Advanced Topics in Probability and Random Processes
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Lecture - 21
Continuous Time Markov Chain – 2

We will continue with Continuous Time Markov Chain, in the last lecture we covered about the dynamic behaviour of continuous time Markov chain, particularly we discussed about Kolmogorov forward and backward equation; so, Kolmogorov backward equation first equation.

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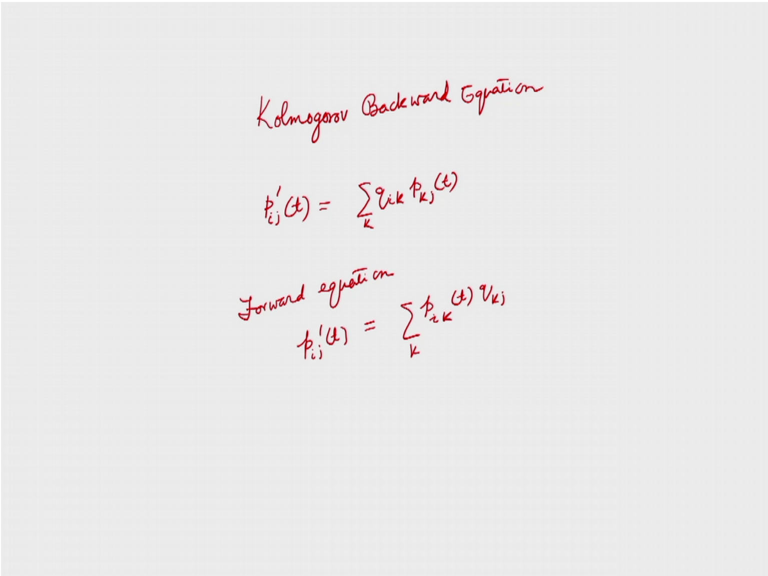

So, it says that P ij dash t first derivative of the transition probability from i to j, that is equal to we can write it as q i k summation over k into p k j t. Similarly, this is the backward equation forward equation also we can write down. P i j dash t so, this will first go to t that is p i k t into q k j. So, that way these 2 equation we are established.

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Example Poisson process $\{N(t)\}$

Suppose the Poisson process has entered state i at time 0. It remains in same state until the next arrival with $T_i \sim \exp(\lambda)$. Once an arrival takes place, the state become i+1

Thus, for $j \neq i$, $P_{i,j} = \begin{cases} 1, & j = i+1 \\ 0 & \text{otherwise} \end{cases}$

$p_0(0) = 1$

Further, $N(0) = 0$ with probability 1. The transition rates are given by

$q_{i,j} = \lambda P_{i,j}$

$\Rightarrow q_{i,i+1} = \lambda, q_{i,i} = -\lambda, q_{i,j} = 0, \ j \neq i, i+1$

Now, let us see some example will consider the Poisson process. So, this is a very important process, Poisson process N t, suppose the Poisson process has entered to state i at time is equal to 0. So, at time 0 it is in state i suppose, it remains in the same state until the next arrival with T i which is exponentially distributed. We know that in the case of continuous time Markov chain. So, the state will enter to a state i and it will remain in that state i in a time random time which is distributed as exponential distribution.

Once an arrival takes place this state becomes i plus 1, it was in state i, it will becomes i plus 1 because, 1 arrival and it is a counting process an so, the count will go up that way this state will become i plus 1. Thus for j not equal to i, we can write that P ij that is their probability of the embedded Markov chain that is equal to 1 if j is equal to i plus 1 and is equal to 0 otherwise because it takes only i plus 1 so, that way it is 1 for i plus 1 and for rest it is 0. So, this is the transition probability for the embedded Markov chain.

A further we left with assume that the counting process N t has started with 0 count, N 0 is equal to 0 with probability 1. Now, this we can write this means that probability of 0 at time 0 is equal to 1, probability of any other state at time 0 will be equal to 0. Now, the transition rates are given by q ij is equal to lambda times P ij because, this is lambda ij nu here. So, that way q ij is equal to lambda into P ij therefore, q i i plus 1 will be lambda, q i i will be minus lambda. Because, we know that any for any continuous time strain q i i is by definition that is equal to minus nu i in discuss minus lambda and q ij is equal to 0 for j not equal to i and i plus 1.

