Advanced Topics in Probability and Random Processes
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Lecture – 24
Martingle Process-2

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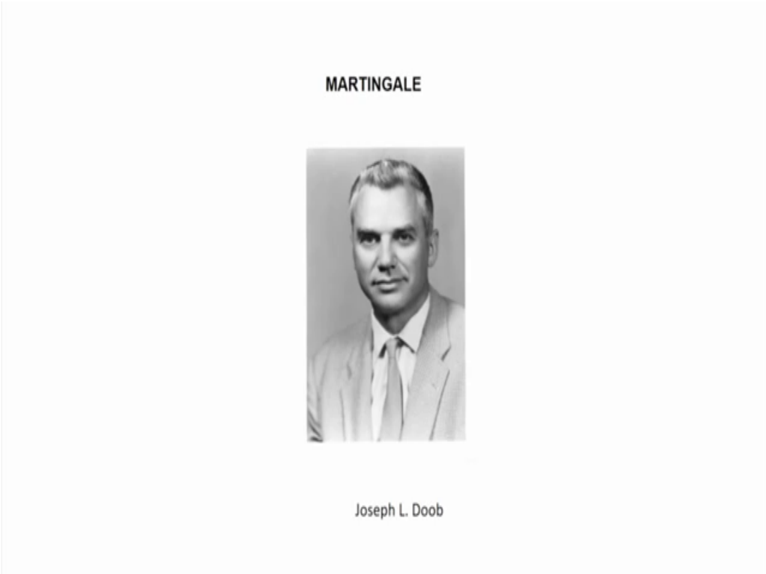

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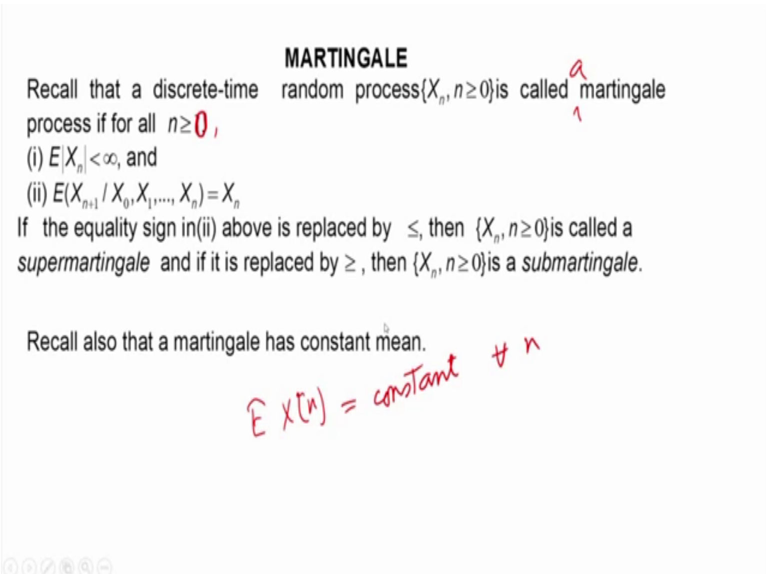

In the last lecture we introduced martingale. Let us recall the definition of martingale or a discrete time random process X n, n greater than equal to 0 is called a martingale process. If for all n greater than equal to 0 E of mode of X n is less than infinity. So that means, average value of the magnitude of X n should be finite. And second thing is conditional expectation of X and plus 1 given X 0 X 1 up to X n is equal to X n. Also we define supermartingale and submartingale. In the case of supermartingale this equal this sign is replaced by less than equal to inequality.

Similarly, in the case of submartingale, this equality sign is replaced by greater than equal to inequality. So, that you know that last class we have just told that it is less than inequality, but it should be less than equal to inequality. Similarly, here also it is greater than equal to inequality. Also proved very important property of martingale that, a martingale has a constant mean; that means, E of X n is equal to constant for all n. So, this is an important property; that means, martingale process is stationary in mean. So, this property is used in deriving 2 important inequalities, other inequalities are also there, but we will not discuss those inequalities, but to basic inequalities we will discuss.

