# Advanced Topics in Probability and Random Processes
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## Lecture – 06
## Infinite Sequence of Events-II

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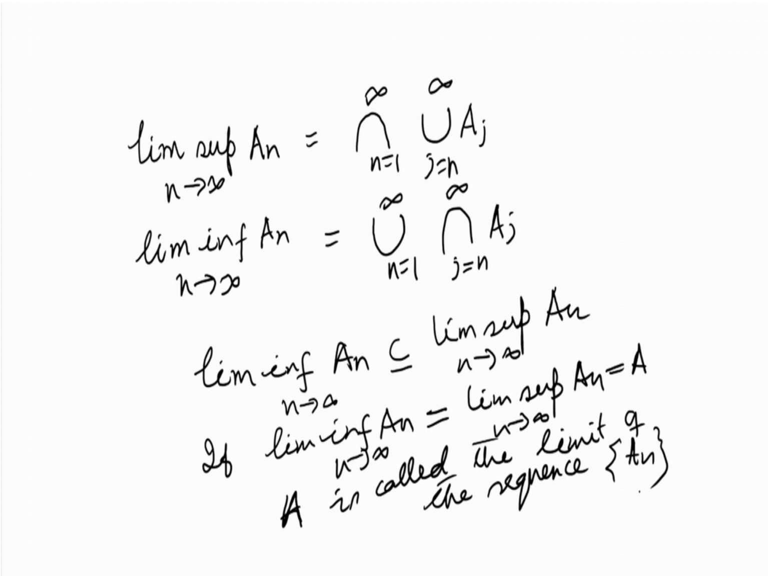


So, we will continue with Infinite Sequence of Events. We talked about two limits lim sup A n that is equal to intersection n is equal to 1 to infinity union A j, j is equal to n to infinity. Similarly, we define lim inf A n, n tends to infinity that is the union of n from n is equal to 1 to infinity of the intersection A j, j going from n to infinity.

So, these two limits are generally not equal and we have discussed that this lim sup lim inf of A n is a subset of lim sup of A n. And if this two limits are equal, then we say that this sequence A n has a limit and that limit is an event suppose and will be given by this lim inf of A n equal to lim sup of A n is equal to A that is if the limit exists lim inf of A n as n tends to infinity is equal to lim sup of A n as n tends to infinity is equal to A suppose, then A is called the limit, A is called the limit of this sequence of events A n ok.

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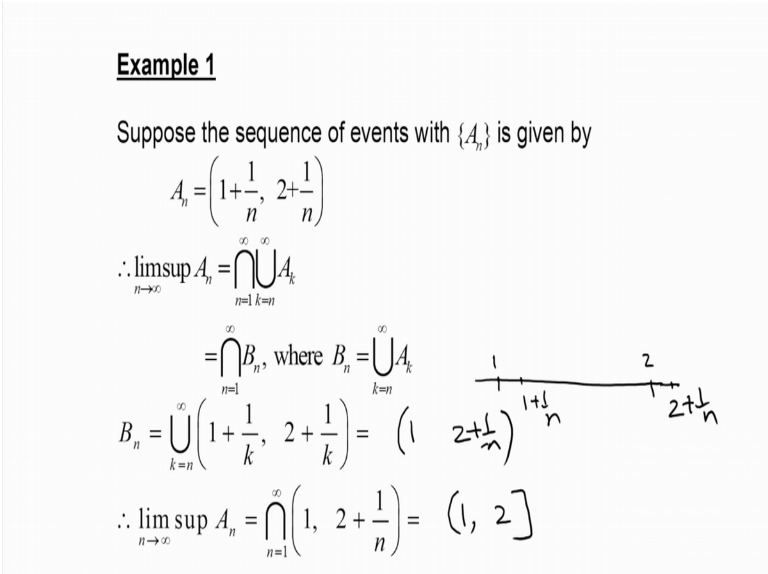


With define lim sup of A n and lim inf of A n; let us see one example. Suppose A n is equal to it is an open interval from 1 plus 1 by n to 2 plus 1 by n. So, first we will find out lim sup of A n. So, let us call this event as B n, then lim sup of A n is the intersection of B n, this B n is a decreasing sequence. So, B n is given by this union k is equal to n to infinity of the interval 1 plus 1 by k 2 plus 1 by k. So, we see that a first element of this interval, suppose, if I consider the sequence. So, B n will be; so, we have defined B n like this.

