Advanced Topics in Probability and Random Processes

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Lecture – 05
Infinite Sequence of Events

Infinite Sequence of Events, in this lecture I will discuss the concepts of limiting events in the probability space and their probabilities. These concepts are essential in understanding the convergence of sequence of random variables; we will start with the continuity theorem in probability.

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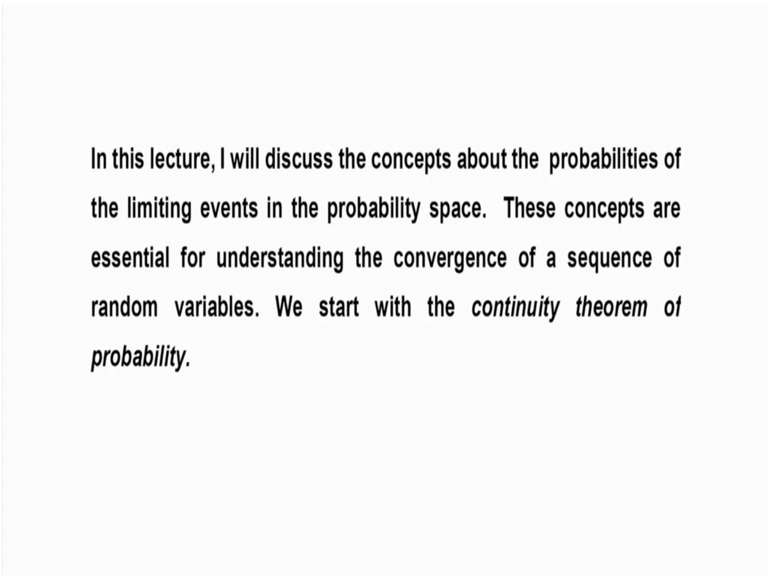

So, therefore we have to discuss about the limiting events in the probability space and how to assign probabilities and this starting point will be the continuity theorem of probability.

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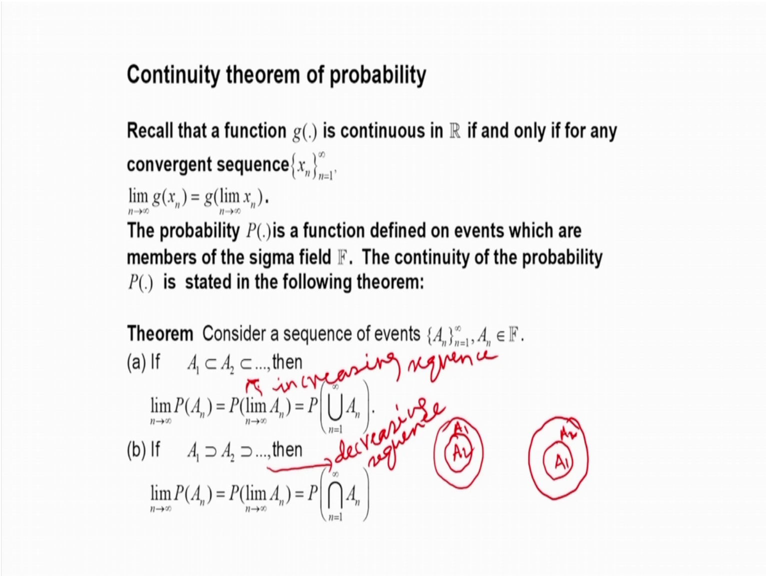

Continuity theorem of probability, first discuss what is continuity of a function. Recall that a function g is continuous in R if and only if for any convergent sequence x n a limit of g x n as n tends to infinity is same as the function g of limit of x n as n tends to infinity. So, limiting value of the function is same as the value of the function at the limiting point. The same concept can be now extended to probability because probability P is also a function. It is defined on the events. Now, what are events? They are the members of the sigma field F. So, that way we define probability on the sigma field F.

