Advanced Topics in Probability and Random Processes
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Lecture - 22
Continuous-time Markov Chain

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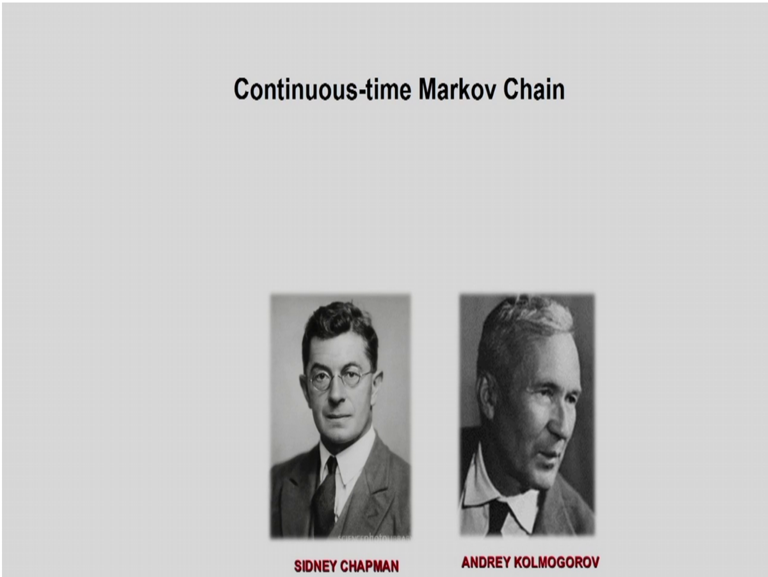

We will continue with continuous time Markov's chain. In the last lecture we discussed birth death process.

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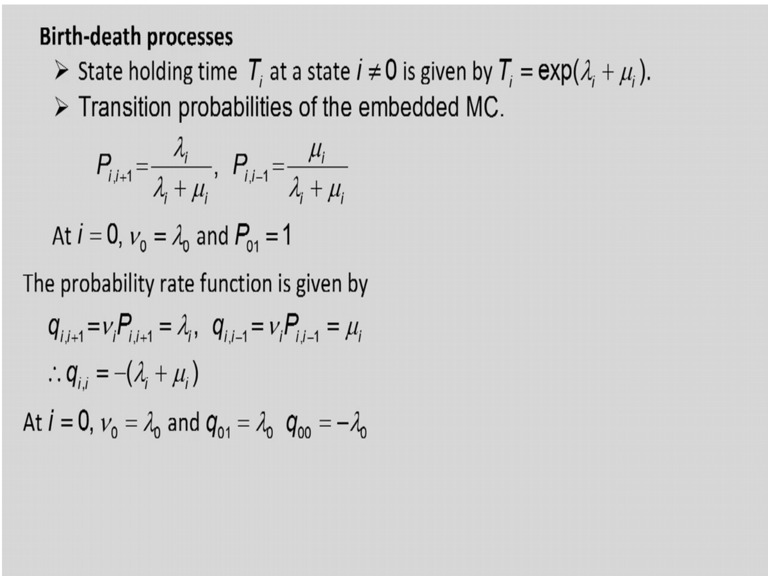

So, we, so, that state holding time t at state i not equal to 0 is given by t i is equal to exponential lambda i plus mu i, that is a exponential distribution with parameter lambda i plus mu i where lambda i is the birth rate mu i is the death rate. And we also found the transition probabilities of the embedded Markov's chain.

So, there are only 2 transition there may be a birth and there may be a death because of death p i plus 1 is lambda i divided lambda i plus mu i. Similarly, p of i i minus 1 is mu i divided by lambda i plus mu i and at state i is equal to 0 there is no death therefore, nu naught is equal to lambda not only. And p 0 capital p 0 one is equal to 1. So, if there is a birth then state will go up by 1. Now, we can compute the probability rate function. So, q i i plus 1 it will be by definition new i into p i i plus 1 that is equal to now, if I put p i i plus 1 into new i; new i is lambda i plus mu i. So, I will get lambda i.