So, B n; if we carry out suppose first element this and that is one ; this is suppose 1, this is 2, then 1 plus 1 by n. So, this is suppose 1 plus 1 by n and upper limit is 2 plus 1 by n this is 2, this is suppose 2, this is 1, this is 2. So, this is suppose 2 plus 1 by n. Now if I consider the union, then 2 plus 1 by n. So, as n tends to infinity this 2 plus 1 by n will come to the left and this one also 1 plus 1 by n also come to the left. So, that way we can find out this limit, this is equal to 1, suppose if we have as n tends to infinity that interval will be increasing.

So, we will have 1 n 2 plus 1 by n that as the limit open interval. So, now, we have to take the intersection of this sequence of events. So, if I take the intersection, so, it is 1 and then 2 plus 1 by n. So, as and when as increasing as n is increasing, so, this sequence will become smaller and smaller and we will have the limit that is a semi open interval 1

2 with closed closer at right hand side. So, it is a semi interval semi open interval which is closed at the right hand side.

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$$\lim_{n\to\infty} \inf A_n = \bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty} A_j$$

$$C_n = \bigcap_{j=n}^{\infty} A_j$$

$$= \bigcap_{j=n}^{\infty}\left(1+\frac{1}{j},\ 2+\frac{1}{j}\right)$$

$$= \left(1+\frac{1}{n},\ 2\right]$$

$$\therefore \lim_{n\to\infty} \inf A_n = \bigcup_{n=1}^{\infty}\left(1+\frac{1}{n},\ 2\right]$$

$$= (1,\ 2]$$

$$\boxed{\therefore \lim_{n\to\infty} \inf A_n = \lim_{n\to\infty} \sup A_n}$$


So, that way we got the lim sup of A n; similarly, we can find out the lim lim inf of A n. Now this is by definition this is the union of this intersection and this intersection if we call as C n, this intersection we will call as C n. So, then C n will be an increasing sequence; therefore, we can find out limit that way we define the limit.

So, similarly, like the like in the previous case, here also first element will be suppose 1 here, 1 plus 1 by n because it is starting from n and this is the first point suppose first left hand side, right hand side is 2, this is 2, suppose 2 plus 1 by n 2 plus 1 by n, now we will take in the intersection of this sequence. So, for different n; now this will come left hand side.

And this one also will come to the left hand side and if I take the intersection of all the intervals considered, then we will get this semi open interval one 1 by n 1 plus 1 by n 2 with which is closed at the right hand side. And the now we have to find out the union of union of C n. So, if I take the union of these intervals for n is equal to 1 to infinity, I will get again a semi open interval 1 2 which is closed at the right hand side. So, in this case we have said that lim inf is 1 2 interval 1 to 2 and lim sup A n also interval 1 to 2.

So, that the limit exists here limit of this sequence exists and the limit is 1 2 interval 1 to 2. So, we have defined two limits lim sup of A n and lim inf of A n and lim sup of A n as n tends to infinity.