So, in each element of the sigma field is an event on that this probability function is defined. So, the therefore, the concept of continuity of probability P can be now stated in the following theorem. Now, we will consider a sequence of events that is A n starting from n is equal to 1 to infinity, where A n is a member of the sigma field F. First case part A if A 1 is a subset of A 2 is the subset of A 3 like that that is A 1 is an increasing sequence A n is an increasing sequence. So, we start with suppose A 1 this is A 1 which now it is a subset of A 2.

So, this is A 2 like that we have we have a an increasing sequence of event. This is a increasing sequence, this is a increasing sequence now in that case so this is an increasing sequence of event. Now in this case the limit of the probability of A n as n tends to infinity that is limit of the probability function is same as probability of limit of A n as n tends to infinity.

Now, this limiting is that limiting event is same as the union of A n from n is equal to 1 to infinity. So, this is the first part of the theorem. Second part, part b is for the decreasing sequence. So, A 1 is there then A 1 is a superset of A 2 like that. So, this is A 1, this is A 2 like that. So, this is a this sequence is decreasing sequence.

So, we have a decreasing sequence of events. In that case the limiting probability P of A n as n tends to infinity is same as probability of the limiting event limit A n, n tends to infinity and which is equal to probability of this event intersection of all A n's and starting from 1 to infinity. So, this is the continuity theorem we will try to prove this, essentially it says that probability at the limiting point is same as the limit of the probability. So, probability satisfies the continuity notion in function in a function.

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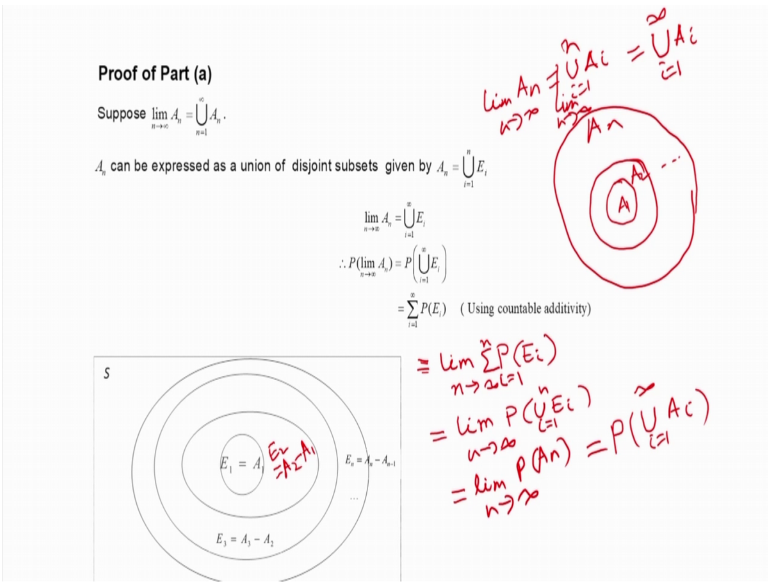

Proof of part a suppose that is limit of A n as n tends to infinity this is equal to union of A n from n is equal to 1 to infinity because, this is a nested sequence of event. So, this is like A 1 A 2 like that and somewhere this way we will get A n. Now that we know that this A n will be equal to union of A i, i is equal to 1 to n. Therefore, if I considered a limit of A n as n tends to infinity that is the limit of this right hand side which will be same as this is if I consider a limit there. So, that will be the union A i, i going from 1 to infinity ok.

So, that way this is the event limiting event this limiting event is given by limit of A n as n tends to infinity is equal to union of A n, n going from 1 to infinity. Now, you refer to this diagram, now A n can be expressed as a union of disjoint subsets. How do I get is that is, suppose if you assume that E 1 is equal to A 2.

Now, this is A 2 therefore, I will define E 2 is equal to suppose this is my E 2, E 2 will be equal to what that is this is my A 2 set difference of A 2 and A 1.So, this is E 2 so now E 1 and E 2 are disjoint. Similarly, this part will be E 3, E 3 will be A 3 minus A 2. So, set difference of A 3 and A 2. So, that way now what we can see there that A n suppose this is my A n, this A n can be considered as the union of the disjoint events E i, i going from 1 to infinity. This is E 1 this is this part is E 2 this part is E3 like that. So, union of the disjoint events E i, i going from 1 to n, that is my A n. Therefore, limit of A n as n tends to infinity that is equal to union of E i, i going from 1 to infinity.