Similarly, q i i minus 1 is equal to new i into p i i minus 1. And if I substitute p i i minus 1 from here and nu i from lambda i plus mu i we get that q i i minus 1 is equal to mu i. So, that way rate at which state goes down by one that is mu i and state goes up by 1 that is equal to lambda i therefore, q i, I know that that is equal to minus nu i that is equal to lambda i minus of lambda i plus mu i. And at state i is equal to 0 nu 0 is equal to lambda 0 therefore, q 0 1 is equal to lambda 0 q 0 0 is equal to minus lambda naught.

So, that way we can get the rate probability for the birth death process.

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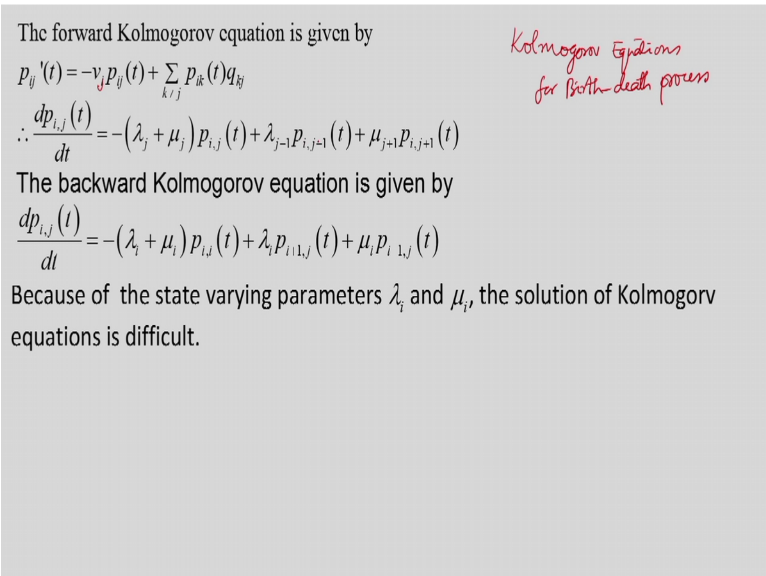

Let us discuss the Kolmogorov equations for birth death process the forward Kolmogorov equation is given by p i j dash t is equal to minus new j p i j t plus summation over k naught equal to j p i k t q k j. This is the forward Kolmogorov equation now for birth death process. There will be transition from state j plus 1 and j minus 1 only. So, that way we can consider this one also we know that nu j is equal to lambda j plus mu j. So, therefore, this derivative if I write it in normal notation d p i j t is equal to minus lambda j plus mu j p i j t plus lambda j minus 1 into p i j minus 1, t plus mu j plus 1 into p i j plus 1 t. So, this is the forward Kolmogorov equation.

Similarly, the backward Kolmogorov equation is given by d p i j t d t that is equal to minus again nu i will be there here, minus lambda i plus mu i into p i I t plus lambda i into p i plus 1 t t mu p i minus 1 j t. This is the backward Kolmogorov equation. So, we know how to derive forward Kolmogorov equation and backward Kolmogorov equation. So, according to that principle we get this 2 equation.

Now, this 2 differential equations apparently they look very simple, but because of the state varying parameter lambda i and mu i. These solution of Kolmogorov equation is difficult. So, you consider either of the equation the solution is difficult because this lambda i mu i are step building parameters, which this step they will be their values may be different.

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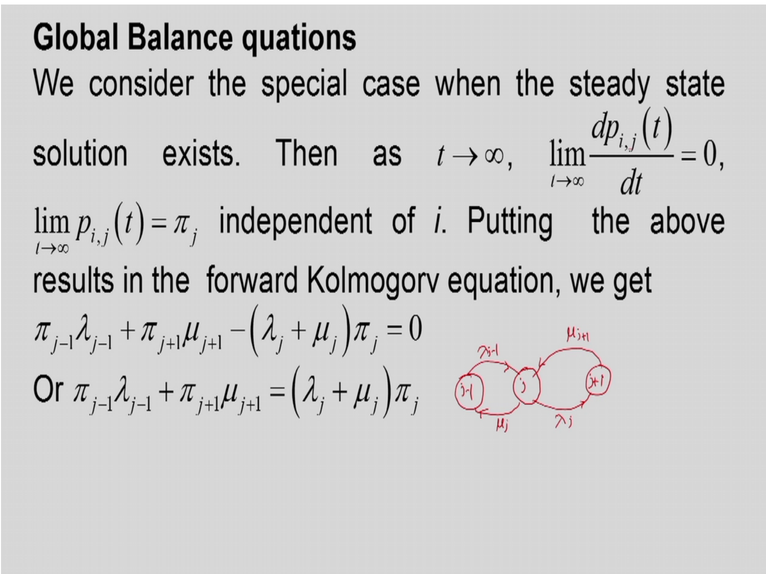

