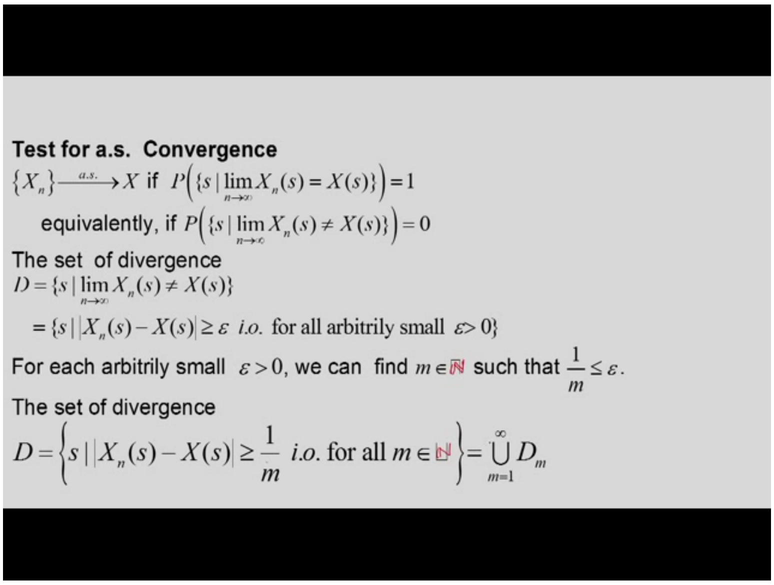

Now, how to test almost sure convergence, X n converges almost sure to X, if probability of this event s such that limit of X n s equal to X s, the probability of this event is equal to 1 or equivalently, if this limit is not equal to X s, if we consider that event that probability should be equal to 0. Now let us define a set of divergence where X n s is not equal to s.

So, that event is we are calling this event as D. Now, that event is equal to, the set as such that mod of X n minus s mod of X n s minus X s is greater than equal to epsilon and now infinitely of 1 for all arbitrarily small epsilon greater than 0. So, what we are defining that the set of divergence is this set comprising of element such that mod of X n s minus X s is greater than epsilon infinitely often for all arbitrarily small epsilon greater than 0.

So, if suppose there is only divergence in the case of finite number of n, then we can always consider the sequence starting with that number where it is diverging suppose after that the sequence we can consider. So, that way this is important that we define the event s such that mod of X n s minus X s is greater than equal to epsilon for in for infinitely many n because if it is finitely many n then we can always go beyond those numbers to find out the convergence sequence.

Now, for is arbitrarily small epsilon we can find a positive integer m, m belonging to set of natural number, this is the set of natural number such that 1 by m is less than equal to epsilon. So, any epsilon, you consider we can get a, positive integer like this such that 1 by m is less than equal to epsilon. So, that we can replace epsilon by 1 by m what is the idea because if it is in terms of m we can enumerate.

So, this set of divergence now we can define as D is equal to the set comprising of element s such that mod of X n s minus X s is greater than equal to 1 by m. Instead of epsilon we are writing greater than equal to1 by m infinitely often for all m will belonging to n, that is equal to suppose now m is we can count m therefore, we can write in terms of suppose Union of Dm m going from 1 to infinity because all m we have to consider. So, that way we have define a set of divergence where the random sequence diverges from Xs. So, that we are considering and that is a union Dm because for each time we have to define that divergence set; so, union Dm m going from 1 to infinity.