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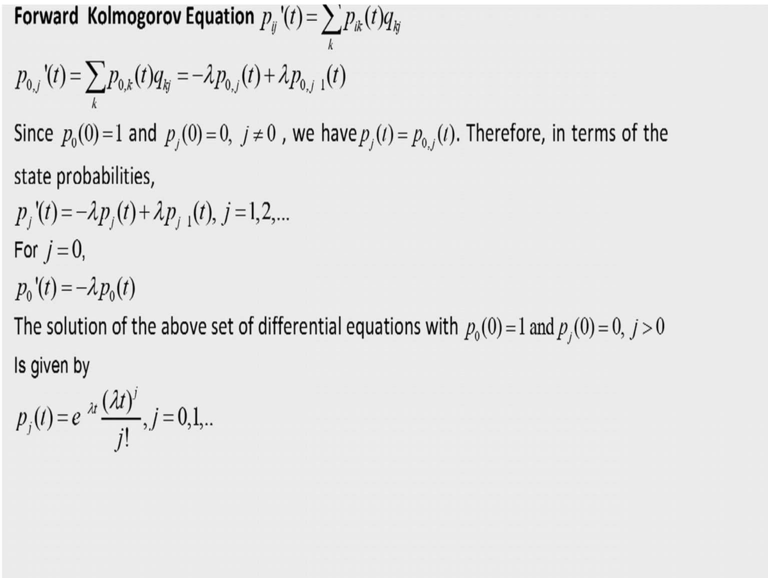

So, now we know that forward Kolmogorov equation is p ij dash t is equal to summation over k p i k into q k j that is the forward Kolmogorov equation we are writing. So, in this case we are starting with state 0. So, that way p 0 j dash t that will be equal to summation over k p 0 k t into q k j that is equal to we have already found out the value for q k j so, that rate is minus lambda times p 0 j t plus lambda times p 0 j minus 1 t. So, this is the differential equation for the Poisson process.

Now, since p 0 is 0 is equal to 1 and p j 0 is equal to 0 for j not equal to 0 we have we can write this transition probability as the state probability therefore, p j t will be equal to p 0 j 0 j t. So, in terms of state probability we can write p j dash t that is the first derivative of the state probability is equal to minus lambda times p j t plus lambda times p j minus 1 t for j is equal to 1, 2 etcetera. And for j is equal to 0 so, this term will not be there so, it will be p 0 dash t is equal to minus lambda p p 0 t.

Now, this is a set of 2 linear differential equations and these equations can be solved with the initial condition to find out the state probability p j t. So, what is the initial condition? P 00 is equal to 1 and p j 0 is equal to 0 for j greater than 0.

So, that way with this initial condition first we can solve this, this is a fairly simple linear and differential equation. And with this solution now this solution this equation can be solved with the principle of mathematical induction. So, that way this solution will be given by p j t. So, therefore, p j t will be equal to e to the power minus lambda t into lambda t to the power j divided by factorial j for j is equal to 0, 1 etcetera.

Now, this is the Poisson distribution. So, in the case of Poisson process this state probability is a Poisson random Poisson probability with that average term lambda t. So, a Poisson process is a very important random process and it can be derived using some other postulates. Basically, here we have derived the Poisson process you assuming as a continuous time Markov chain, but it can be derived using some postulates, this postulate also has a state.