Because, we are considering sigma algebra therefore, these limits whether it is limit here limit here do these uncountable unions limit will exist. Now, we have to find out the probability of these limiting events. So, probability of limit of A n as n tends to infinity, this will be same as probability of this event probability of union of E i, i going from 1 to infinity. Now, probability satisfies the accountability countable additively axiom. What does it mean, if E i's are disjoint then the probability of the countable union is equal to sum of the probabilities.

So, therefore, this is equal to sum of the probabilities P E i, i going from 1 to infinity. Now, this part I can write as the limit what does this upper limit infinity means, it is limit n tends to infinity probability of E i summation is there i is equal to 1. So, this infinity here we can write it as a limit that is equal to limit of P sum of P of E i, i is equal to 1 to n. So, as n tends to infinity. So, this upper limit infinity we are writing it as the limit of n as n tends to infinity. Now this part I can write as because probability because these even E i events are these disjoint.

So, I can write this as limit of union of probability of the union E i, i is equal to 1 to n as n tends to infinity. Now what is this quantity? So, this is same as n therefore, what it will be limit of P of A n as n tends to infinity. So, what we started with that is probability of limit of A n as n tends to infinity that the same as limit of probability of A n as n tends to infinity. So, therefore, this is the continuity theorem that is we are considering a sequence of increasing events A 1, A 2, A 3 etcetera they are increasing events.

So, A 1 is a subset of A 2. A 2 is a subset of A 3 like that we have got an increasing sequence of event. In this case the probability of the limit, probability of limit A n as n tends to infinity is same as limit of the probability of A n as n tends to infinity. Of course, this probability of A n means now, we can write this as limit of because A n is this. So, this is same as probability of union i is equal to 1 to infinity of the event A i ok. So, that way we can define this continuity theorem, we will go to the second part of the theorem.

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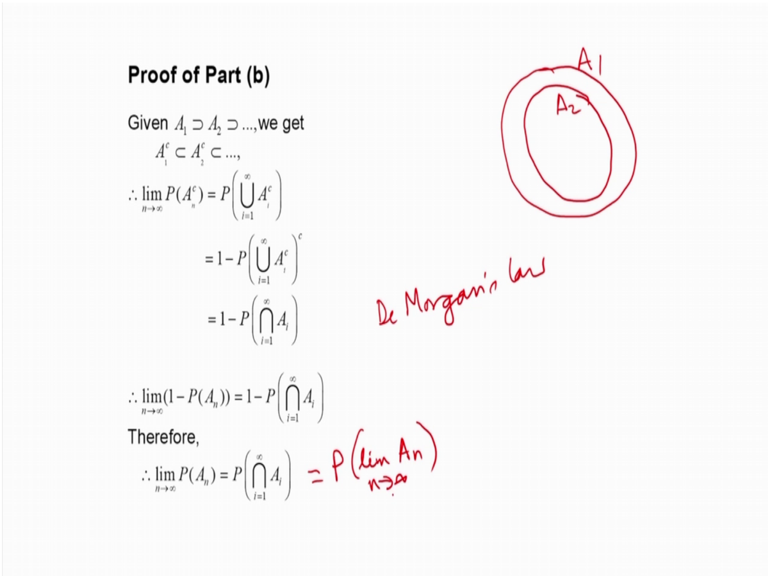

So, second part says that if A i is a superset of A 2 A 2 is a superset of A 3 like that we have it a decreasing sequence. Suppose somewhere here A 1 and somewhere here A 2 like that. So, in that case therefore, if A 1, A 2 etcetera are decreasing events decreasing sets then their complement will be increasing.

So, for example, A 1 complement will be outside this and A 2 complement will be outside this A 2. So, therefore, definitely A 1 complement will be a subset of A 2 complement. So, that way we can get an increasing sequence. Therefore, we can apply the continuity theorem with which will which we derived in the part a, to this case that is the sequence of A i complement. So, what we will get them probability of A n c limit of depth limit as n tends to infinity probability of A n complement.