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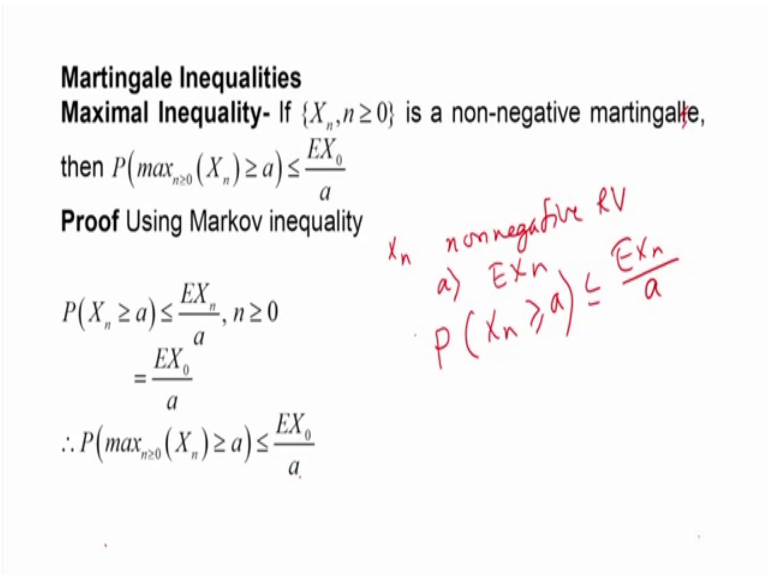

Martingale inequalities, maximal inequality if X n, n greater than equal to 0 is a non negative martingale; then, probability of maximum of X n n greater than equal to 0, greater than equal to a is less than equal to E of X 0 divided by a.

So, X n is a non negative martingale that is important and probability of maximum of X n that, X that is greater than equal to a is bounded by E of X 0 divided by a. So, this is a statement of Markov inequality in the case of martingale process. Here we recall that

Markov in equality stay there if X n is a nonnegative random variable and a is a number greater than E of X n then, detail probability that X n greater than equal to a is bounded by E of X n divided by a. So, that way we get this result probability that X n greater than equal to a is bounded by E of X n divided by a for all n greater than equal to 0.

Now, we know that E of X n is the equal to E of x 0 that is martingale is a constant mean process. So we can write that probability of X n greater than equal to a is less than equal to E of X naught divided by a. Now this is true for all n greater than equal to 0 therefore, it will be true for maximum of action n greater than equal to 0. Therefore, we can write probability of maximum of action and greater than equal to 0, greater than equal to a is bounded by E of X naught divided by a. So, this is known as the maximal inequality.

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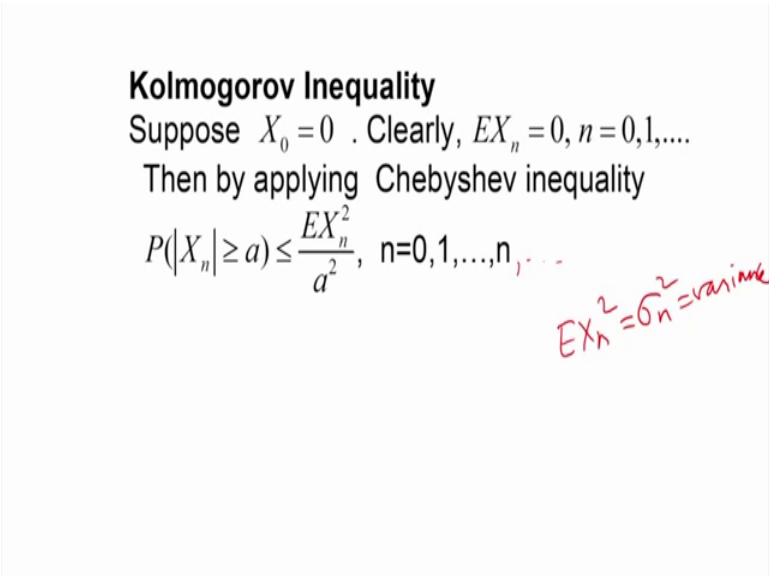

Another inequality is Kolmogorov inequality. Suppose, X 0 is equal to 0 clearly E of X n is equal to 0 because X 0 is equal to 0 then E of X n is equal to 0. Therefore, E of X n square will be the variance. So, E of X n square is equal to sigma square is the variance; sigma m square is the variance. Why, because X 0 is equal to 0 because of that E of X 0 is equal to 0 and E of X n is equal to 0 for all n. Then, by applying Chebyshev inequality now, probability of mod of X n greater than equal to a is less than equal to now, sigma is here in this case E of X n square divided by a square for n is equal to 01 etcetera.

