Advanced Topics in Probability and Random Processes
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Lecture – 23
Martingale Process – 1

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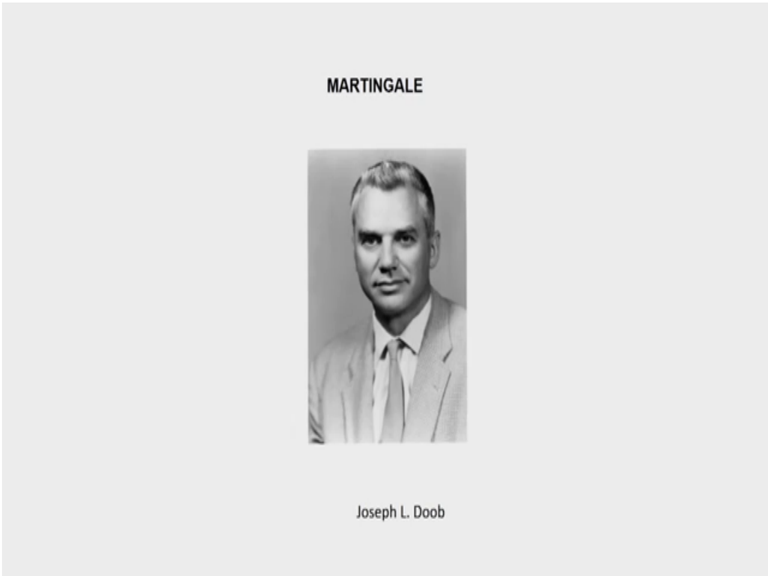

We will talk about Martingale. It is one of the important random processes and one of the pioneer of this process is Joseph L Doob, a famous American mathematician. In the case of Markov chain we exploited the conditional independence between random variables. Martingale uses conditional expectation. So, that is the difference between these two process in the case of Martingale we exploit the conditional expectation between random variables. We will start with the definition of conditional expectation.

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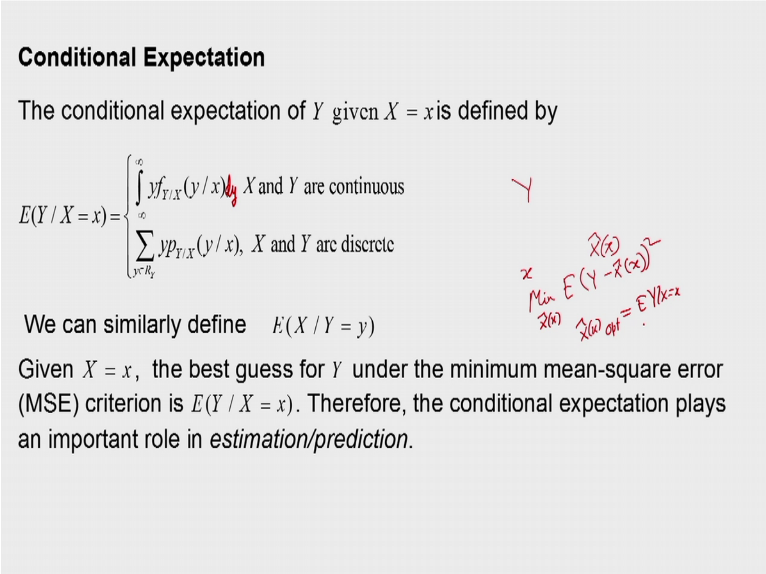

Let us recall the definition. The conditional expectation of a random variable Y given X is equal to small x is defined by this relationship. If X and Y are continuous then in the case of it is integration y conditional PDF of y given x dy. So, this is the definition for a conditional expectation of Y given X is equal to x when X and Y are continuous. Similarly, when X and Y are discrete in that case this is defined by the sum that is yp y x this is the conditional PMF of Y given that X is equal to small x and where y belongs to the lens of y because y they are random variable it will have some range. So, for all those y we have to consider this sum.

So, that way we define the conditional expectation and similarly we can define conditional expectation of X given that Y is equal to small y there is an interpretation of conditional expectation which makes it practically very useful. Let us see this interpretation given X is equal to small x the best guess for y under the minimum mean square error criterion is E of Y given X is equal to small x. Suppose, we have an observation small x and Y is a related random quantity and we want to guess what is the best value for Y. Now, we can get this suppose this guess will be a function of X, so, that way we will consider suppose this is the guess for Y.