So, what does it means this A n complement is this is we can write it as probability of union of A i complement from i is equal to 1 to infinity. So, this is the continuity theorem. So, probability of this part A n complement as n tends to infinity is same as the limit in event that A i complement that limit is that event is there that sequence if I considered in limit. So, probability of that limit is same as the limit of the probability.

Now this part I can consider if I consider the complement of this event union A i complement i is equal to 1 to infinity if i consider the complement of this event then 1 minus probability of complement of this event. Now if I apply the De Morgan's theorem De Morgan's law then what I will get law. So, this will be complement will be that is union is here complement is there. So, that that will be intersection and then complement of A i complement will be A i. So, therefore, probability limit of this probability of A n complement is equal to 1 minus probability of this event that is intersection of i is equal to 1 to infinity. So, now, we can apply this is because it is limit probability of A n compliment. So, now this is equal to 1 minus P of A n.

So, therefore, limit of n as n tends to infinity of this event of this quantity 1 minus P of A n is same as 1 minus probability of the intersection of A i, i going from 1 to infinity. So, just I am equating these and this for this I am substituting 1 minus P of A n. Therefore, what we get we get that this is probability of A n as n tends to infinity this limiting probability is same as the probability of this limiting event. Now, this is a decreasing sequence therefore, this is same as probability of limit a n as n tends to infinity.

So, because this intersection of A i this is a decreasing sequence. So, intersection if there are n events is there their intersection will be A n itself and as n tends to infinity we will get the infinite sequence of event. So, intersection of infinite sequence of event is given by limit of A n as n tends to infinity. So, therefore, we established the a second part that is if A i s are a decreasing sequence in that case also the continuity theorem holds. What is the continuity theorem? That is limit of the probability of A n as n tends to infinity is same as the probability of the limiting event limit of A n as n tends to infinity.

So, this is the continuity theorem and for applying the continuity theorem we require the sequence of events to be either increasing or decreasing, but what happens in the case of a general sequence of events that we will be discussing next.

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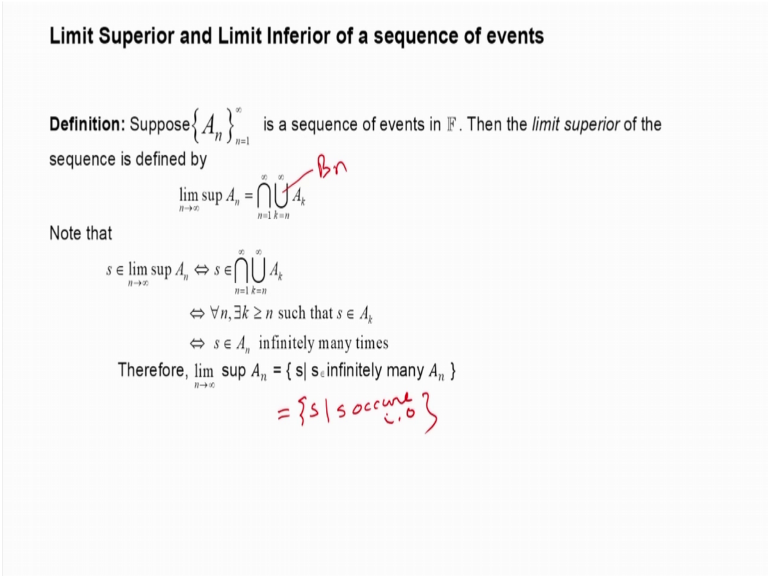

Limit superior we will define limit superior suppose a n n going from 1 to infinity is a sequence of events in F. Then the limit superior of the sequence is defined by that is limit lim sup n tends to infinity of a n lim sup is symbol for limit superior as n tends to infinity. Now, we will consider the sequence this sequence this sequence was our B n. So, B n is a decreasing sequence they are therefore, their intersection in finite intersection is defined and that infinite intersection is known as the limit superior.

So, therefore, limit superior of A n as n tends to infinity that will be intersection n is equal to 1 to infinity of the event union of a k, k going from n to infinity. So, this is the limit superior as n tends to infinity and now let us see what this event imply suppose s belong to lim sup A n as n tends to infinity s is a member of this limiting event, this implies that an implied by that s belongs to this by definition s belongs to this.