We will consider a special case when this study state solution exists. So, then as t tends to infinity this d p i j t dt because this steady state this will become 0. And p i j t limit of p i j t as t tends to infinity pi j independent of state i. So, that is the special case we will consider and putting the above results in the forward Kolmogorov equation we get a set of equation what is known as the global balance equations.

So, we will get the global balance equations what are these equation that is if I consider the forward Kolmogorov equation. For example, so this is the forward Kolmogorov equation. So, here I will put 0 here. And this is equal to pi j this is pi j minus 1 and this is again equal to pi j plus 1. So, that way I will get a equation that is 0 is equal to minus lambda i plus mu j into pi j pi j lambda j minus 1 into pi j minus 1, plus mu j plus 1 into pi j plus 1. That way we will get a global balance equation that we have written here.

So, pi j minus 1 into lambda j minus 1, plus pi j plus 1 into mu j plus 1 minus lambda j plus mu j into pi j is equal to 0. Or if I write this in the right hand side, pi j minus 1 into lambda j minus 1 plus pi j plus 1 into mu j plus 1 is equal to lambda j plus mu j into pi j. So; that means, if we are in a suppose state j then there is state j minus 1 and there is state j plus 1 j plus 1 j j minus 1. Now what this global balance equation says that suppose this probability study state probability at this state is pi j minus 1. Now, probability of suppose this quantity pi j minus 1 lambda j minus 1 that is you can denote it by this. So, this is the pi j minus 1 that is pi j minus 1 is this state probability into lambda j minus 1 that will be the probability rate that will a probability rate at which the strain entered to state j from state j m minus 1.

And let us interpret this probability that is pi j plus 1 is there. And from pi j plus 1 if there is a state it will go to state j. So, that way this is mu j plus 1. So, pi j plus 1 into mu j plus 1 this is the rate at which probability rate at which this state will change from j plus 1 to j. So, that way this is also a probability rate probability that it enters into state from j plus 1. Similarly, if we interpret these equation that is pi j into lambda j. So, in state j there may be a birth. So, we can write pi j into lambda j. And there may be a death also that is pi j into mu j. What the this global balance equation says, it says that at state j that probability rate of entering that is pi j into lambda j minus 1 plus mu pi j plus 1 into mu j plus 1 that is the probability rate of entering, is equal to probability rate of leaving the state. That is pi j into lambda j plus pi j into mu j. So, this is the global balance equation.

So, this global balance equation we can write for is step j plus 1 etcetera.

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This equation can be solved on the basis of following information.

(1) $\sum_{j=0}^{\infty} \pi_j = 1$

(2) At j=0, there cannot be further death

$\therefore \lambda_0 \pi_0 = \mu_1 \pi_1$

$\Rightarrow \pi_1 = \frac{\lambda_0}{\mu_1} \pi_0$

Substituting the value of π_1 in the global balance equation for state 2, we get

$(\lambda_1 + \mu_1)\pi_1 = \lambda_0 \pi_0 + \mu_2 \pi_2$

$\Rightarrow \pi_2 = \frac{\lambda_1}{\mu_2} \pi_1 = \left(\frac{\lambda_1}{\mu_2}\right)\left(\frac{\lambda_0}{\mu_1}\right)\pi_0 = \left(\frac{\lambda_0}{\mu_1}\right)^2 \pi_0$

In the same manner, $\pi_j = \left(\frac{\lambda_0}{\mu_1}\right)^j \pi_0$

Now, how to solve the global balance equation? Because, if we can solve this global balance equation then we can find the steady state probability at different states.

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This equation can be solved on the basis of following information.