Now, what we want to do is that minimize Y minus X set up x. So, this is the mean square error, now this we have to minimize with respect to all the possible function except X. So, minimize this would minimize this function over all possible X set of x.

So, if we carry out the minimization, then we arrive at this relationship that best value of X set x optimum is equal to E of Y given X is equal to small x. So, therefore, if we have observed X then what is the best guess for Y that is the conditional expectation. That is why conditional expectation is very important; and when I plot suppose this conditional expectation as a function of X because it is a function of X this is known as regression of Y on x.

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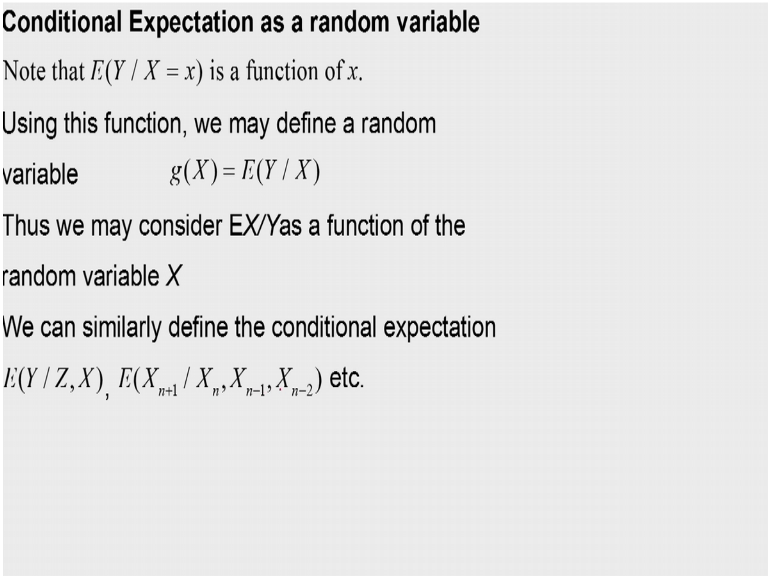

Conditional expectation is a random variable. Note that E of Y given X is equal to x is a function of x. So, therefore, using this function we can define a random variable g X that is equal to E of Y given X when this part is random. So, all possible values of X if we have to consider, then this is a function of a random variable and that to a conditional expectation will be now a random variable in terms of X. We can similarly define conditional expectation here we have defined conditional expectation of Y given X, but we can define conditional expectation of a random variable given multiple random variables.

Suppose E of Y given that X. Similarly, E of X n plus 1 given X n, X n minus 1, X n minus 2 etcetera because this definitions are important for us.

Now, we will discuss one important theorem what is known as the total expectation theorem.

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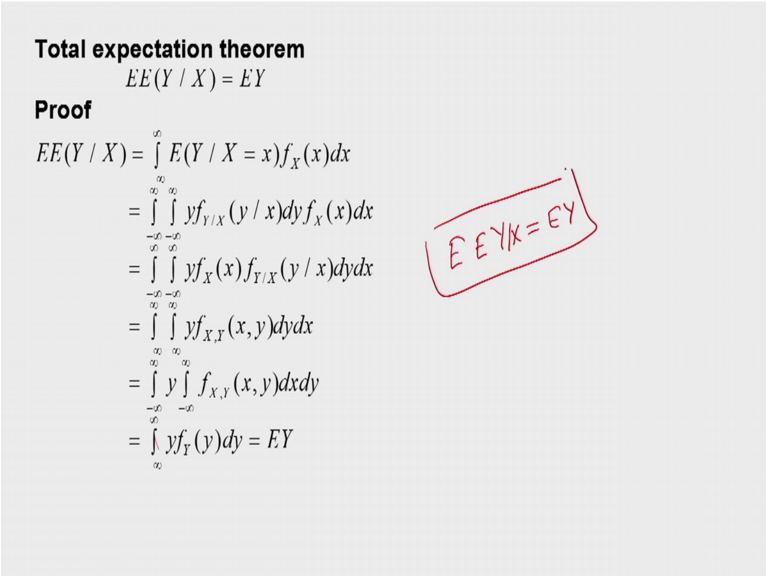

Now, it states that expected value of E of Y given X is equal to E Y because E of Y given X is a function of X. So, if we take the expectation with respect to X then we will get EY. Proof: we will prove this, E of Y given X by definition because that outer expectation is with respect to x therefore, we will write it as integration f x of x dx of E of Y given X is equal to small x.