Now let us interpret this there is an intersection here there is an union here. So, for all n this intersection means this s will belong to all the events or all these sets. So, for all n and union means there exists at least one k greater than equal to n such that s belongs to A k. So, we will find what is s as if s belongs to this limiting event what does it means this is for all n there exists A k greater than equal to n such that s belongs to A k.

So, therefore, corresponding to all n at least there is one A k where s will be occurring. So, that way how many times s will occur s belongs to A n in infinitely many times because for all n, we are finding an event where s is occurring therefore, this particular s is occurring infinite number of times. So, that way this event that is this lim sup of n as n tends to infinity this is a limiting event what does it imply this is the shape consisting of sample point s such that s belongs to infinite many A n. So, that sometimes we write this as s s occurs infinitely often i o. So, that way also we right write. So, that way we have interpreted the lim sup of A n next we will go to lim.

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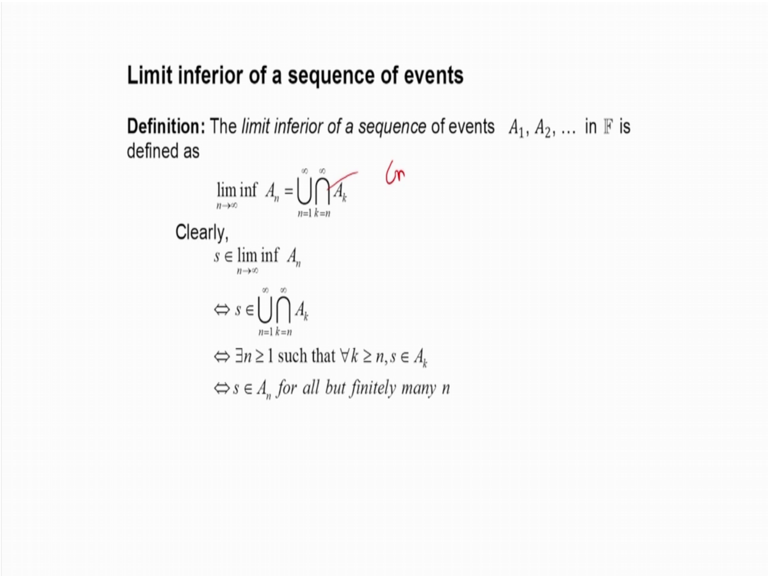

In limit inferior of a sequence of events, the limit inferior of a sequence of event A 1, A 2 etcetera in F is defined as now this is lim inf as n tends to infinity of A n. Now, I know that the sequence intersection of A k, k going from n to infinity this is an increasing sequence. So, this is an increasing sequence this is an increasing nested sequence. Therefore, we can find out the limit that union if I take the union of this sequence then the this sequence is now a monotonic sequence therefore, I can find out the union and that union will be equal to suppose if I consider this union now this union.

We can define problem we can apply the continuity theorem let us just I will start again the limit inferior of a sequence of events A 1, A 2 etcetera in F is define as lim in F lim inf A n as n tends to infinity is equal to that is union n is equal to 1 to infinity of and then intersection A k, k is equal to n to infinity. So, we have considered this event as C n. So, this is the C n is a now it is an increasing sequence and we consider the infinite or union of this increasing sequence that is the lim inf of A n as n tends to infinity.

So, this is the union of this intersection. So, you consider the intersection from k is equal to n to infinity of all even A k and then of those intersection you consider the A union starting with an integral to 1 to infinity. Now what does it mean? What does this limit inferior means? This is also a limiting event, what does it mean? Suppose s belong to lim inf of A n, this implies that and implied by what does it say because, it is a definition s belongs to union from n is equal to 1 to infinity of intersection k is equal to n to infinity of the event A k. So, this is intersection of the event A k, k is equal to n to infinity and then for all n you are considering n taking the union.