So, this inequality is known as Kolmogorov inequality; sometimes these inequalities are helpful in deriving some important bounds.

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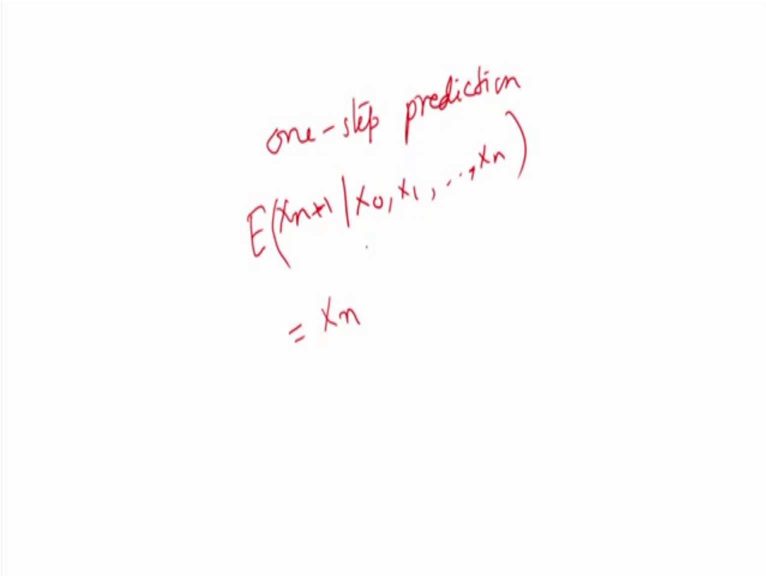

So far we discussed one step prediction. So, what we discussed that E of X n plus 1 given X 0 X 1 up to X n is equal to X n. This is the martingale property.

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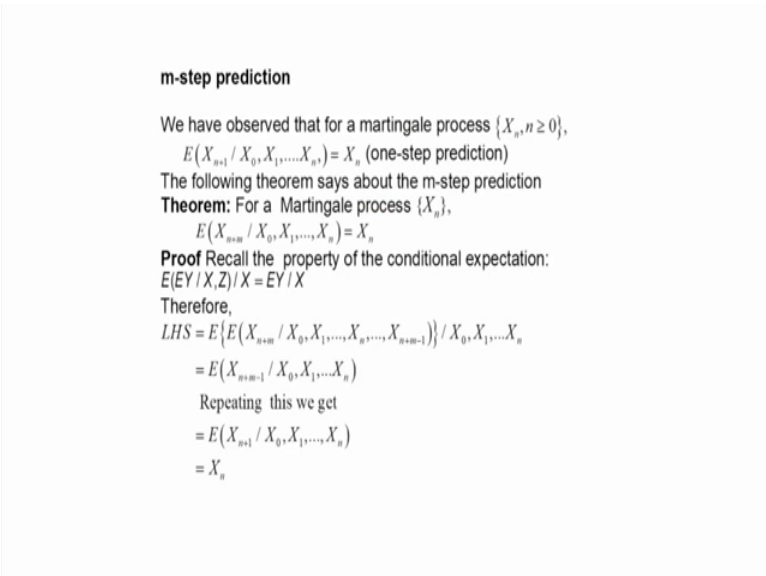

Now, let us see what happened to m step prediction. Suppose, we want to predict beyond X of n plus 1 we have one important result. So for a martingale process X n expected value of X of n plus m given X 0 X 1 up to X n is equal to X n for a martingale process X n. So, m step prediction is also equal to X n. So, to prove this, let us apply this theorem to this important result here. So, left hand side will be expected value of E of X of n plus

1 given X 0 X 1 up to X n, then we introduced the term X n plus 1 up to X of n M 1 plus 1. So, like that here we are introducing new terms given X 0 X 1 up to X n.

Because, we are interested to find out this quantity therefore, we are again taking the conditional expectation with respect to X 0 X 1 up to X n. This is same as E of X of n plus m minus one given X 0 X 1 up to X n.

So, left hand side, this is the quantity. So, we introduce additional members here and now we introduce the martingale property. So, if I have to predict this conditional expectation of X n plus m giving X n plus M minus 1 then, this will be same as X of n plus m minus 1 given X 0 X 1 up to X n.