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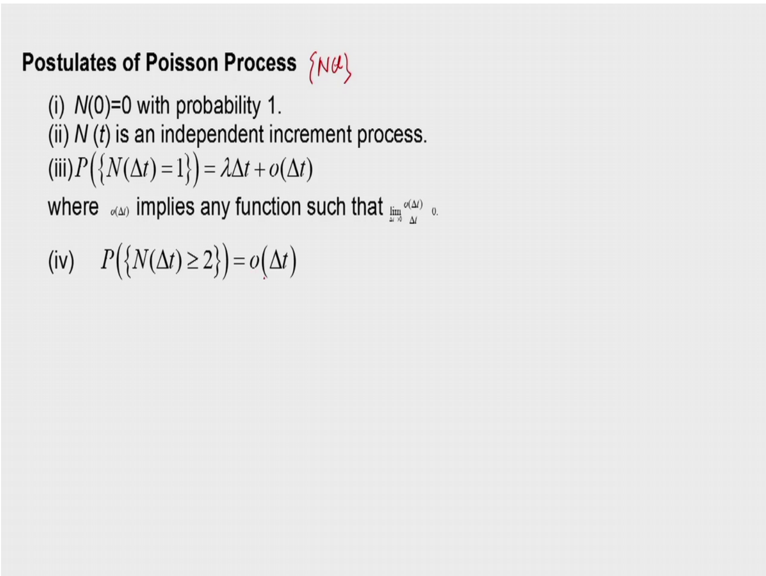

That Poisson process N t suppose postulates of Poisson process N t. First of all N 0 is equal to 0 with probability 1, second postulate is N t is an independent increment process increment are independent. Now, probability of arrival, single arrival at time delta t, this is the probability of N delta t is equal to 1. This is equal to lambda times delta t plus o delta t. So, this quantity is a small quantity so, that limit of o delta t by delta t as delta t tends to 0 will become 0. So, this is the implication of o delta t.

And 4 postulate is that the number of arrival more than 2 in small interval delta t that is very small that is o delta t. So, with these 4 postulates also we arrive at the same set of differential equations and derive the state probability for the Poisson process.

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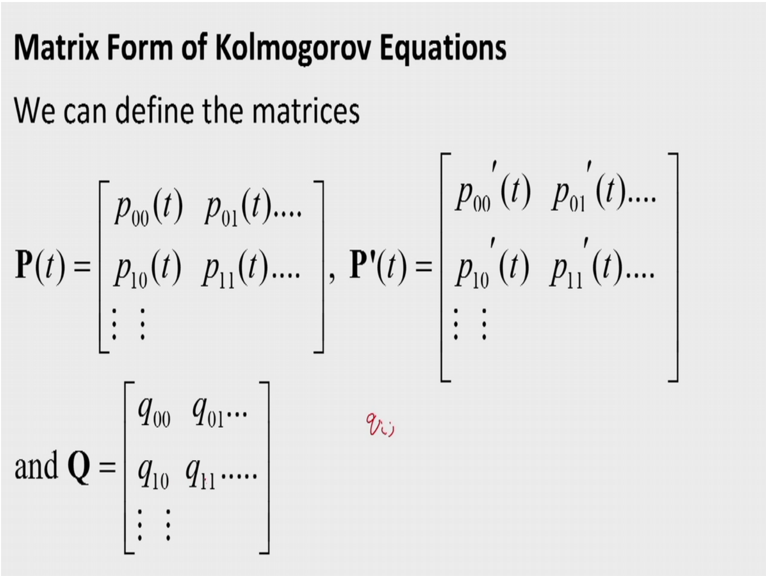

Matrix form of Kolmogorov equations, we can define the matrices P t. That is the step transition probability matrix suppose this is the for e state we have a low p 00 t, p 01 t etcetera like p 10 t, p 11 t like that. Similarly we can find out the corresponding derivative that we call it as derivative matrix. So, that way this will be p 00 dash t, p 01 t like that corresponding to each state we will have a rho.

And also we define the rate matrix so, Q, Q is the rate we have already defined that q ij so, it is the matrix of q i j. That is the probability rate at which strain leaves state i to enter state j. So, it comprises of q 00 corresponding to each state we will have a rho q 00, q 01 like that q 10, q 11 etcetera.

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Matrix Form of Kolmogorov Equations

Tin matrix form, the Kolmogorov backward and forward equations can be written as

$$\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$$

and $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$

Example A certain system has two states – under operation state 1 and under repair state 0. The duration of operation and repair are exponential RVs with rate parameters λ and μ respectively.

Find $P(t) = \begin{bmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{bmatrix}$

So, backward equation now backward and forward equation we can write back equation P dash t is equal to Q times P t and P this is the backward equation, forward equation is P dash t is equal to P t into Q. So, Q is the probability rate matrix; so, consisting of the q i j elements which we have defined earlier.