Now, this quantity we can write as integration from minus infinity to infinity y f of Y given X y at point y dy then this is f x dx. Now, this quantity suppose if I separate out this dy dx now we have the product of f x of x and this conditional PDF. Now, this quantity is nothing, but the joint PDF. So, that way we arrive at double integration of y into joint PDF at x y dy dx. Now, note that this is there is no function of x therefore, we can integrate with respect to dx this quantity and that will give us the marginal density f y of Y and what we are left with y f y of y dy that is equal to integration of depth over minus infinity to infinity and that is equal to E of Y.

So, what we have observed an expected value of conditional expectation of Y given X. So, this is a random variable if I take the expectation of this with respect to X then I will get E of Y. So, this is an one important result.

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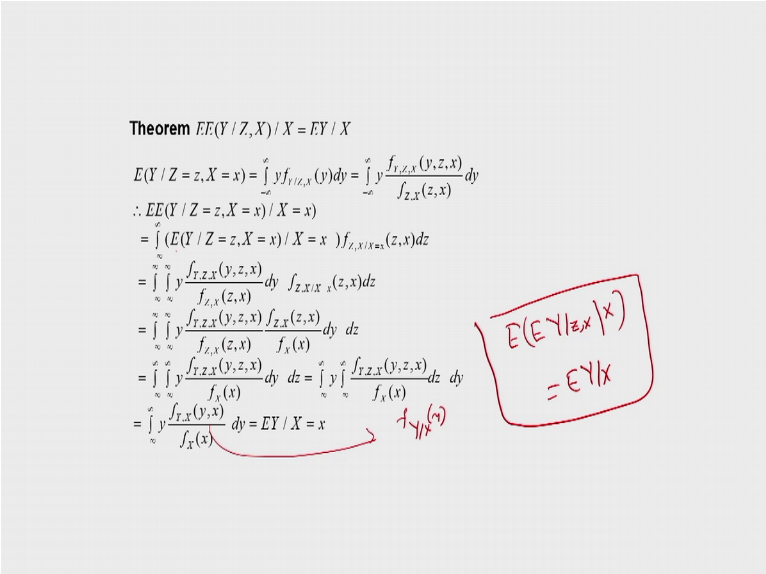

One similar result that we will be using later on, now here conditional expectation with respect to random variable, so, E of again this is conditional expectation conditional expectation of Y given Z, X. So, but this one is conditional expectation with respect to X.

Therefore what we can say that, conditional expectation with respect to X of conditional expectation of Y given Z, X is equal to conditional expectation of Y given X. So, this is the result we will establish then later on we will use this result. Now, let us first E of Y given Z, X. So, for a particular value of z and particular value of x this can be written as the integral minus infinity to infinity y small y and then conditional PDF into dy.

Now, we know that conditional PDF is joint PDF by marginal PDF. So, that way we have this joint PDF f Y, Z, X at point y, z, x divided by the marginal PDF which is nothing, but the joint PDF of z and x.

Now, we will write this expression. So, this conditional expectation now E of Y given Z is equal to small z, X is equal to small x given X is equal to small x. So, what we will get because now it will be conditional expectation with in terms of x. So, that way and this term is a function of Z and X. So, that way now conditional PDF will be that is conditional PDF of Z, X given x is equal to small x that way we can write. So, what we have established. So, this quantity into this conditional PDF and then integrate with respect to z. So, this is the conditional expectation.

Now, we will substitute this quantity that is equal to integration of y this is joint PDF at y, z, x divided by f Z, X of x z, x dy into this conditional PDF. So, we have used this relationship here. Now, we will bring dy this side and then carry out the manipulation we see that f z, x given x is equal to x that we can write as f z, x at z is at point z x divided by f x of x. So, that this and this will get cancel. So, what we will have this term integration y this joint PDF divided by f x of x dy dz.