So, that way what we have shown that, this quantity, conditional expectation of X of n plus m given X 0 X 1 up to X n is equal to conditional expectation of X of n plus M minus 1 given X 0 X 1 up to X n. Again we can introduce term up to suppose X of n plus 1, X of n plus 2 etcetera up to X of n plus m minus 2. Then we can so apply this theorem and we can show that this expression is same as E of X of n plus m minus 2 given X 0 X 1 up to X n.

Repeating in this manner we can show that this is equal to ultimately, it will come down to E of X n plus 1 given X 0 X 1 up to X n and that is equal to X n. So, that way we have proved the remarkable property that m step prediction for the martingale process is also equal to X n and this result gives some additional important properties of the martingale process that we will be discussing next.

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Corollary 1 For a Martingale process $\{X_n\}$,

$E(X_n X_{n+m}) = EX_n^2, m \geq 0$

EEX|Y = EX

Proof

$$
\begin{aligned}
E(X_n X_{n+m}) &= EE(X_n X_{n+m} / X_0, X_1, \ldots, X_n) \\
&= EX_n E(X_{n+m} / X_0, X_1, \ldots, X_n) \\
&= EX_n X_n \\
&= EX_n^2
\end{aligned}
$$

First we will proved at for a martingale process E of X n into X of n plus m is equal to E of X n square. So, now we use the again property of conditional expectation that E of X given Y is equal to E of X. So, this is the property. This property also we proved therefore, here we can write that this is the E of X n into X of n plus m is equal to expectation of the conditional expectation of X n into X of n plus m given X 0 X 1 up to X n. So, first we will take the conditional expectation then, since it will be a function of these quantities, we will take expectation again then we will get the expectation of X n into X of n plus m.

Now, this is conditional expectation with respect to X n. Therefore, this term will result X n itself, given X n this expectation will be X n only. So, we will having only E of X of n plus m, given X 0 X 1 up to X n. So, that way, what we will have now? Now this quantity again I know that this is equal to X n. Therefore, we will have e of X n into X n that is equal to E of X n squared.

So, what we have is that, this is in fact, autocorrelation function E` of X n into X of n plus m is equal to E of X n squared. So, this is one important property of the martingale process.

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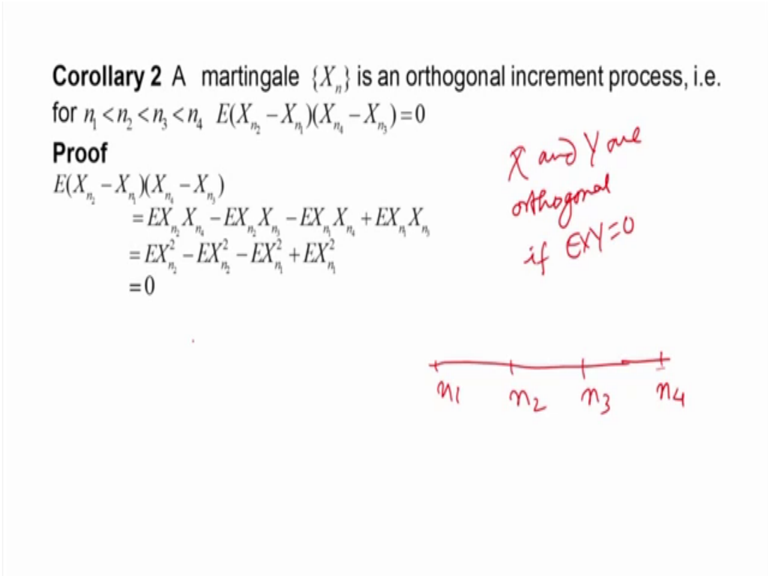

Next property is a martingale X n is an orthogonal increment process. So, what is orthogonal process? Suppose, first of all orthogonal random variable, X and Y are orthogonal, X and Y are orthogonal, if E of X Y is equal to 0. Now, martingale is an orthogonal increment process. What does it say that suppose, we have some this is n 1 n 2 n 3 this is somewhere n 3 suppose this is somewhere n 4, what we have is that this interval and this interval are non overlapping interval. And if you considered a increments in non overlapping interval that is X of n 2 minus X n 1, that is the increment X n 2 minus X n 1 and similarly, increment here is X n 4 minus X n 3. These intervals are non overlapping therefore, the expectation of the increments X of n 2 minus X n one into X of n 4 minus X n 3 that must be equal to 0. This also can be easily proved.