(1) $\sum_{j=0}^{\infty} \pi_j = 1$

(2) At j=0, there cannot be further death

$\therefore \lambda_0 \pi_0 = \mu_1 \pi_1$

$\Rightarrow \pi_1 = \frac{\lambda_0}{\mu_1} \pi_0$

Substituting the value of π_1 in the global balance equation for state 2 (1), we get

$(\lambda_1 + \mu_1)\pi_1 = \lambda_0 \pi_0 + \mu_2 \pi_2$

$\Rightarrow \pi_2 = \frac{\lambda_1}{\mu_2} \pi_1 = \left(\frac{\lambda_1}{\mu_2}\right)\left(\frac{\lambda_0}{\mu_1}\right)\pi_0$

In the same manner, $\pi_j = \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i} \times \pi_0$

The global balance equation can be solved on the basis of following information. First of all, the sum of all state probabilities must be equal to 1. Therefore, summation pi j j going from 0 to infinity is equal to 1. Second information is at state j is equal to 0 there cannot be further death. So, there can be birth only. So, therefore, the balance equation will be equal to lambda 0 pi 0 is equal to mu 1 pi 1.

So, from which we get pi 1 is equal to lambda 0 by mu 1 into pi 0. So, given pi 0, we can pi 1. Now substituting the value of pi 1 in the global balance equation for state 1 we get now what will get because probability rate enter leaving state 1. That will be equal to pi 1 into lambda 1 plus mu 1 is equal to probability rate of entering that is lambda 0 pi 0 plus mu 2 pi 2. From this again we get that pi 2 is equal to lambda 1 by mu 2 into pi 1. Because here we can cancel lambda 0 pi 0 is equal to mu 1 pi 1 that we cancel. So, from which we get this relationship. Pi 2 is equal to lambda 1 by mu 2 into pi 1 and again if we substitute pi 1 will get this is equal to lambda 1 by mu 2 into lambda 0 by mu 1 multiplied by pi 0.

So, we have got the value of pi 2 in terms of pi 0. In the same manner we can continue and we can show that pi j is equal to product of lambda i minus 1 divided by mu i, i going from one to j multiplied by pi 0. So, this is the value of pi j in terms of pi 0. Now will substitute this value of pi j in this equation that sum of total probability is equal to 1.

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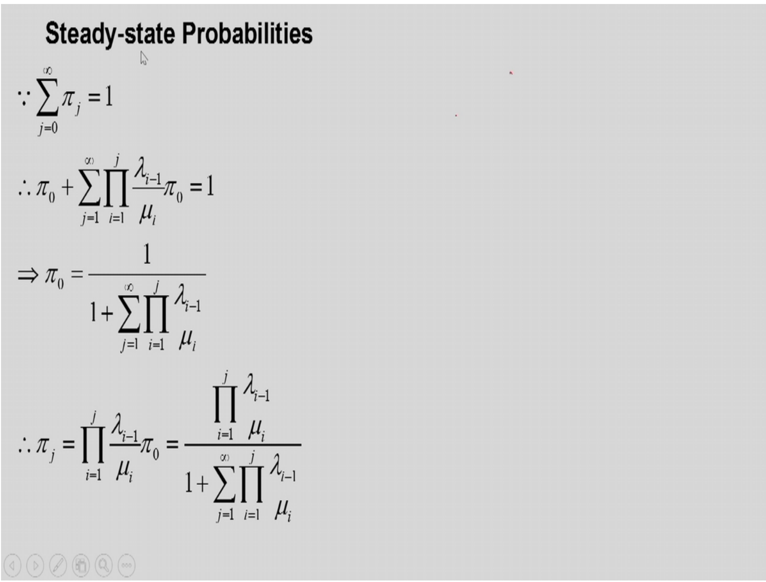

We get pi 0 plus this probability of pi j, j going from 1 to infinity. So, that must be equal to 1 and we have substituted pi j s product of lambda i minus 1 divided by mu i into pi not i is equal to i going from 1 to j. So, from this we get that pi not is equal to 1 by 1 plus that is pi not if I take common I will get pi is equal to 1 divided by 1 plus summation j going from 1 to infinity of the product lambda i minus 1 divided by mu i going from 1 to j.

So, now substituting this value now in pi j expression will get pi j that is equal to product of lambda i minus 1 minus mu i, i going from one to j into pi naught ok. And we have already got the value of pi naught. So, therefore, pi j is equal to product of lambda i minus 1 divided by mu i, i going from one to j divided by 1 plus summation j going from 1 to infinity of the product of lambda i minus 1 my mu i, i going from 1 to j. So, this is the expression for pi j. What is this steady state probability at state j that is given by this? So, it depends on this successive ratio of lambda i minus 1 divided by mu i.