Again we see that this term there is no other term involving Z therefore, we can integrate this term with respect to z, so that we will get the marginal PDF of f marginal PDF f Y, X. So, that if we integrate this quantity with respect to dz what will left with f Y, Z of f point y, x. So, that is what we have written here divided by f X of x, now only one integral dy. So, that way this is the result and this is and this is nothing, but the conditional PDF this quantity is conditional PDF f of Y given X at point y.

So, if we integrate y into this conditional PDF with respect to dy then we will get E of Y given X is equal to small x, ok. Now, considering this as a function of random variable we will get this result E of conditional expectation of Y given Z, X this given X is equal to E of Y given X. So, this is the result we have established.

So, with this background now we will go to definition of martingale.

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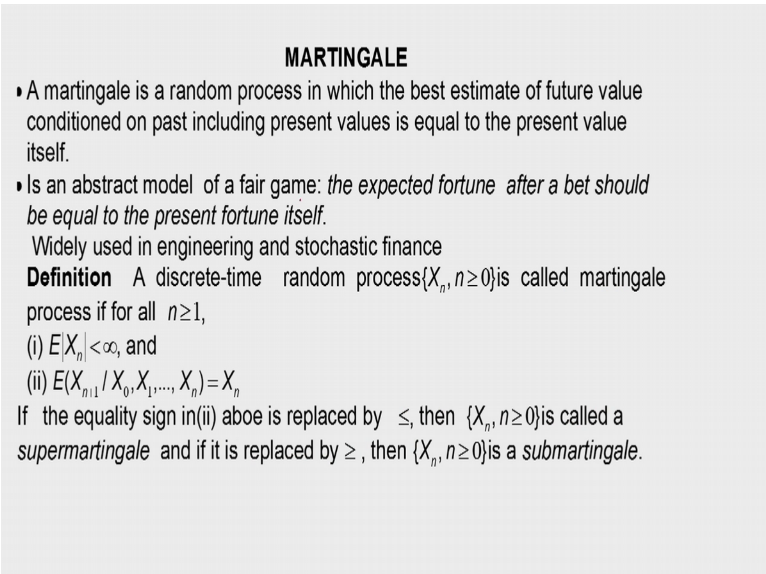

Now, a martingale is a random process in which the best estimate of future value conditioned on past including the present is equal to the present value itself. So, we know about the past values, we know about the present values, what is the best estimate for the future value. So, if the process is martingale then that predicted value is the present value itself. This is an abstract model of a fair game the expected portion after a bet should be equal to the present portion itself. So, that is the model of a fair game because there is a probability of losing and winning half half. So, in that case the expected value expected portion after the bet should be equal to the present portion itself. This martingale is widely used in many engineering applications and particularly stochastic finance etcetera.

So, that way martingale is a very important process and here the future prediction is the present value itself. We will now formally define a martingale process a discrete time random process X n and greater than equal to 0 is called a martingale process if for all n greater than equal to 0. E of mod of X n is finite that is mod of X n has finite average value. Number 2 – the conditional expectation of X n plus 1 given X 0, X 1 up to X n E is equal to X n.

So, this is the important property the conditional expectation of the future value given the past values and the present value is the present value itself. So, E of X n plus 1 given X 0, X 1, up to X n is equal to X n. So, this is the martingale property. Now, if the equality sign here there is a equality sign that it is equal to X n is replaced by less than X n that is less than inequality if we replace this equality sign by less than inequality, then the process X n is called a super martingale.

So, in the case of a supermartingale process, this conditional expectation is less than X n. And if we replace this equality by greater than inequality then this process is called a submartingale. So, we have defined martingale, supermartingale and submartingale. In the case of martingale this is exactly equal to X n in the case of supramarginal it is less than X n and in the case of submartingale it is greater than X n.