We will consider another example a certain system has two states, one of type of states under operation state 1 and under repair state 0. The duration of operation and repair are exponential random variables with rate parameters lambda and mu respectively. So, we have to find out P t matrix that is equal to p 00 t, p 01 t, p 10 t, p 11 t this is the matrix we have to find out.

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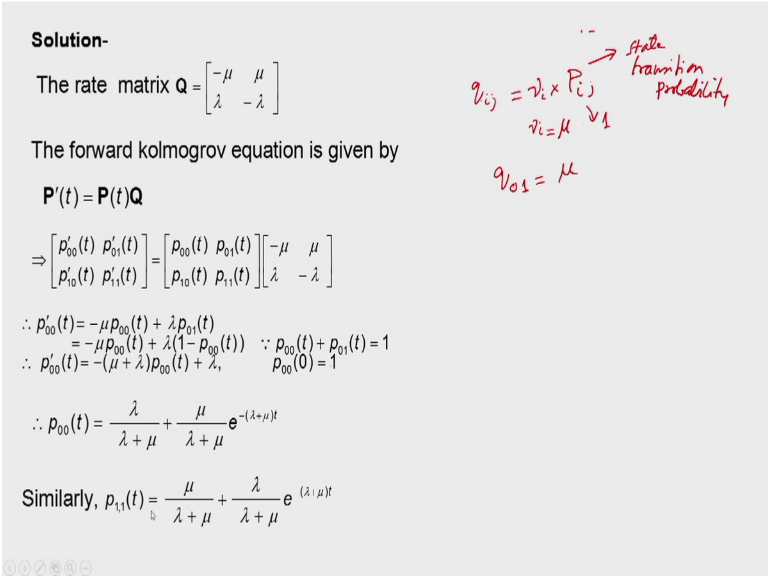

Now, in this case we will solve this problem, the rate matrix Q is equal to minus mu mu lambda minus lambda. So, how we get this? We know that q i j is equal to mu times mu i times capital P ij, where this is the state transition probability for the embedded Markov chain.

So, now in this case I know that suppose it is under repair. So, that way 0 is the repairing state so, that way nu i is equal to mu. And this quantity p ij once it is repaired it will be going to operation state with probability 1 therefore, p ij is equal to 1. Therefore this is mu and this quantity is 1. So, that way this q i, q 01 will be equal to mu. So, that way q 01 is equal to mu and q 0 0 that I know that that is minus nu i so, that way this is minus mu.

Similarly, this matrix this element will be lambda and this element will be minus lambda so, rate matrix Q is given by this. Now, if we write the forward Kolmogorov equation then this differential matrix will be state transition probability matrix multiplied by this Q matrix, rate matrix ok. So, from this matrix equations we can get equation for corresponding to e state for example, first we will get p 00 dash t will be equal to minus mu times p 00 t plus mu times p 01 t. Now, we have also since these are transition probabilities sum of p 00 t and p 01 t is equal to 1 p 00 t plus p 01 t is equal to 1.

So, this is one condition also we know that this is initially several p 00 is 0 p 00 at time 0 is equal to 1. So, with this initial condition I can solve this because I from here I will substitute p 01 t as 1 minus p 00 t. So, I will get this differential equation which can solved now. So, we will get the solution this is a first order linear differential equation therefore, solution is given by this. Similarly, if we consider suppose p 10 t and p 11 t we will get this solution p 11 t is equal to mu, where lambda plus mu plus lambda by lambda plus mu into e to the power minus lambda plus mu t. So, that way we have got p 00 t and p 11 t and remaining elements we can find out p 01 t that is equal to 1 minus p 00 0 t similarly, p 10 t will be equal to 1 minus p 11 t.