Now, how we can prove, we will use the earlier result that is equal to suppose, first I will multiplied by X n 2 into X n 4 similarly, minus E of X n 2 into x of n 3. Now, if I consider this E of X n 1 into X of n 4 plus E of X n 1 into X of n 3. So, that way this we expand, now we know what is E of X n 1 into X of n 4 because, here n 4 is greater than n 2 therefore, it will be E of X n 2 square. Similarly, this quantity also will be E of X n 2 square so; this and this will get cancel. This quantity will be E of X n 1 squared and this is plus E of X n 1 squared.

So, what we will have is that, this is equal to 0. So, what we have got is that a martingale process is an orthogonal increment process. Earlier we discussed independent increment process for example, Poisson process is an independent increment process, but a martingale process is an orthogonal increment process.

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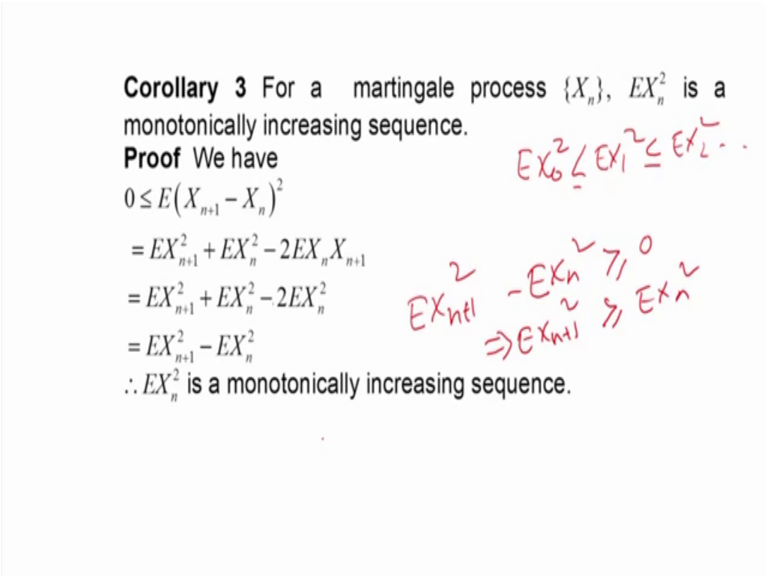

We have another corollary, this is also important for a martingale process X n, E of X n square that is mean square value is a monotonically increasing sequence. So, E of X n square it is a deterministic quantity. Now sequence of E of X n square is a monotonically increasing quantity; that means, E of X 0 square is less than equal to E of X 1 squared is less than equal to E of X 2 square like that.

So, to prove this let us consider this quantity, what happen? The difference between E of X of n plus 1 and X n. So, what we get? 0 is less than equal to E of X of n plus 1 minus X n whole square because, whole square we are taking. Therefore, this is a positive quantity, so 0 is less than equal to this. Now, we if we expand it, this will be E of X n plus 1 square E of X plus E of X n square minus twice E of X n into X of n plus 1.

Now, we will apply the property of the martingale that we have derived earlier that is E of X n into X of n plus 1 is equal to E of X n square. So, that way E of X n plus 1 square plus E of X n square minus twice E of X n squared. Now, this minus 2 n plus 1 will get minus E of X n square. So, what we have established that, this quantity E of X n plus 1 square minus E of X n square is greater than equal to 0 because, 0 is less than equal to here and this will imply that E of X n plus 1 squared is greater than equal to E of X n square. Therefore, what it says that E of X n square is a monotonically increasing sequence.

So, this is important that, the mean squared value away martingale process is monotonically increasing with respect to n. Through mean square value away a martingale process is monotonically increasing with respect to n.