So, we have derived this solution for the birth death process with the assumption of 0 death. That is at state 0 there will be no more death. And also this summation of the state probabilities must be equal to 1 from there we got a very elegant solution for the global balance equation, which gives pi j. Now we will apply this equation in the simple example.

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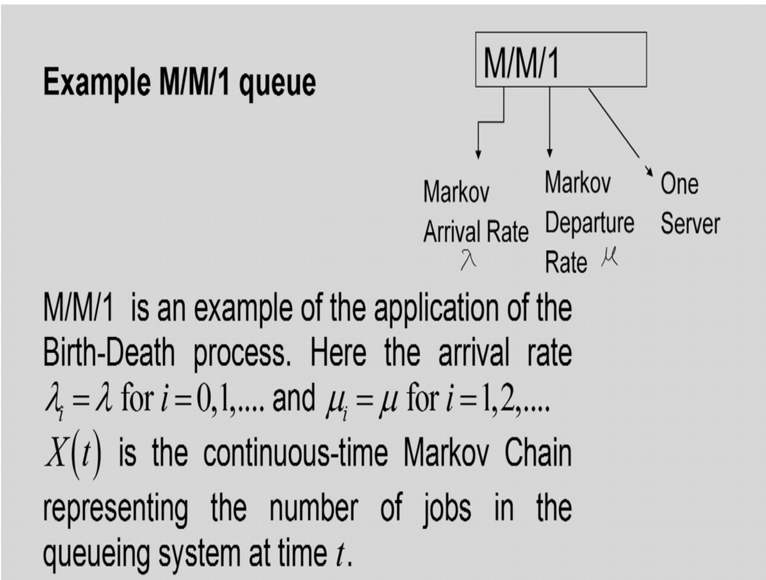

will consider one what is known as the m m 1 queue, m m 1 queue here it means that it is queuing system service system, where arrival rate is Markov's that is exponentially distributed. Similarly, departure rate is also Markov's that is also exponentially distributed and there is single server in the system.

Now, this m m 1 system is an example of the application of the birth death process. Here the arrival rate lambda is equal to lambda o assumed as constant arrival rate. So, arrival rate lambda is constant and mu i is equal to mu that is also constant. So, x t now that way x t is a continuous time Markov chain representing the number of jobs in the queuing system at any time t. So, we have a continuous time system, where arrival rate is constant and given by lambda departure rate is mu that is also constant and this is a one server system.

We can model this as a birth death process because arrival is birth and departure is death.

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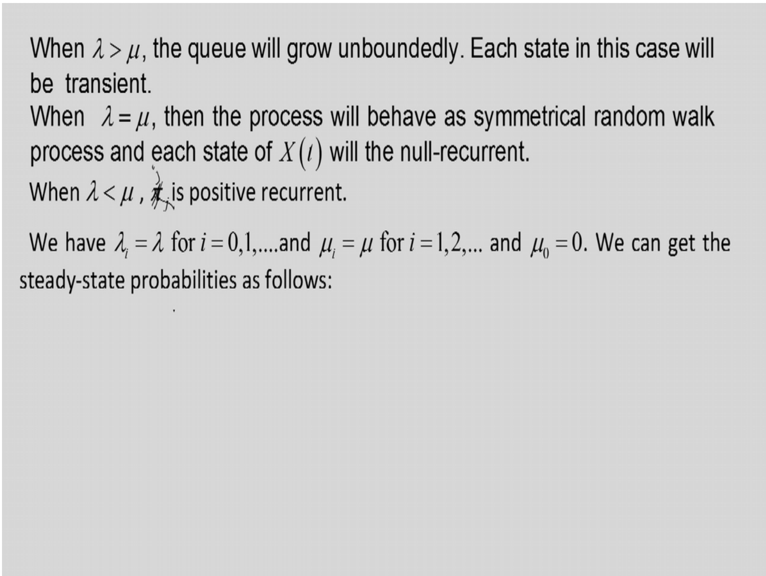

When lambda is greater than is mu the queue will grow unboundedly. Because we want to we are interested to find out the steady state probabilities, but if lambda is greater than mu d q will grow unboundedly is state in this case will be transient when lambda is equal to mu. So, that is special case then the process will behave as a symmetrical random walk. So, symmetrical random walk; that means, it will go there will be single birth with probability same probability suppose lambda and there will be single death with the same probability death that is lambda is equal to mu.