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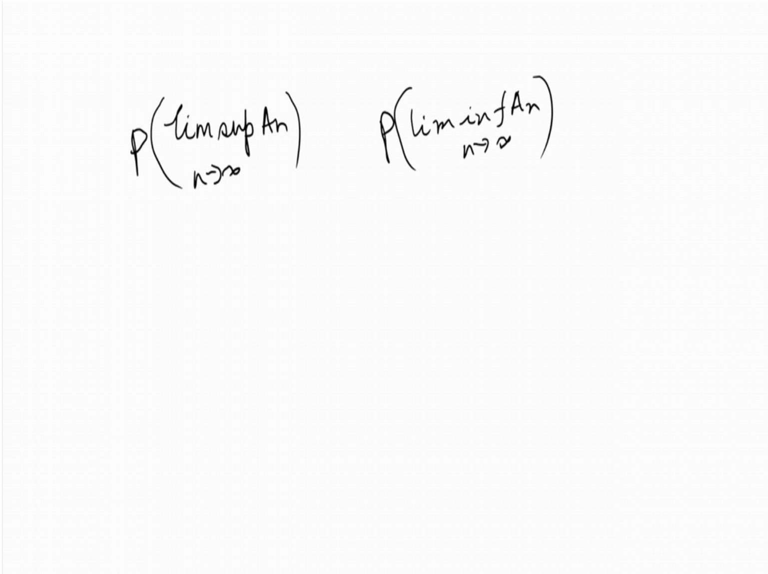

And we have defined two limits lim sup of A n as n tends to infinity and lim inf A n as n tends to infinity. Now we can apply the continuity theorem to find out the probability of these two events, how to find out the probability of this event, suppose and probability of these events. So, that because now these limits are limits of some a monotone sequence, therefore, we can apply the continuity theorem.

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**$P(\limsup_{n\to\infty} A_n)$ and $P(\liminf_{n\to\infty} A_n)$**

Applying the continuity theorem,

$$P(\limsup_{n\to\infty} A_n) = P(\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty} A_j)$$

$$= \lim_{n\to\infty} P(\bigcup_{j=n}^{\infty} A_j)$$

and

$$P(\liminf_{n\to\infty} A_n) = P\left(\bigcup_{n=1}^{\infty}\bigcap_{n=n}^{\infty} A_k\right) = \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} A_k\right)$$

So, we want to find out the probability of lim sup of A n and probability of lim inf of A n. So, we will apply the continuity theorem; so, probability of lim sup of A n that is the probability of this sequence intersection from n is equal to 1 to infinity of the union is A j is equal to n to infinity.

Now, I know that this sequence is a decreasing sequence. So, for the decreasing sequence, we can apply the continuity theorem. So, this limit probability of this limiting event is equal to limit of this probability of union of A j, j equal to n to infinity. So, that way we can find out the lim sup of A n and similarly we can find out probability of lim inf of A n that is by definition probability of union n is equal to 1 to infinity of the intersection A k, k is equal to k is equal to n to infinity.

Now, I know that this event this sequence of event if I consider this, this is an increasing sequence. Therefore, for this also we can apply the continuity theorem and therefore, this limit probability of this limiting event will become limit of the probability of intersection A k k going from n to infinity. So, that way we can find out the probability of the limiting events lim sup of A n and probability of the lim inf of A n as n tends to infinity.

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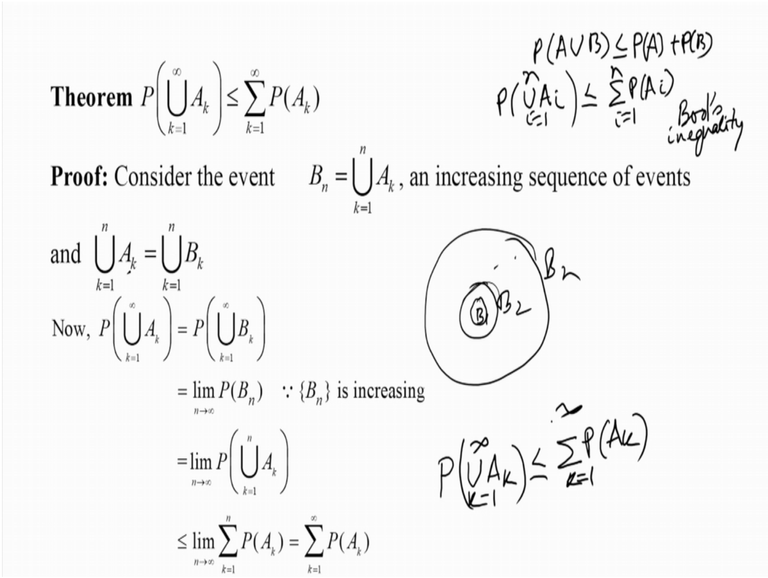