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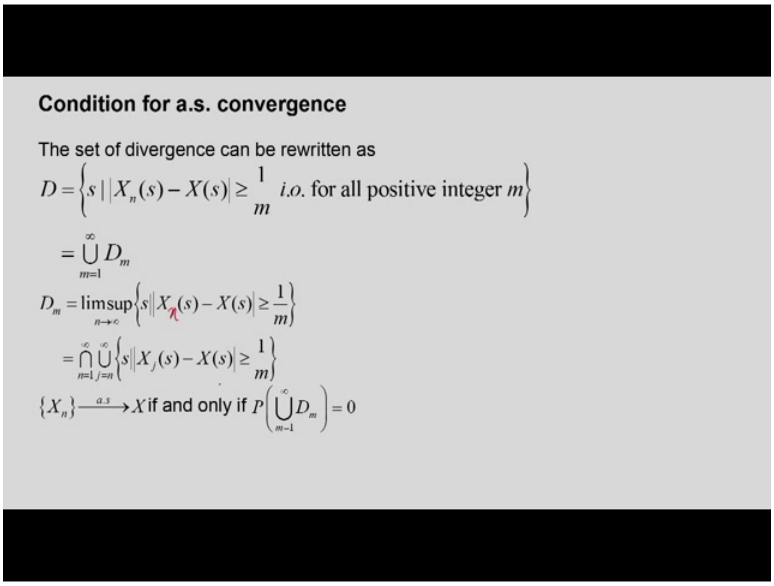

Now we get a condition for s convergence convergence in terms of D. Suppose this set of divergence can be rewritten as which we have already done that it is a Union of Dm m going from 1 to infinity. Now this infinitely often that is the, that is the lim sup of s such that mod of X, X n s minus X s is greater than equal to1 by m as n tends to infinity, that is the event of this, that X n minus X n s minus X s absolute value of that is greater than 1 by m infinity often that we are writing in terms of lim, lim sup.

Therefore, now this is equal to as from definition of lim sup intersection n going from 1 to infinity of Union there going from n to infinity of the event s such that mod of X s minus X s is greater than equal to 1 by m. So, this is the basic event where it is greater and this deviation is greater than 1 by m then we are considering the lim sup of that event sequence.

Obviously, now X n will converge to X almost sure if and only if probability of Union of Dm is equal to 0. So, we have the condition now if and only if condition that X n will converge to X almost sure if and only if probability of union of Dm m going from 1 to infinity is equal to 0, this is a very strong condition.

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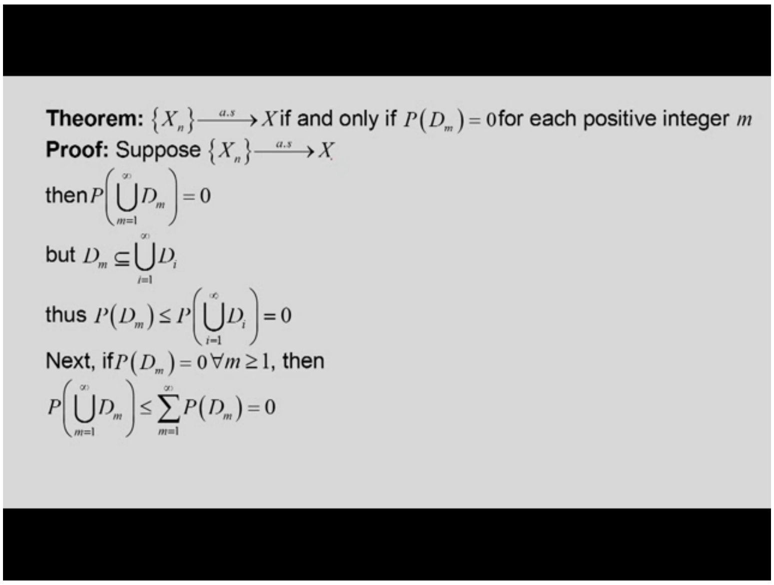

Now, we will because here we have the union, union of Dm m going from 1 to infinity, this condition we want to simplify. We will state the theorem the sequence X n converges almost sure to X if and only if probability of Dm is equal to 0 for each positive integer m. So, this is a simpler condition because we have to consider P of Dm instead of the probability of union Dm.

Now, as the proof suppose X n converges to X, almost here then probability of union of Dm m going from 1 to infinity must be equal to 0 but we also know that this Dm, any Dm any member is a subset of the union, union D i, i going from 1 to infinity. Therefore, we can write that probability of Dm because it is a subset it is a less than equal to probability of this union but, we know that probability of this union is equal to 0. Therefore, probability Dm must be equal to 0. So, that way if X n converges almost sure to X, then probability of Dm must be equal to 0.