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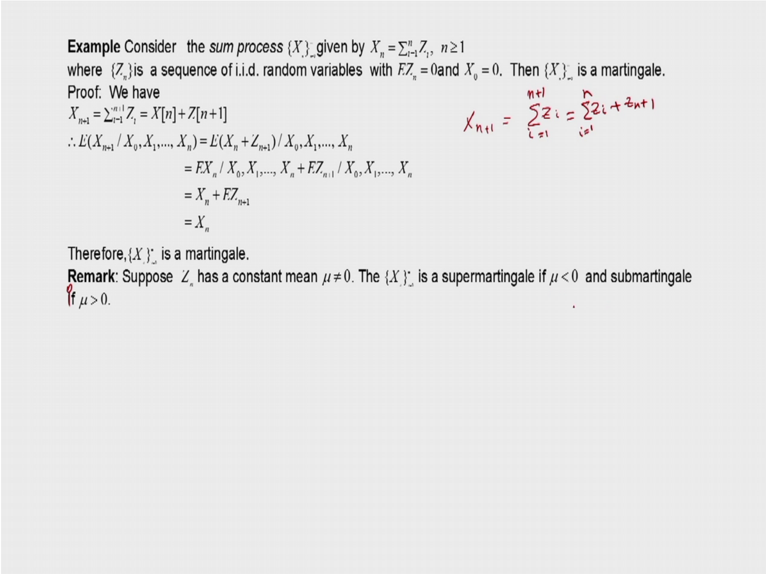

We will consider one example consider the sum process X n; n going from 0 to infinity. So, here we define X n is equal to summation Z i; i is equal to 1 to n for n n greater than equal to 1 and x 0 is equal to suppose 0. Now, we have to prove that this X n is a martingale. So, how do I prove? So, we will prove this. So, we see that X of n plus 1 is equal to summation Z i, i going from 1 to n plus 1. So, this I can write because I can write this as summation up to n then it will become X n plus Z of n plus 1.

So, what we have written X of n plus one is equal to summation Z i, i is equal to 1 to n plus 1 then this is equal to I can write Z i, i is equal to 1 to n plus Z of n plus 1. So, that way I get that X n plus Z of n plus 1. Now, let us find out the conditional expectation of X n plus 1 given X 0, X 1 up to X n. Now, we have a sum therefore, it will be conditional expectation of this sum given this random variables.

So, now first one we can write E of X n given X 0, X 1 up to X n plus E of Z n plus 1 given X 0, X 1 up to X n. Now, in the first case because we are given X n therefore, expected value of X n will be X n itself and second case now this Z n plus 1 is independent of all past values, because this quantity X 0, X 1, X n they are function of Z 0, Z 1 up to Z n. So, in that way Z of n plus 1 is independent of this quantity therefore, we can write it as E of Z n plus 1 and this quantity is 0 E of Z n is given to be 0 therefore, this is equal to X n.

So, therefore, if we have a suppose some process where each increment has 0 mean E of Z n is equal to 0 and they are identically distributed independent and identically distributed in that case this process will be a martingale process. Now here Z n is 0 mean suppose instead of that Z n has a constant mean mu not equal to 0 in that case X n is a supermartingale if mu is less than 0 and submartingale if mu is greater than 0. So, that way in this example itself if Z i's are supposed nonzero mean in that case it can be either submartingale or supermartingale if mean is greater than 0 then it will be a submartingale and if mean is less than 0 it will be a supermartingale.

So, we gave the example of martingale, supermartingale and submartingale.

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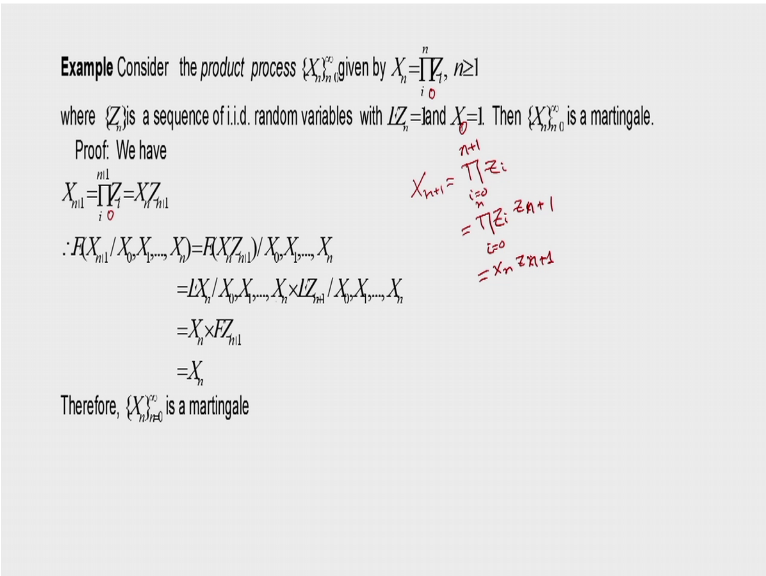