We will establish one important theorem. We know that suppose in a simple case probability of A union B is less than equal to probability of A plus probability of B and A general form of this theorem is probability of; suppose if I consider A union of A i, i going from 1 to n. So, this we can establish through mathematical induction that this is less than equal to summation of P of A i i is equal to 1 to n, this inequality is known as the bulls inequality bulls inequality. So, similar to that now we can establish on inequality for the infinite sequence here it is a finite sequence.

So, we applied the principle of mathematical induction and we can establish this, but how to prove this probability of union A k k going from 1 to infinity is less than the infinite sum of the probability of A k sequence k going from 1 to infinity let us consider the event B n is equal to union A k k going from 1 to infinity 1 to n.

So, it is a sequence in terms of n. So, it is B n is a sequence in it is an increasing sequence because when I consider B of n plus 1 this union will go up to n plus 1 that way it is in increasing sequence of events, we also observe that union of A k k going from one to n this is same as A union of B k because this B n, we have defined as the nested union of nested events. So, that way it is same as union of B k, k going from 1 to A n therefore, if I have to consider the infinity sequence then this infinite union of A k will be same infinite union of B k sequence.

Now, let us find out this probability of union of A k, k going from 1 to infinity. So, this probability is same as the probability of the event union of B k k going from 1 to infinity. Now this is an increasing sequence of event, therefore, we can apply the continuity theorem. So, this will be same as limit of the B n because this B k sequence, it is a like B 1, this is about B 1 B 2 like that; this is a increasing sequence of events suppose, this is B n then this probability of union of A k, k going from 1 to infinity.

This is same as probability of union of B k, k going from 1 to infinity and this is if I apply the continuity theorem this is same as probability limit of the probability B n as n tends to infinity. So, what we got therefore, this left hand side is equal to limit of probability of B n as n tends to infinity. Now probability of B n is same as probability of union of A k, k going from 1 to n. Therefore, this left hand side is same as limit n tends to infinity probability of union of A k, k going from 1 to n

So, now here we can apply the Boole’s inequality. So, we can write less than equal to limit n tends to infinity probability of A k, k is equal to 1 to n. And now this n going to infinity that will come here. So, this is the notation. So, we can write that probability of union of A k k going from 1 to infinity that is less than equal to probability of A k, k going from 1 to infinity. So, what we have establish probability of union of A k, k going from 1 to infinity. This infinite union countably infinite union that is less than equal to the infinite sum of the individual probabilities; k is equal to 1 to infinity.

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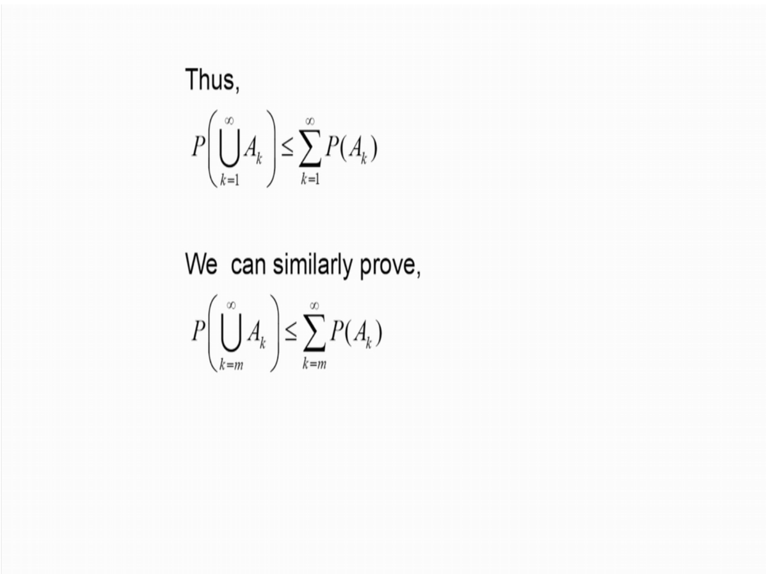

So, this result, we have established. We can also show that similar way, we can show that this result. Here it is k is equal to k going from 1 to infinity, but we can start with any m k equal to m, then this union of this sequence also probability of union of A k, k going from m to infinity that will be less than equal to the infinite sum of the probability sequence probability of A k k going from m to infinity.