So, this is the lim inf. So, therefore, if s belongs to lim inf of A n as n tends to infinity, then, what does it means? They are exist some n greater than equal to 1 at least one n greater than equal to 1 such that for all k greater than equal to n s belongs to A k. So, there is an n; so, there is an n. So, because it is a union they are exist in n and at least one n, there exists at least one n greater than equal to 1 such that for all these subsequent values of k s will be a part of A k.

So, therefore s will occur how many times? Because there if there is an n and subsequently for k greater than equal to n, s will be occurring in all the events; so, how many events it will be occurring? Because, it is for all k greater than equal to n therefore, it will occur infinitely many. But how many, but they are existent n after that only it will occur. Therefore, s will occur in A n for all but finitely many n; you have to consider a screwed thumb n up to where it is not occurring.

Because, we have to consider their existence n greater than equal to 1; this n maybe 1 this n maybe 2, etcetera so, that way as may not occur in some finite number of events, but subsequently it will occur for all events. So, that way this lim inf is also a limit, what does it say? It represent the event for which s occurs in A n for all bar finitely many n. May be first few event it may not occur, but rest of the event it will occur in all. So, that way we define the limit inferior, we observe that in the case of limit inferior also that s will occur infinitely many times.

Because it may not occur finitely many events in faintly many events, but subsequently it occurs. Therefore, it occurs in infinitely many events therefore, next theorem we get a theorem will not prove it here.

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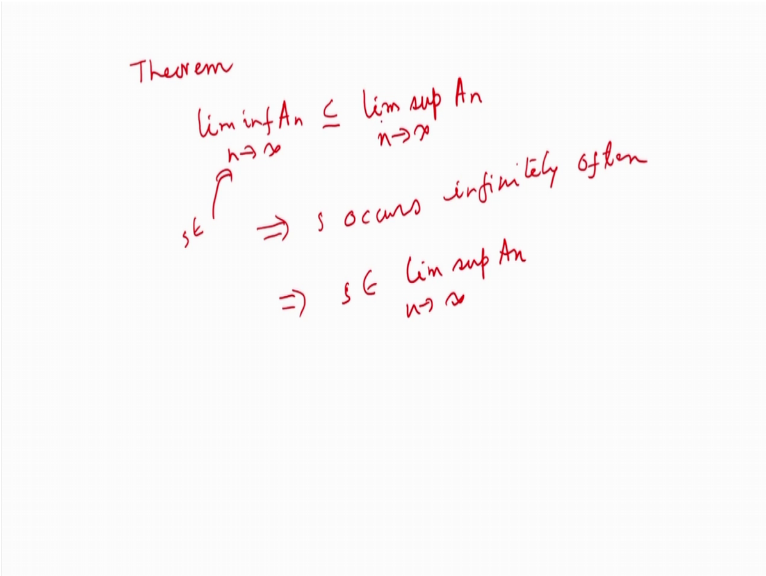

But I will state the theorem lim inf A n as n tends to infinity. So, this event is a smaller element it is a subset of the event lim sup n tends to infinity of A n. So; that means, if s belongs to this if s belongs to this event then it implies that s occurs infinitely often and therefore, I imply that s belongs to only s belong to lim sup A n as n tends to infinity. So, this implies that lim inf of a n is a subset of lim sup of A n. Now, if I have to find out the probability of this event, probability of this event will be less than equal to the probability of this event.

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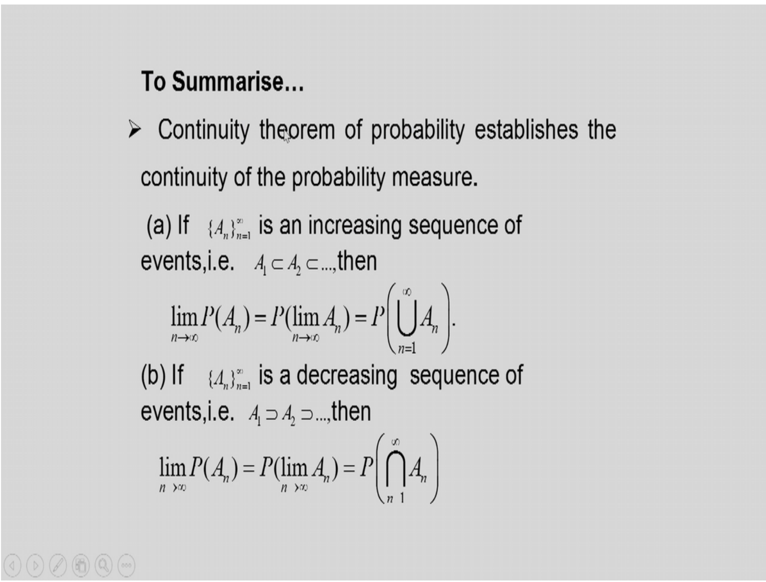