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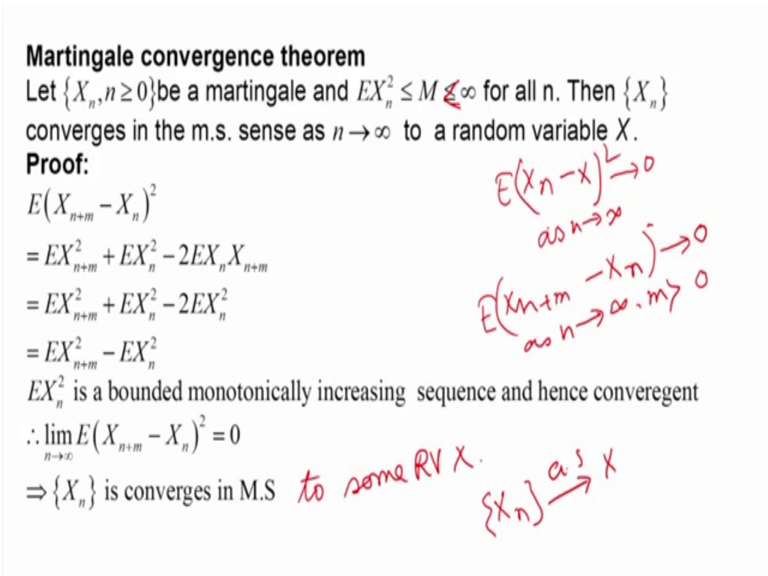

Now, we will come to the martingale convergence theorem, very important theorem, it is important in establishing many results in various areas science and engineering. Let X n n greater than or equal to 0 be a martingale and E of X n square is bounded, so it is a less than equal to m where m is less than infinity, m is less than infinity for all n. Then X n converges in the mean square sense as n to infinity to a random variable x.

So, X n converges in mean square sense to a random variable X. So, what we want to essentially we want to prove that, E of X n minus X whole square that converges to 0as n tends to infinity, as n tends to infinity. So, to prove this, we will use the quasi criterion for mean square convergence. So, what it says that, if you can show that E of X n plus m minus X n whole squared. So, if this quantity goes down to 0, as n tends to infinity and m greater than 0 for all m greater than 0 then, this sequence X n will be mean square convergence. This is a if an only if criterion. So, here without knowing the random variable X we can prove that, this sequence is convergence; this is the quasi convergence criterion for mean square convergence. We will prove this, E of X of n plus m minus X n whole square is equal to E of X of n plus m squared plus E of X n squared minus twice E of X n into X of n plus m and that is equal to E of X n plus m square plus E of X n square minus.

Now, this quantity is e of X n square so, that way we have E of X n X of n plus m square minus E of X n square. Now is of this quantity is a bounded quantity and monotonically increasing quantity. Because, bounded because of this and we know that E of X n square is a monotonically increasing sequence. Therefore, both will converge to some limit and because of that this quantity will become 0 as n tends to infinity, which implied that E of x n plus m minus X n whole square is equal to 0. Therefore, X n converges in mean square sense to sum random variable X random variable X.

So, this is a very powerful result. In fact, it can be shown that a martingales bounded martingale sequence, that is if E of n square is bounded. Then a martingale sequence convergence not only in mean square sense, but it converges almost here also. So that means, this X n converges almost shear to some random variable X, but we will not prove this, but this is a very powerful result. And therefore, if it is a martingale sequence and E of X n square is bounded by m. Then, this martingale sequence will converge in almost shear sense in mean square sense in probability sense to a random variable X. So, this is the martingale convergence theorem.

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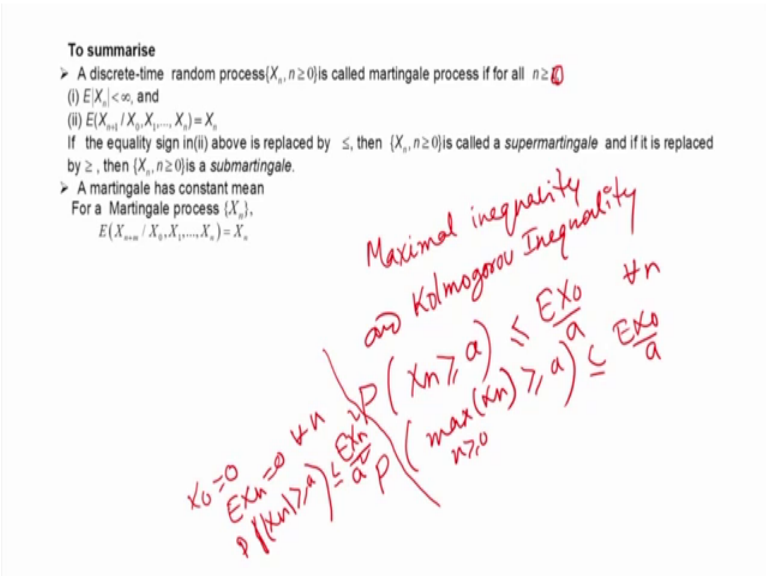