So, we established one important result and the corollary of this result is this. So, first result is this that probability of infinite union of any sequence of events is always less than equal to the sum of the that infinite sum of the corresponding probability sequence and the corollary is if we start from k is equal to m, then also this inequality is satisfy.

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**Theorem Borrel Cantelli lemmas: Let $\{A_n\}$ be a sequence of events in $(S, F, P)$.**

(a) If $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P\left(\limsup_{n\to\infty} A_n\right) = P\left(A_n i.o.\right) = 0$

(b) If the events in $\{A_n\}$ are mutually independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P\left(\limsup_{n\to\infty} A_n\right) = P\left(A_n i.o\right) = 1$

Now, we will discuss on important theorem it has two parts and they are known as Borrel Cantelli lemmas, this theorem tells about the probability of this event lim sup of A n as n tends to infinity also we know that it is the event of those sample points which occurs in A and for infinitely many n. So, that we added A n infinitely often now let us state the theorem let A n be a sequence of events in S, P, F; that is probability space S of P. Now if summation P of A n and going from 1 to infinity is less than infinity that is it is convergent this series is convergent then probability of lim sup of A n is equal to 0.

So, if this sum is less than infinity, then this probability will be always equal to 0. Now part B if the events a and r mutually independent this is a requirement for part B events in A n are mutually independent and the sum of this probability of A n, n going from 1 to infinity that diverges that is equal to infinity means at that diverges then probability of lim sup of A n is equal to 1. So, these two lemmas are very important as far as application is concerned.

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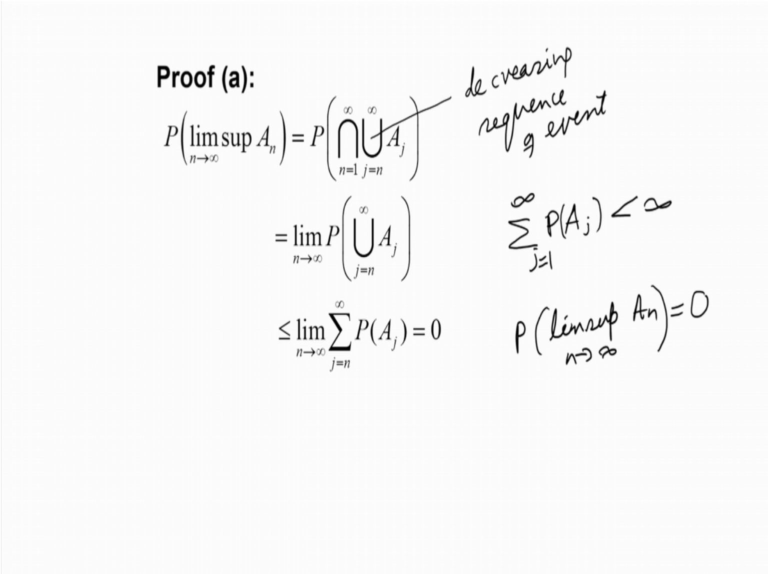


Next we will prove this two lemmas we will first consider part a probability of lim sup of A n as n tends to infinity by definition that is equal to probability of the intersection n going from 1 to infinity of the union of events either they starting with n to infinity.

Now, I know that this sequence union A j, j going from n to infinity this sequence this part is decreasing sequence. So, this is a decreasing sequence of event. So, because of that we can apply the continuity theorem events. So, this is same as probability a limit of the probability of union is A j, j going from n to infinity as n tends to infinity. So, this we got by using the continuity theorem and now we apply the earlier inequality we developed. So, therefore, this is A union this union is less than equal to the sum of the respective pro probabilities as n tends to infinity.