Let us summarize the lecture continuity theorem of probability establishes the continuity of the probability measure. So, this continuity is of the probability of the sequence of increasing or decreasing events. So, we have two parts that is part a if A n is an increasing sequence that is A 1 is a subset of A 2 like that. In that case limit of P A n as n tends to infinity, limit of probability as n tends to infinity is probability of limit of A n as n tends to infinity. So, a limiting probability is equal to probability of the limiting events and in this case this limiting event is union of A n from n is equal to 1 to infinity.

So, second part is about the decreasing sequence of events that is A 1 is a superset of A 2 is a superset of A 3 like that, in that case limit of P A n limit of the probability that is same as probability of the limiting event A n as n tends to infinity. And, in this case the limiting event is given by intersection of A n from n is equal to 1 to infinity. So, these two results establishes the continuity theorem of probability.

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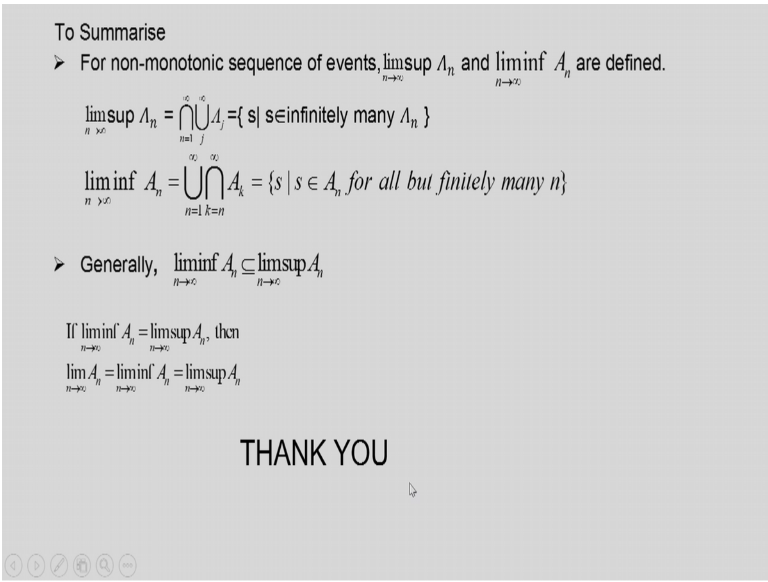

Next we, what happens for non monotonic sequence of events. So, for that we define two limits lim sup of A n as n tends to infinity and lim inf of A n as n tends to infinity. So, we define two limits lim sup of A n and lim inf of A n; lim sup of A n is equal to that is intersection n going from 1 to infinity of the union j going from j is equal to n to infinity A j.

So, this j is j is equal to n to infinity. So, this is same as the set of elements s which occurs infinitely many A n. So, s belongs to infinitely many A n. So, this is the lim sup similarly, we define lim inf A n that is union from n is equal to 1 to infinity of the intersection k going from n to infinity of A k. So, if we interpret this event this means as such that s belongs to A n for all, but finitely many n. So, since it belongs to A k, k is equal to n to infinity.

So, s belongs to A n for all, but finitely many n. We also establish the n equality that is generally lim inf of A n is a subset of lim sup A n, but when these two limits are equal then we can define the limit of this sequence A n that is limit of A n, n tends to infinity is equal to lim inf of A n as n tends to infinity is equal to lim sup of A n as n tends to infinity.

Thank you.