So, in that case it can be shown that this chain will be recurrent, but it is it will be null recurrent. That is average time of recurrent will be infinity. Now, steady state probability is will exists only for lambda less than mu. And in that case this state j will be positive recurrent. And also given that this m m 1 queuing system lambda i is equal to lambda or I is equal to 0 1 up to infinity and mu i is equal to mu for I is equal to 1 2 etcetera and mu 0 is equal to 0. So, with this conditions we can find out steady state probabilities let us see how to find it.

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Steady State Probabilities

$$\sum_{j=1}^{\infty}\prod_{i=1}^{j}\left(\frac{\lambda_{i-1}}{\mu_i}\right)=\sum_{j=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^{j} = \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}$$

$$\therefore \pi_j = \frac{\left(\frac{\lambda}{\mu}\right)^{j}}{1+\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}}, \quad j = 0,1,2,\ldots$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{j} = (1-\rho)\rho^{j}, \quad \rho = \text{utilization factor}$$

$\rho = \frac{\lambda}{\mu}$

$\pi_j = (1-\rho)\rho^{j}$

So, we know that this is the (Refer Time: 18:43) expression in the steady state probability that is the summation j going from 1 to infinity that is we are discussing this expression. So, that way we have this expression. So, summation j going from 1 to infinity of the product i going from one to j lambda i minus 1 divided by mu i.

Now, this is I know that lambda i minus 1 is equal to lambda n mu i is equal to mu. So, that way this is the summation from j is equal to 1 to infinity of lambda by mu to the power j. This product will be lambda by mu to the power j. This is a geometric series now and lambda by mu is less than 1, therefore, we have a infinite sum that is convergent and it is given by this is first term z is equal to lambda by mu divided by 1 minus common ratio that is lambda by mu.

So, this is lambda by mu divided by 1 minus lambda by mu. So, this summation is like this. Therefore, we can find out pi j that is steady state probability of state j will be equal to now we know that that is lambda product of lambda i minus 1 divided by mu i. So, that way this will become lambda by mu to the power j and 1 plus product of the same quantity.

So, therefore, pi j will be given by lambda by mu to the power j divided by 1 plus lambda by mu into 1 minus lambda by mu. Because this sum we know that is equal to lambda by mu divided 1 minus mu. So, will get here 1 plus this sum is replaced by lambda by mu divided by 1 minus lambda by mu. So, this is for j is equal to 0 1 2 etcetera. Now if I simplify this I will get that this is equal to 1 minus lambda by mu into lambda by mu to the power j. That is equal to 1 minus rho into rho to the power j now this quantity rho is equal to lambda by mu this is known as the utilization factor and therefore, this pi j steady state probability at state j is equal to 1 minus rho into rho to the power j.

And we observe that this is a pi j is equal to 1 minus rho into rho to the power j. So, this is the geometric probability. So, geometric probability distribution function. So, that way pi j is equal to 1 minus rho into rho to the power j. This is this probability mass function is distributed as the geometric distribution with parameter rho. So, this is and this is the steady state probability for the m m 1 queuing system. So, one in other words this is the probability that there are j jobs in the system eventually and this probability is equal to 1 minus rho into rho to the power j.