So, limit n tends to infinity of the sum P of A j, j going from n to infinity now we are given that summation of P of A j, j going from 1 to infinity is convergent it is less than infinity. Therefore, if it is a converging series then any tail sum if we consider; so, that tails sum will become 0. Therefore, this sum probability of A j, j going from n to infinity this sum is equal to 0

So, therefore, what we have established that if this happens then probability of lim sup of A n as n tends to infinity. So, probability of these limiting events will be always equal to 0. So, this result this is the part a of Borrel Cantelli lemma.

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**Proof (b):** Suppose $A = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty} A_j$

$$A^C = \left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right)^C = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k^C \qquad \text{De Morgan (}$$

$$\therefore P\left(A^C\right) = P\left(\bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty} A_j^C\right)$$

$$= \lim_{n\to\infty} P\left(\bigcap_{j=n}^{\infty} A_j^C\right) \quad \text{(By the continuity theorem)}$$

$$= \lim_{n\to\infty}\prod_{j=n}^{\infty} P(A_j^C) = \lim_{n\to\infty}\prod_{k=n}^{\infty}(1 - P(A_j)$$

We will prove the part b. So, here we have we have considered sequence of independent events now the event lim sup of A n as n tends to infinity let us call this event as a suppose. So, we want to show that probability of this event P of A is equal to 1.

So, we will consider the complement of this event a complement. So, probability of this complement should be equal to 0 if we can prove that P of ac is equal to 0, then probability of a will be equal to 1. So, we are defining a complement as the complement of this intersection n going from 1 to infinity of the event union of event A k, k going from n to infinity. Now we have to take the complement. So, we can apply De Morgan law. So, we will get that this will be this intersection will become union n union will become intersection and this will be A j complement here also this will be k A k.

So, probability of a complement is probability of union n going from 1 to infinity of the infinite intersection of A j complement j are going from n to infinity. So, this is the probability a complement that is equal to probability of union n going from 1 to infinity and intersection of A j, j going from n to infinity intersection of A j complement j going from n to infinity.

Now, this intersection sequence this is an increasing sequence, therefore, we can apply the continuity theorem again; so, this will be same as limit n tends to infinity probability of this event that is intersection of A j complement j going from n to infinity, we have assumed that the events are independent. Therefore, this probability of the intersection will be equal to the product of the probabilities. So, that way, this probability is equal to limit n tends to infinity of the product probability of A j complement j going from n to infinity.

And now we use the result that probability of A j complement is 1 minus P A j. So, that way we establish the result.

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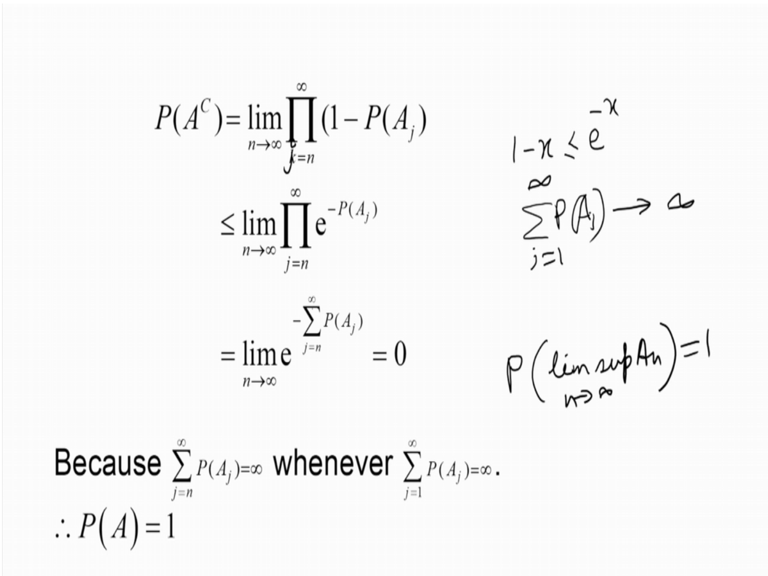


What is what we have established that is probability of a complement is equal to limit of the product of this sequence that is 1 minus P A j, j going from n to infinity. So, this is P A probability of a complement; now we can use one result that 1 minus x 1 minus x is always less than equal to e to the power minus x. So, this we can prove easily.