Now, next if probability of Dm is equal to for all m greater than equal to 1 then probability of a union of Dm, I know that that is less than equal to sum of D corresponding probability, this inequality we proved earlier therefore, and I know that each of this probability is equal to 0 therefore, probability of this union will be equal to 0.

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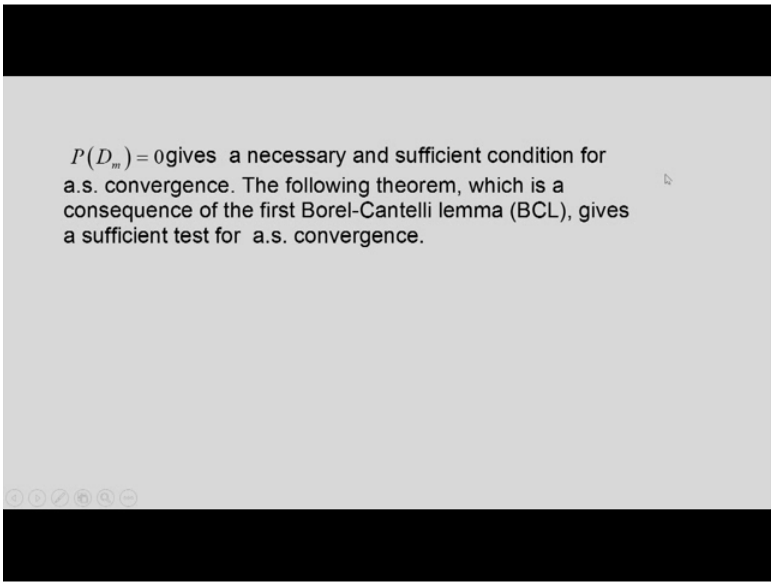

So, that way we have proved that X n converges to X if and only if probability of D m is equal to 0 for each positive integer m so, this is a condition that is, if and only if condition probability of Dm is equal to 0 gives a necessary and sufficient condition for almost your convergence. So, we can find out this probability to prove almost your convergence but this is still a, difficult task because we have to consider Dm then Dm is, Dm for each m and we know that Dm is defined through that lim sup operation.

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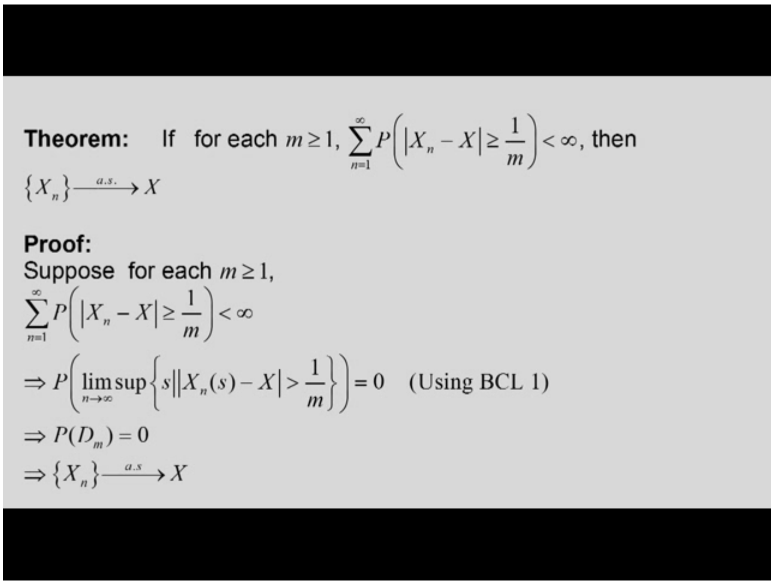

Now, we will state a theorem that is a which is a consequence of first Borel- Cantelli lemma that gives a sufficient s for almost sure convergence that we will discuss now., theorem if for each m greater than equal to 1, now we are considering this sum probability of mod of X n minus X greater than equal to m. So, this is for each n, we will consider and then sum up; so, therefore infinite sum of probability of mod of X n greater than equal to 1 by m.