Let us summarize the lecture, a discrete time random process X n, n greater than equal to 0 is called a martingale process, if for all n greater than equal to 0, E of mod of X n is finite and conditional expectation of X of n plus 1 given X 0 X 1 up to X n is equal to X n itself is the present value itself. And also we remembered at if we replace this equality by less than equal to inequality then, X n is a supermartingale and if it is replaced by greater than n greater than equal to n equality, then X n will be a submartingale.

So, we also the find the supermartingale and submartingale. And a martingale has a constant mean. So, that way proved in the earlier class, but because of that we could establish 2 important inequalities that is that maximal inequality and Kolmogorov inequality. So, what does it say, probability that X n if X n is a non negative process, probability of X n greater than equal to a is less than equal to E of X 0 divided by a. This is for all n therefore, we this is 240 maximum of this quantity, so that way probability of maximum of X n over n greater than equal to 0, greater than equal to a is less than equal to E of X 0 divided by a and the kolmogorov inequality was for martingale which X 0 is equal to 0.

So, Kolmogorov inequality was for a martingale process, which x 0 is equal to 0 and because of that, E of X n is equal to 0 for all n for all n and therefore, probability of mod of X n greater than equal to a. So, this probability is less than equal to E of X n square divided by a square. So, this is the Kolmogorov inequality. First, we derived the maximal inequality which is an extension of Markov inequality and the Kolmogorov inequality is a special case of Chebyshev inequality.

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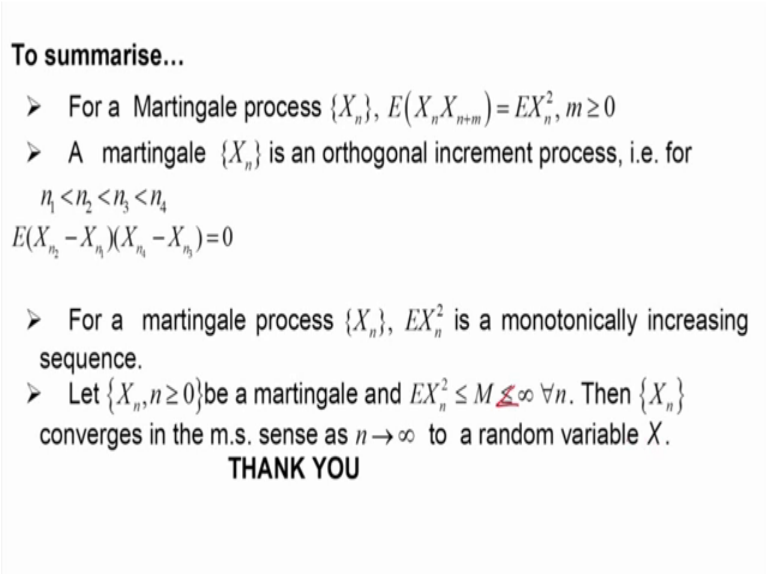

Ah then we establish one important result basically, first we consider a m step prediction, so this is another important result for a martingale process X n E of X of n plus m given X 0 X 1 up to X n is equal to X n that is m step prediction is also equal to present value.

So, this result resulted into several important results, first of all E of X n into X of n plus m for m greater than equal to 0 is equal to E of X n square. Then, we establish the that martingale is an orthogonal increment process, that is for n 1 less than n 2 less than n 3 less than n 4 E of X of n 2 minus X of n 1, that is the increment, X of n 4 minus X n 3, that is another increment. So, joint expectation of these increments is equal to 0.

So, that way a martingale process is an orthogonal increment process. For a martingale process X n E of X, X n square is a monotonically increasing sequence. So, this using this result we proved a very important result that is a martingale sequence for which E of X n square is less than equal to m less than infinity then X n converges in mean square sense to a random variable X. So, a martingale sequence with mean square value bounded by some number m converges in ms sense to a random variable X. We also told that this convergence is also in the almost shear sense, but that we did not prove it, but this is a very powerful result.

Thank you.