Consider the product process X n and going from 0 to infinity given by we will consider another process X n is equal to product of Z i, i going from 0 to n suppose. Z n is a sequence of IID random variable suppose and we assume that E of Z n is equal to 1. So, each sub Z n has the same expected value 1, nonzero expected value 1 and X 0 is equal to 1.

So, in that case we can write X of n plus 1 that is equal to product of i from i is equal to 0 to n plus 1 then I can write similarly like in the previous case X of n plus 1 is equal to product of Z i, i going from 0 to n plus 1 that I can write as product of i going from 0 to n Z i into Z i. Now, this is my X n and this is Z i. So, that a Z of n plus i. Now, the Z of n plus 1 this is Z of n plus 1. So, if I take the conditional expectation now E of X n plus 1 given X 0, X 1 up to X n. Now, that will be conditional expectation of X n into Z of n plus 1 given X 0, X 1 up to X n.

Now, there are two product of two terms and we know that Z of n plus 1 is independent of X n. So, that way we can write E of X n given X 0, X 1 up to X n multiplied by E of Z of n plus 1 given X 0, X 1 up to X n. Now, I know that given X n this quantity will be X n itself into E of Z of n plus 1. Now, Z of n is a sequence which unity mean therefore, this quantity will be 1. So, we will get X n itself. Therefore, this sequence X and product sequence which is a product of IID random variables with unity mean, X n is a sequence of product of random variables with unity mean then this sequence is a martingale.

So, we saw that the sum process as well as this product process are martingale.

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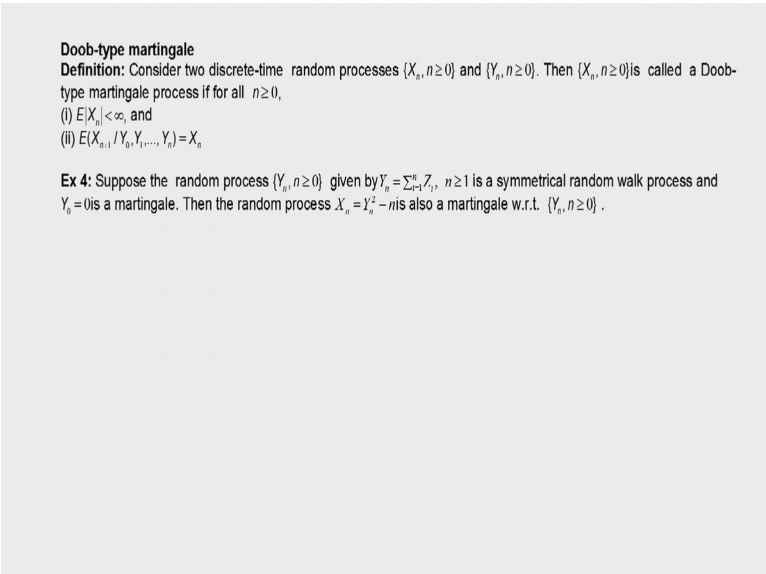

Next we will define what is known as Doob-type martingale. Consider two discrete-time random processes X n; n greater than equal to 0 and Y n; n greater than equal to 0 then this process X n is called a Doob-type martingale process with respect to this auxiliary process Y n; n greater than equal to 0. If for all n greater than 0 this sequence X n sequence is as finite absolute mean E of X n, E of mod of X n is less than infinity and E of X n plus 1 given Y 0, Y 1 up to Y n this is with respect to that Y n process E of X n plus 1 given Y 0, Y 1 etcetera up to Y n is equal to X n itself.