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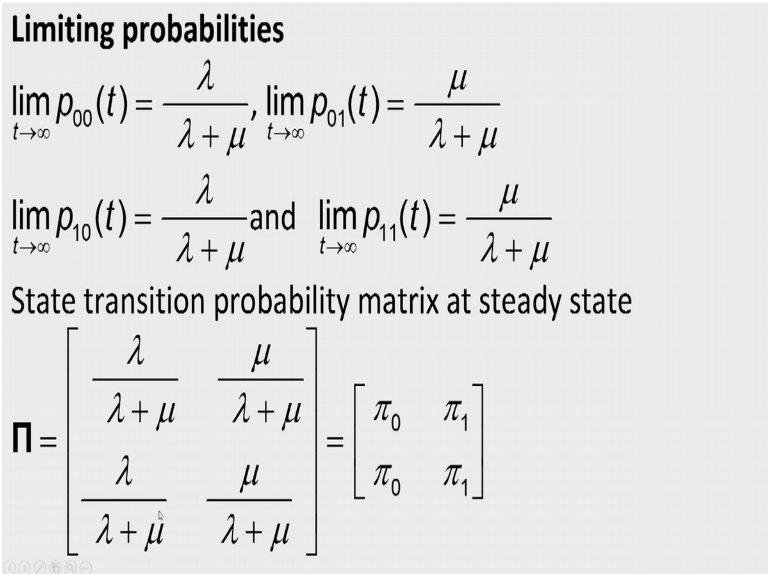

So, that way we have the solution; now let us see what happened here we have seen that this is exponential with negative term. So, that way this term will always decay we will have these steady state probabilities. So, what we will get is that let us find out limit of p 00 t as t tends to infinity. So, this quantity will be lambda by lambda plus mu and similarly p 01 t that will be equal to mu by lambda plus mu.

Again if we see p 10 t that is equal to lambda by lambda plus mu and p 11 t as t tends to infinity that will be also mu by lambda plus mu; that means, for state 1, this probability limit of p 01 t is also mu by lambda plus mu and limit up p 11 t also it is mu by lambda plus mu.

Similar is the case for 0 p 00 t as t tends to infinity is lambda by lambda plus mu, p 10 t also as t tends to infinity lambda plus lambda by mu. That way we get the state that is limit of this state transition probability matrix, that is equal to lambda by lambda plus mu, mu by lambda plus mu and second row is lambda by lambda plus mu, mu by lambda plus mu. That we can denote it by pi 0 pi 1, pi 0 pi 1. Now, since this is pi 0 pi 1 and pi 0 pi 1 just like in the case of CTMT we can show that the limiting state probabilities will be pi 0 pi 1. So, it is it will be independent of the initial state probabilities.

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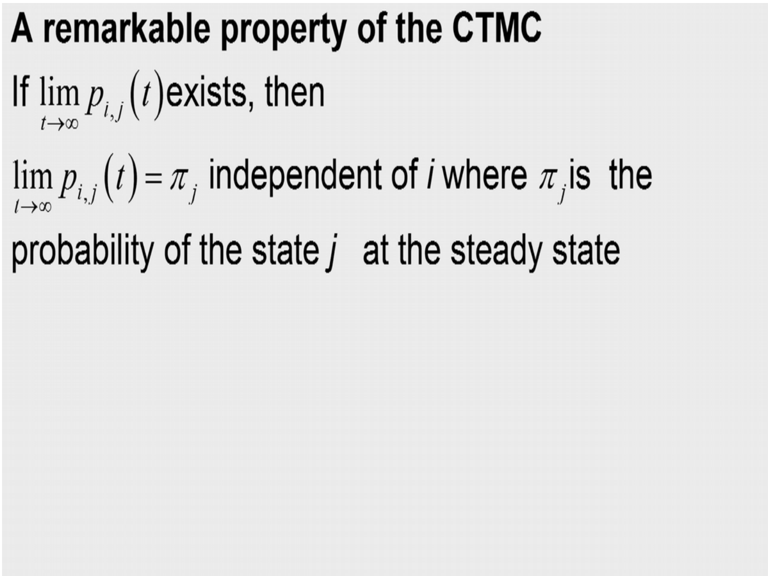

We will state a remarkable property of this CTMC, if limit of p i j t as t tends to infinity exists then a limit of p i j t t tends to infinity is equal to p i j, it is independent of i where p i j is the probability of this state at the steady state. So, steady state probability will be equal to p i j. So, this is markable if this limit exists p i j t as t tends to infinity then that limit will be the state probability of state j.