So, if this infinite sum, sum of the infinite series is less than infinity, then X n converges almost sure to X. So, we have to consider the probability of deviation for each m and then sum up, if this sum is less than infinity then X n converges almost sure to X. Suppose, for each m greater than equal to 1this sum is convergent. So, what does it mean? Now, we can apply the Borel - Cantelli lemma if the sequence of event.suppose I can consider this as a sequence of event, if the probability of this sequence if the sum of the probability of this sequence is less than infinity, then a probability of lim sup of that event sequence is equal to 0, that is the Borel Cantelli lemma 1.

So, what we will get here this implies that probability that lim sup of, the event s such that mod of X n s minus X, that is greater than 1 by m as n tends to infinity is equal to 0, that is the conclusion. We draw with the help of Borel-Cantelli lemma 1. Therefore, but this is by definition this is the Dm therefore, probability of Dm is equal to 0 the probability of Dm is equal to 0, then X n converges almost sure to X.

So, that way we can apply the Borel- Cantelli lemma 1 and that is first part of Borel -Cantelli lemma to prove the, prove the convergence of a sequence and thus we can apply this condition that is probability then mod of X n minus X greater than equal to 1 by m, that is the sum from n going from infinity, if that sum is convergent then X n converges to X almost sure but this is your sufficient condition unlike probability of Dm is equal to 0 that is a necessary and sufficient condition but this is an this is a sufficient condition only.

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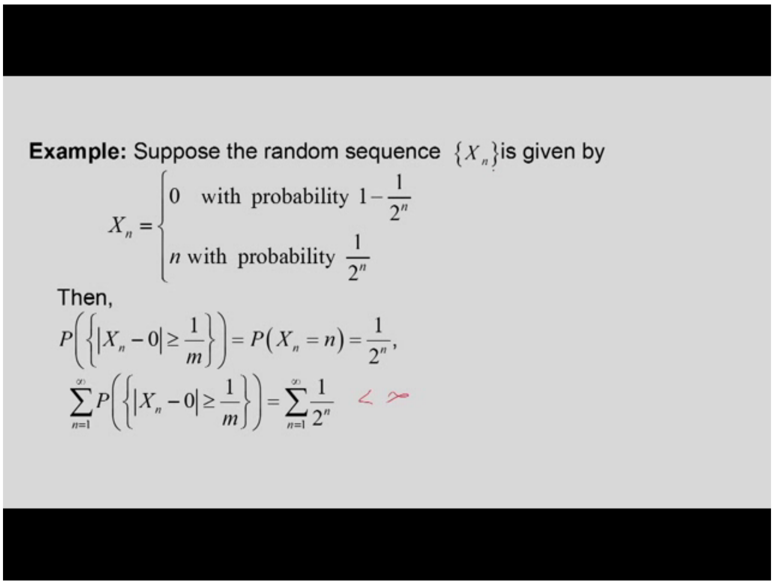

We shall consider one example suppose the random sequence X n is given by X n is equal to 0 with probability 1 minus 2 to the power n and equal to n with probability 1 by 2 to the power n. So, we see that, this, this probability will become 1 as n tends to infinity and correspondingly this probability will become 0. So, therefore, X n approaches 0.

Now, whether this is the limit we get in the almost your sense let us see. So,, we see that probability of X n mod of X n minus 0 greater than equal to 1 by m that is equal to probability of X n is equal to n because for n only it is this quantity is becoming different from 0 here it is exactly 0. Therefore, only n we have to consider probability X n is equal to n that is equal to 1 by 2 to the power n.

So, we have to consider this sum now, sum of these probabilities. So, if we consider sum up from n is equal to 1 to infinity that is equal to sum of 1 by 2 to the power m as n going from 1 to infinity and we know that this sum is less than infinity this is a geometric series we can find out this sum. So, this sum is less than infinity. Therefore, this sequence X n converges almost sure to X. So, that way, we can prove whether a sequence convergence to a random variable X in the almost sure sense.