So, this result we will be using here therefore, what we will get that one minus P A j that will be less than equal to therefore, we can write that this limit is less than equal to limit n tends to infinity a product of the sequence e to the power minus P A j because this is our 1 minus x that is less than e to the power minus x. So, that way this is the product of the sequence e to the power P of A j, j going from n to infinity now this product exponential. So, that way, we can take this sum here. So, limit n tends to infinity e to the power summation of P A j, j going from n to infinity with this minus sign.

Now, again we are given that these sequence this sequence summation of P of A j, j going from 1 to infinity that is diverging sequence. Then if we consider any tail part j going from n to infinity, this sum also will become infinity. Therefore, this will be e to the power minus infinity that will be equal to 0.

So, this is the result we used that is if summation P A j, j is equal to 1 to infinity is equal to infinity, then this any tail some if we consider that will be also infinity. So, we have got P of a complement is 0 therefore, P of a will be equal to 1 ; that means, what we have established probability of lim sup of lim sup of A n as n tends to infinity that is equal to 1 what what is the constraint that this sequence A n sequence must be mutually independent and summation of probability of A j, j is equal to 1 to infinity if this sum diverges then probability of this limiting event is equal to 1.

Therefore we have established the border Borrel Cantelli lemmas it has two part; what are two parts?

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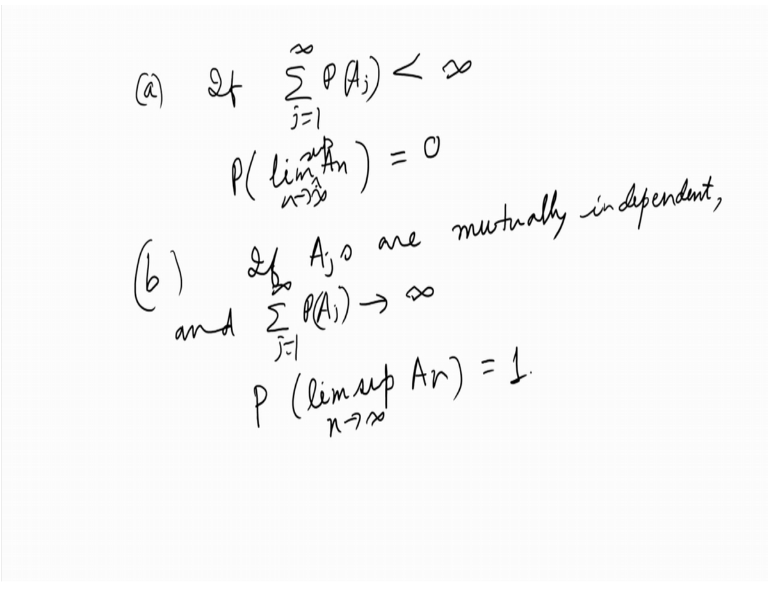


So, let us recapitulate. So, a; first one is that is. So, if summation of P of A j ,j going from 1 to infinity if this is diverging, this is converging then probability of the event lim sup of A n as n tends to infinity that is equal to 0 and part B if A j s are mutually independent.

We know what is mutually independent any combination of events, if we consider they are independent, they are joint probabilities product of the individual probability

So, if A j s are mutually independent and summation of P A j, j going from 1 to infinity if that diverges then probability of lim sup of A n. This is lim sup lim sup of A n as n tends to infinity that is equal to 1. So, these results will be used at the background for the next topics next topic will be considering that is convergence of a sequence of random variables.

Thank you.