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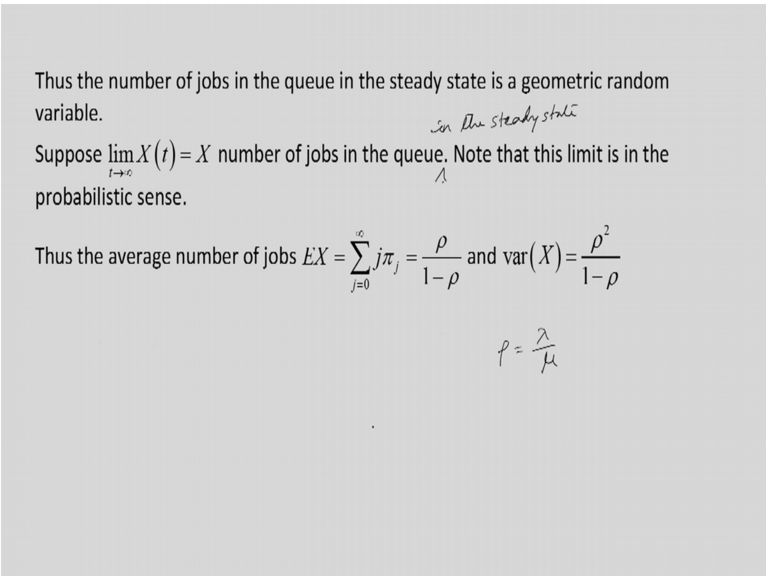

Thus the number of jobs in the queue in the steady state is geometric random variable. Now, suppose x t that is continuous time Markov's chain as t tends to infinity t is equal to x, that there is a limit that is the number of jobs eventually in the queue. So, number of jobs in the queue in the steady state in the steady state. Now note this limit this is a random variable and this is a random process. So, this limit is the in the probabilistic sense only may be main square sense or in probability like that that way we get a a limiting random variable x that represent the number of jobs in queue in the steady state. Now because the distribution of this random variable is geometric we can find out the average number of jobs that is x is equal to summation j pi j, j going from 0 to infinity.

In the same manner we can find out that mean of the geometric distribution is equal to rho divided by 1 minus rho. Similarly, variants of x is rho square divided by 1 minus rho. So, average number of jobs in the system is rho divided by 1 minus rho where rho is equal to lambda by mu this is the utility factor. So, this example illustrates that we can apply the birth death process to analyse m m 1 queue. Similarly, there are different queuing system will not discuss those things, but this gives an simple example of application of birth death process.

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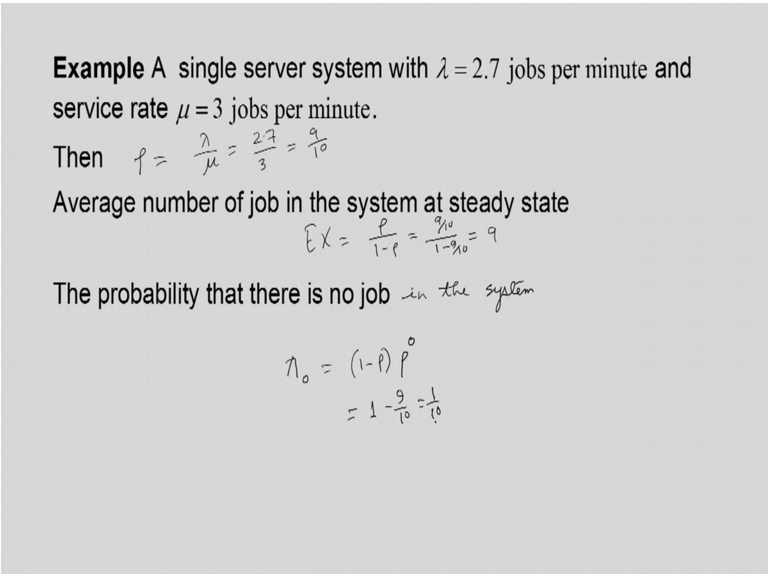

We will consider one numerical example suppose we have a single server system with lambda is equal to 2.7 jobs per minute and service rate mu is equal to 3 jobs per minute rho will be equal to lambda by mu that is equal to 2.7 divided by 3 that is equal to 9 by 10

So, average number of job in the at the steady state. So, that is equal to we know that e of x is equal to rho divided by 1 minus rho that is equal to 9 by 10 divided by 1 minus 9 by 10 that is equal to 9. So, the average number of jobs in the system at the steady state will be 9. Suppose, now you can find out the probability that there is no job in the system that is that counter is empty. So, this probability will be equal to, I know that pi 0 will be equal to 1 minus rho to the power rho to the power 0. That is equal to 1 minus 1 by 1 minus 9 by 10 that is equal to 1 by 10.

So, this is the probability that there will be no job in the system the counter will be empty therefore, pi 0 is equal to 1 by 10.