So, this is the definition of Doob-type martingale and now we will give one example suppose the random process Y n is given by Y n is summation Z i is a symmetrical random walk process and Y 0 is equal to 0 that Y n itself is a martingale process. Then the random process X n is equal to Y n square minus n is also a martingale with respect to Y n, n greater than equal to 0. So, now because symmetrical random walk. So, in that case this is a sum process with average of the increment 0 therefore, it is a random walk process. Now, we are considering another process X n which is equal to Y n square minus n. We have to prove that this is a Doob-type martingale with respect to Y n.

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Proof: We have

$$X_{n+1} = Y_{n+1}^2 - (n+1)$$
$$= (Y_n + Z_{n+1})^2 - (n+1)$$
$$= Y_n^2 + Z_{n+1}^2 + 2Y_n Z_{n+1} - (n+1)$$
$$\therefore EX_{n+1} / Y_0, Y_1, \ldots, Y_n$$
$$= E(Y_n^2 + Z_{n+1}^2 + 2Y_n Z_{n+1} - (n+1)) / Y_0, Y_1, \ldots, Y_n$$
$$= Y_n^2 + EZ_{n+1}^2 + 2Y_n EZ_{n+1} - (n+1)$$

(Using indepence of Z_{n+1} with each $Y_i, i = 0,1,\ldots,n$)

$$= Y_n^2 + 1 + 0 - (n+1)$$
$$= Y_n^2 - n$$
$$= X_n$$

Therefore, $\{X_n\}$ is a Doob-typemartingale.

Let us see the proof X of n plus 1 is equal to by definition Y n plus 1 square minus n plus 1 that is equal to now we know Y n plus 1 is Y n plus Z n plus 1 whole square minus n plus 1 minus of n plus 1. Now, this whole square we can we can expand Y n square plus Z n square plus twice Y n into Z n plus 1 minus n plus 1. Now, we have to take the conditional expectation of E of X X n plus 1 given Y 0, Y 1, up to Y n.

So, that way we will get E of Y n square given Y 0, Y 1 up to Y n E of Z n plus 1 square given Y 0, Y 1 up to Y n and like that this contains it twice of E of Y n into Z n plus 1 given Y 0, Y 1 up to Y n and minus this quantity, ok. Now, given Y 0, Y 1, up to Y n E of Y n square will be Y n square itself. Similarly, E of Z 1 plus 1 square because we have considered a symmetrical random walk process which it takes 1 with probability half and minus 1 with probability half. So, that way this E of Z n square Z n plus 1 square is equal to 1 and now Y n is independent of independent of Z n plus 1.

So, that way conditional expectation of the product we can write that it is conditional expectation of Y n into E of Z n plus 1 and conditional expectation of Y n given Y n is Y n itself and E of Z n plus 1 is equal to 0 minus n plus 1. So, that way we will get Y n square minus n that is equal to X n. So, therefore, X n and going from 0 to infinity is a Doob-type martingale. So, we gave an example of Doob-type of martingale.

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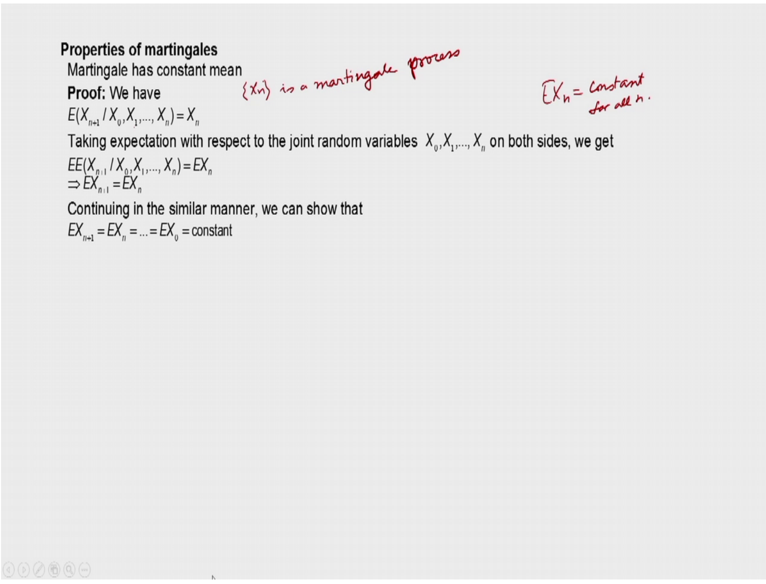

Next, we will discuss properties of a martingale process. Now, there are different interesting properties first we will prove a simple property that martingale has a constant mean that is if X n is a martingale E of X n is equal to constant for all n for all n. How we can prove this? We know that since X n is a martingale process therefore, what we will have E of X n plus 1 given X 0, X 1, up to X n is X n itself.