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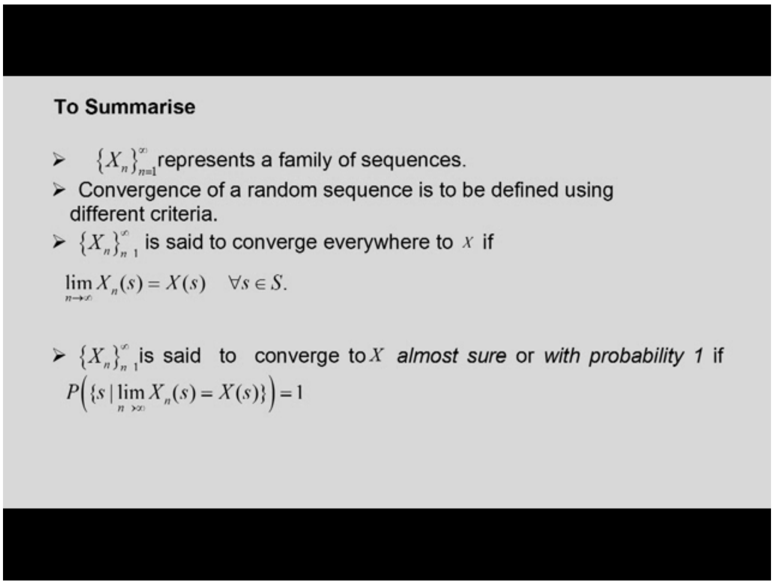

Let us summarize the lecture, recall that this X n this is sequence of random variable, it represents a family of sequence. So, unlike in the case of sequence of real numbers, the sequence of real random variables represents a family of sequence. Convergence of a random sequence is to be defined using different criteria the sequence is said to converge everywhere to X or point wise point, point wise converge to X if limit of X n s is equal to X s for all s belonging to S, that is the convergence everywhere.

Now this sequence X n n going from 1 to infinity is said to converge to X almost sure or with probability1 if probability of this event that is S such that limit of X n s equal to X s that is equal to 1. So, this is the, definition of almost sure convergence which we have introduced here and this is a strong mode of convergence.

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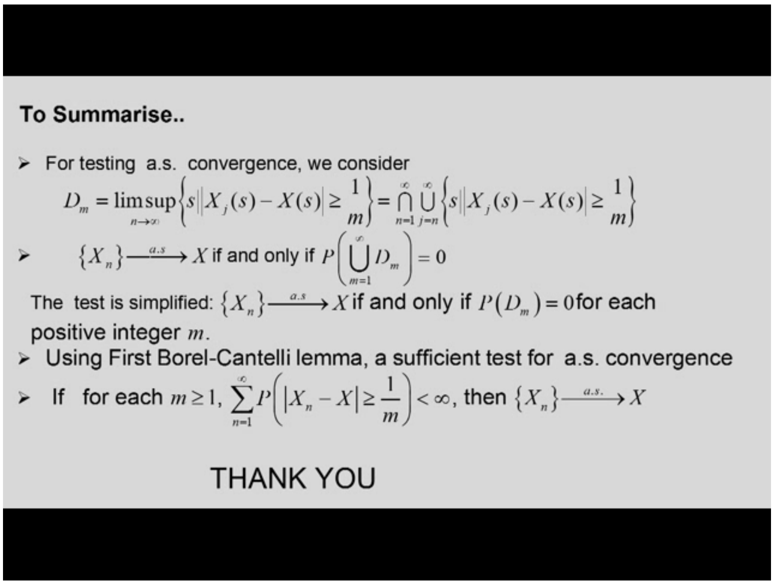

And then, for testing almost sure convergence we consider this event Dm, what is the Dm lim sup of the event s such that X j s minus X s mode of that should be greater than equal to 1 by m. So, lim sup of this event and this is given by this definition of lim sup only we are considering here. Now, we have a necessary and sufficient condition X n converges almost 0 to X if and only if probability of union of Dm m going from 1 to infinity is equal to 0. The test is simplified that is X n converges almost sure to X if and only if probability of D m is equal to 0 for each positive integer m.

Now, using the first Borel - Cantelli lemma, we got a sufficient contest for almost sure convergence if for each m greater than 1, this sum n is equal to 1 to infinity probability of mod of X n minus X greater than 1 by m. So, the, if the sum is less than infinity, then X n sequence X converges almost 0 to X.

Thank you.