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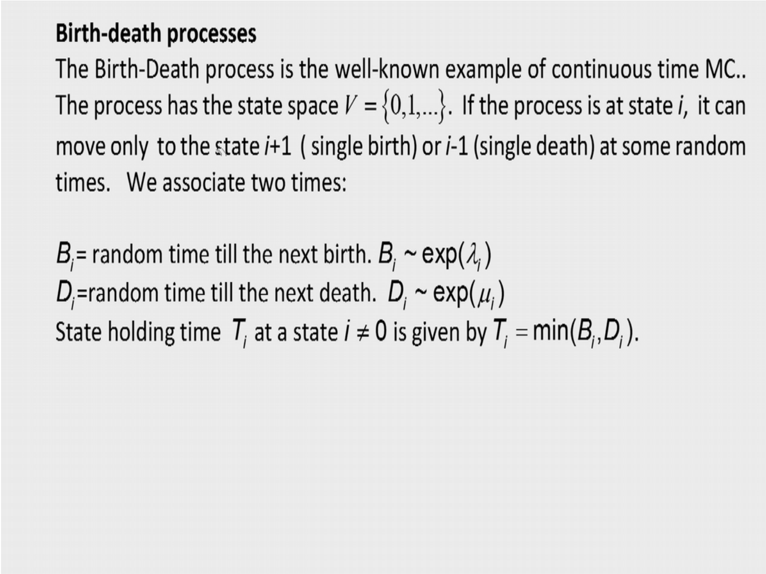

Now, we will consider a special class of continuous time Markov chains that is birth-death processes. The birth death process is the well known example of CTMC. the process has the state space that is starting with 0, 1 etcetera. If the process is at state i it can move only to state i plus 1 that is single birth or i minus 1 single death at some random times. We associate 2 times, that is B i random time till next birth. We assume that B i is exponential with parameter lambda i similarly, D i that is random time till the next death D i is also assumed to be exponential with parameter mu i.

Now, we can find this state holding time t i for this process. So, t i is given by t i is equal to minimum of B i D i because, one state will change if either 1 birth occurs or 1 death occurs. So, whichever time is minimum at the time state change will occur. Therefore, state holding time t i at time i not equal to 0 is given by t i is equal to minimum of B i D i. And at state i is equal to 0 it will be simply lambda i because, there cannot be any death.

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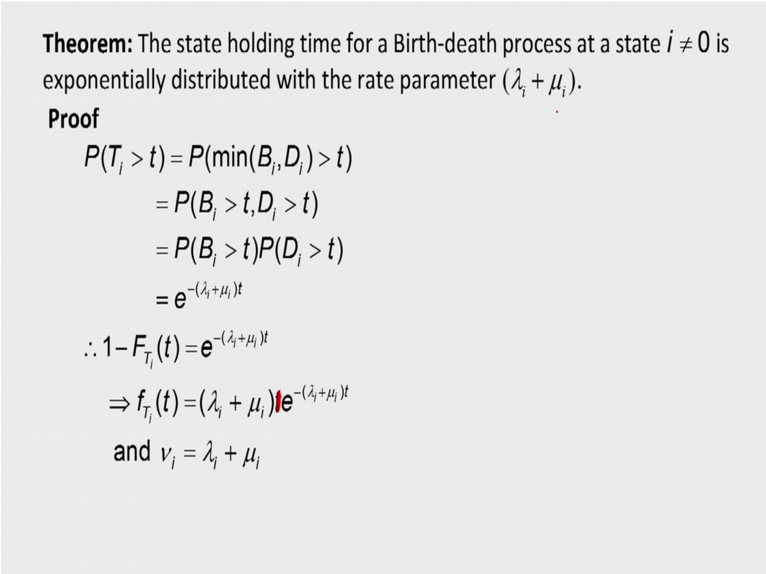

We will state a theorem, this state holding time of a birth death process at state i not equal to 0 is exponentially distributed with the rate parameter lambda i plus mu i. So, state holding time here it is exponential with parameter lambda i plus mu i. We will find out what is probability of T i greater than t that is equal to probability of minimum of B i D i greater than t.