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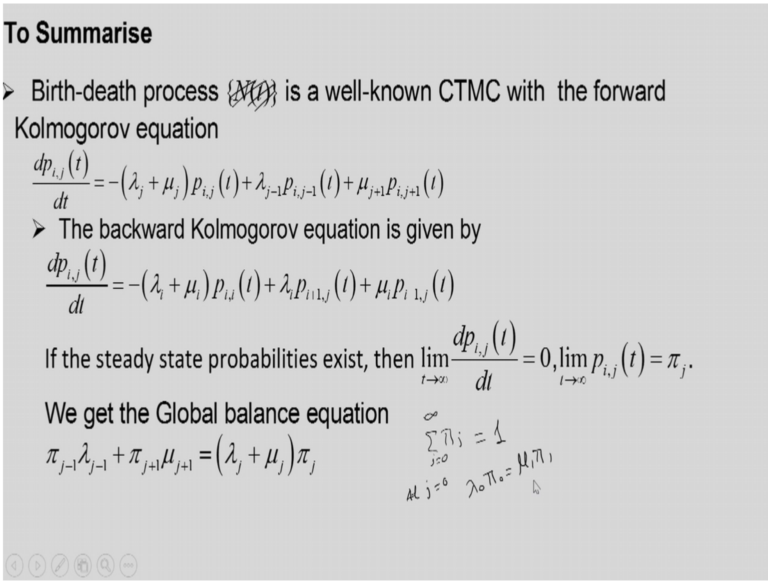

Let us summarize birth death process is a well known CTMC with the forward Kolmogorov equation do i j t dt is equal to lambda j plus minus of lambda j plus mu j into p i j t plus lambda j minus 1 into p i j minus 1 t plus mu j plus 1 into p i j plus 1 t. The backward Kolmogorov equation is similarly given by dp i j dt is equal to minus lambda i plus mu i into p i j t plus lambda i p i plus 1 j t plus mu i into p i minus 1 j t. So, this is the backward Kolmogorov equation.

If the steady state probability is exists then limit of dp i j dt is equal to 0 and limit of p i j t t tends to infinity is equal to pi j. So, using this values we got the global balance equation. That is equal to probability rate of entering a state is equal to probability rate of leaving the state. And that is given by pi j minus 1 into lambda j minus 1 plus pi j plus 1 into mu j plus 1 is equal to lambda j plus mu j into pi j. This is the global balance equation which can be solved with the condition that summation pi j, j is equal to 0 to infinity is equal to 1 and at j is equal to 0 lambda 0 pi 0 is equal to mu 1 pi 1.

So, using this 2 equation we can solve the global balance equation.

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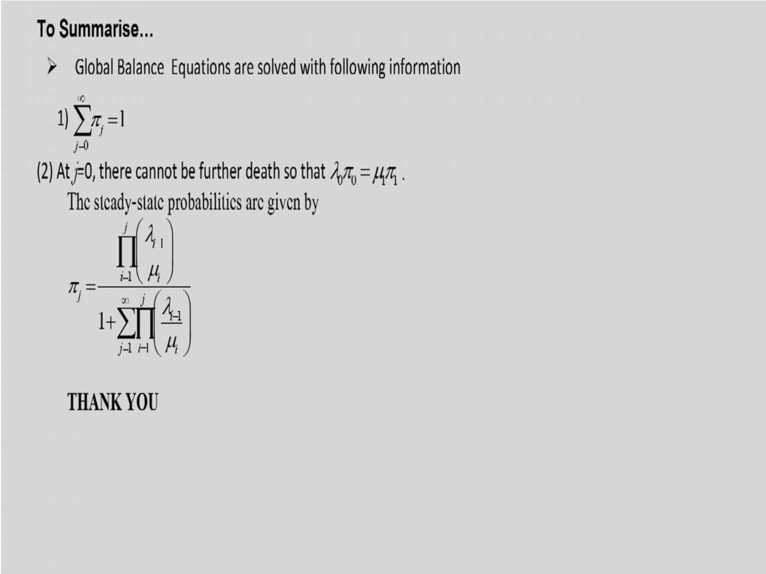

So, these are the conditions. So, we got the solution of the global balance equation using by this relationship. We also discussed one example of birth death system that is m m 1 queuing system.

Thank you.