So, taking the expected value both side now because this is a conditional expectation. Now, if I take the expected value of this, what we will get that is E of this quantity is equal to E of X n right hand side will be E of X n and this left hand side will be the expectation of the conditional expectation.

Now, we discuss about the total expectation theorem. So, according to that theorem this conditional expectation will be E of X n plus itself. Therefore, E of X n plus 1 is equal to

E of X n. So, X of n plus 1 and X n have the same mean. So, continuing in the similar manner we can throw that E of X n plus 1 is equal to E of X n is equal to E of X n plus 1 etcetera up to E of X n, X 0 is equal to a constant. Therefore what we have proved that E of X n is constant for all n for a martingale process X n. So, this is the constant mean property of martingales.

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Let us summarize the class. So, like Markov chain which exploited conditional independence martingale exploits conditional expectation. And, one important result we saw that E of Y given X is equal to X is the minimum mean square estimation of Y given X is equal to small x.

Therefore, this conditional expectation is important because it directly gives the minimum mean square estimation and it is used in prediction. Also we see that conditional expectation of Y given X equal to X is a function of X. Thus it defines a random variable E of Y given X and also we studied the total expectation theorem that is E of Y given X is equal to E of Y.

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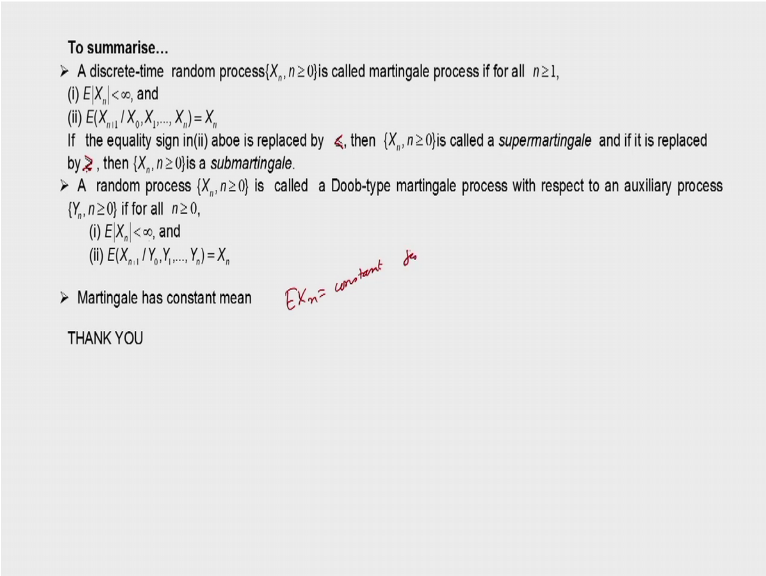

Next we define martingale a discrete time process X n is called a martingale process if for all n greater than equal to n E of mod of X n is finite and E of X n plus 1 given X 0, X 1 up to X n is X n.

So, conditional expectation of future value given values up to present is equal to present value itself if the equality sign is in this expression is replaced by less than inequality then X n is called as supermartingale and if it is replaced by a greater than inequality greater than inequality it is called a submartingale. So, we know the definition of supermartingale and submartingale as well.

Now, we define another type of martingale process what is known as the Doob-type of martingale process. A random process X n n greater than equal to 0 is called a Doob-type martingale process with respect to an auxiliary process Y n; n greater than equal to 0 if for all n E of mod of X n is less than infinity that is finite and conditional expectation of X of n plus 1 given Y 0, Y 1, up to Y n is equal to X n. So, then we started with one interesting property of the martingale process that a martingale has a constant mean what does it say that E of X n is equal to constant for n greater than equal to 0.

Thank you.