If minimum is greater than t that implies that B i is also greater than t, D i is also greater than t. So, that way this will be equal to probability of B i greater than t and D i greater than t and D i greater than t and birth and death are independent. So, that way will birth time and death time are independent. So, that way will write this is probability of B i greater than t into probability of D i greater than t. Now, this I know that this is equal to e to the power minus lambda i t and this quantity is equal to e to the power minus mu i t. So, that way it will be equal to e to the power minus lambda i plus mu i into t.

So, this is the complementary CDF actually therefore, we get that 1 minus f t i t is equal to e to the power minus lambda i plus mu i t. And if we differentiate we will get f t i t is equal to lambda i plus mu i into into e to the power minus lambda i plus mu i t this is distributed exponentially with the parameter nu i is equal to lambda i plus mu i. So, that way we have proved that this state holding time of a birth death process is exponentially distributed with parameter nu i is equal to lambda i plus mu i.

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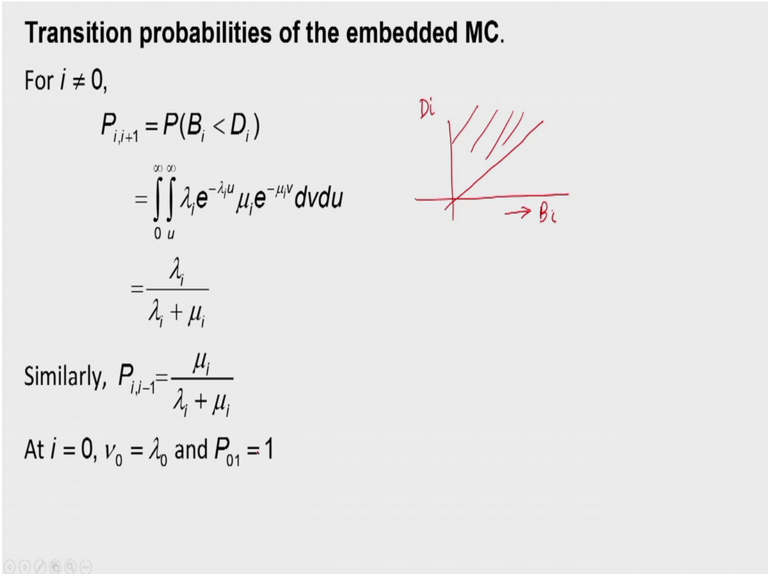

Now, we have to find out the transition probabilities of the embedded Markov's chain. P i i plus 1 because, from state i it can go to i plus 1 or it can go to go down to i minus 1 if there is a death. So, that way probability of i i plus 1 will be same as probability of B i that is birth occurs, before death before single death single birth occurs.

So, that way this we can write now because if I consider this suppose this is my B i, this is my D i. So, that way now will find the transition probabilities of the embedded Markov chain. That is we have to find out P that is capital P i i plus 1 and P of i i minus 1. Now, there will be transition to i plus 1 if there is a birth; that means, birth time is earlier than that time. So, that way this if I put this is the birth axis and this is the date axis then this region will be death time is bigger than birth time.

So, that way we can now integrate because both birth time and death time are independent. So, that we have used this independent property and then integrate suppose d v from u to infinity and d u from 0 to infinity. So, in this region you will this is my u, u will go from 0 to infinity and this is my v, v will go from u to infinity. That way we are integrating and if we and this is an exponential integration is also easy and we will get this is equal to lambda i divided by lambda i plus mu i. So, this is the P i i plus 1.

Similarly, P i i minus 1 will be equal to mu i divided by lambda i plus mu i that is equal to P i i minus 1. Now, at state i is equal to 0 nu 0 is equal to lambda 0 and from state 0 there can be a single birth only that way P 01 is equal to 1. So, these are transition probabilities for the embedded Markov chain.

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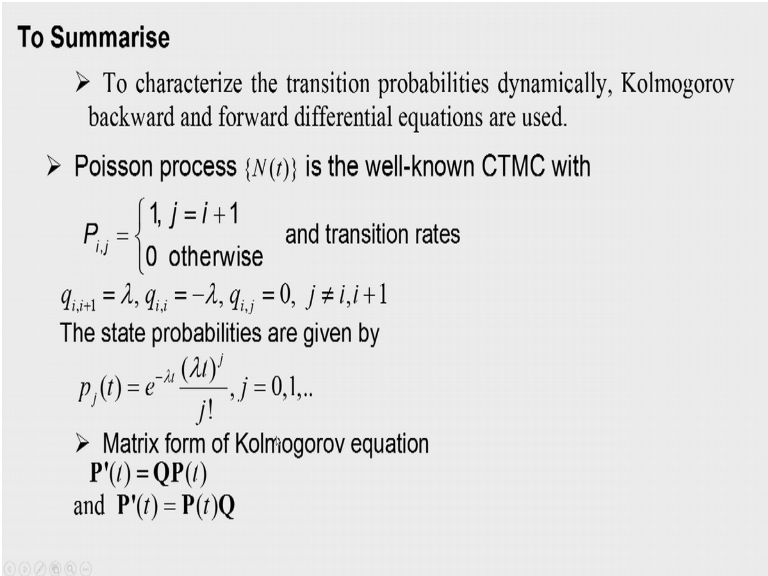

Let us summarize the lecture to characterize the transition probabilities dynamically Kolmogorov backward and forward differential equations are used. And for Poisson process this is the very well known process and for this process from this transition probability embedded Markov chain is given by P i j is equal to 1 for j is equal to i plus 1 is equal to 0. Otherwise and transition rates are given by q i i plus 1 is equal to lambda q i i is equal to minus lambda and q i j is equal to 0 for j not equal to i i plus 1.

So, with the assumption that N 0 is equal to 0 and this transition rates we get the state probabilities at the state j by this expression P j t is equal to e to the power minus lambda t into lambda t to the power j divided by factorial j, j is equal to 0 1 etcetera. So, this is the state probability of the Poisson process. So, what is the probability that there will be j numbers at time t. Also we discussed about matrix form of Kolmogorov equation, that is P matrix, P t matrix, differentiation of P t matrix is equal to Q times P t. Similarly, P dash t differentiation of the probability transition matrix that is equal to P t times Q this is the forward Kolmogorov equation; this is the backward Kolmogorov equation.

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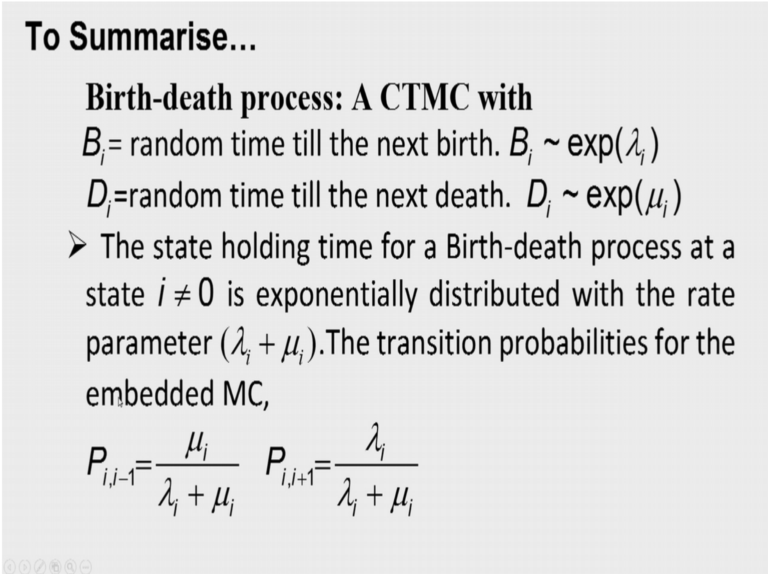

We also introduced birth death process a CTMC with B i birth death random time till the next birth, that is B i is exponentially distributed with parameter lambda i. D i random time till the next death D i is also exponential with parameter mu i. This state holding time for a birth death process at a state i not equal to 0 is exponentially distributed with parameter lambda i plus mu i.

Also the transition probabilities for the embedded Markov chain is given by P i i minus 1 is equal to mu divided by lambda i plus mu i and P i i plus 1 is equal to lambda i divided by lambda i plus mu i.

Thank you.